Name: ______________________________

MATH 3310
FINAL EXAM
FALL 2018


DEPARTMENT OF MATH AND STAT
TEXAS TECH UNIVERSITY

## Part I

*Solve seven (7) of problems 1–8 below; they count for 4% each. Your answers must be written on this sheet of paper. No aids are allowed on this part of the test. When you have turned in this part, you may use books and notes to solve the problems on Part II. For full credit, you must show complete, correct, legible work. Read carefully before you start working.*

1. Give counter examples to prove that the following statements are false:

   (a) For all real numbers $x, y \geq 0$, one has $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   (b) Let $A$ be a set. If $\{x, y\} \in \mathcal{P}(A)$, then $\{x, y\} \notin A$.

2. Mark with a ✓ each statement that is proved by the following argument.

   If $a$ is odd, then one has $a = 2n + 1$ for some $n \in \mathbb{Z}$ and hence

   $$(a+1)^2 = (2n+2)^2 = (2(n+1))^2 = 2(2(n+1)^2)\ .$$

   As $2(n+1)^2$ is an integer, it follows that $(a+1)^2$ is even.

   (a) Let $a$ be an integer. If $(a+1)^2$ is odd, then $a$ is even.

   (b) Let $a$ be an integer. If $a$ is odd, then $(a+1)^2$ is even.

   (c) Let $a$ be an integer. The integer $(a+1)^2$ is even if and only if $a$ is odd.

   (d) None of the above.

3. Prove the following:

   *Claim:* Let $r, i \in \mathbb{R}$. If $r$ is rational and $i$ is irrational, then $r + i$ is irrational.

4. Complete the truth table

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $P \vee Q$ | $(P \vee Q) \wedge \sim P$ | $(P \vee Q) \wedge \sim P \Rightarrow \sim Q$ |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

5. On the set $S = \{w, x, y, z\}$ consider the relation:

$$R = \{(x, x), (y, y), (w, z), (z, w)\}.$$

   (a) Is $R$ reflexive?
   (b) Is $R$ symmetric?
   (c) Is $R$ transitive?
   (d) Is $R$ a function?

6. For $n \in \mathbb{N}$ set $n\mathbb{Z} = \{nz \mid z \in \mathbb{Z}\}$. Among the sets

$$2\mathbb{Z}\,, \quad 6\mathbb{Z}\,, \quad \text{and} \quad 48\mathbb{Z}$$

some are subsets of others. Which ones?

7. Consider the following claim: *Let $n \in \mathbb{N}$. If $n$ is a prime and $n > 2$, then $n$ is odd.*

   (a) What should the assumption be in a direct proof?
   (b) What should the assumption be in a proof by contrapositive?
   (c) What should the assumption be in a proof by contradiction?

8. Let $n$ be an integer. Below you will find two claims. Determine if they are logically equivalent.

   (a) If $n$ is odd, then $(n + 1)^2 + n + 1$ is even.
   (b) The integer $n$ is even or the quantity $(n + 1)^2 + n + 1$ is even.

## Part II

*Solve five (6) of the problems 9–15; they count for 12% each. Your solutions to these problems must be written in a blue book. Turn in your solutions to five problems only!*

*Books, notes, and old homework are allowed aids on this part of the test. Calculators are also allowed, but phones, PDAs, Apple watches etc. are not.*

9. Let $n \in \mathbb{N}$. A relation $\mathrm{R}_n$ on $\mathbb{Z}$ is defined as follows: $x \, \mathrm{R}_n \, y$ if and only if $n | xy$.
   (a) Prove or disprove: $\mathrm{R}_6$ is an equivalence relation on $\mathbb{Z}$.
   (b) Prove or disprove: $\mathrm{R}_7$ is an equivalence relation on $\mathbb{Z}$.

10. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + x^2$.
   (a) Is $f$ 1-to-1?
   (b) Is $f$ onto?

11. Consider functions $f, g\colon X \to Y$ and $h\colon Y \to W$. Prove the following statement.

   *If $h \circ f = h \circ g$ and $h$ is 1-to-1, then $f = g$.*

12. Consider the relation R on $\mathbb{Q}$ given by: $q \, \mathrm{R} \, r$ if and only if $q - r \in \mathbb{Z}$.
   (a) Show that R is an equivalence relation.
   (b) Show that every rational number belongs to the equivalence class $[q]$ for some rational number $q$ with $0 \le q < 1$.
   (c) Show that addition $[q] + [r] = [q + r]$ is a well-defined operation on equivalence classes.

13. Consider the function $f\colon \mathbb{N} \to \mathbb{Q}$ given by

$$f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} \, .$$

   (a) Compute the function value $f(n)$ for $1 \le n \le 4$.
   (b) Conjecture a closed form expression, i.e. a formula, for the function values $f(n)$.
   (c) Use induction to prove the conjectured expression.

14. Set

$$a_1 = 1\,, \quad a_2 = 2\,, \quad \text{and} \quad a_n = \frac{a_{n-1}}{a_{n-2}} \text{ for } n \geq 3\,.$$

   (a) Compute $a_n$ for $3 \leq n \leq 6$.

   (b) Prove that

$$a_n = \begin{cases} 1 & \text{if } n \equiv_6 1 \text{ or } n \equiv_6 4 \\ 2 & \text{if } n \equiv_6 2 \text{ or } n \equiv_6 3 \\ \frac{1}{2} & \text{if } n \equiv_6 0 \text{ or } n \equiv_6 5 \end{cases}$$

   for every $n \in \mathbb{N}$.

15. Let $S$ be a subset of $\mathbb{R}$. A en element $l \in \mathbb{R}$ is called a *lower bound for* $A$ if $l \leq x$ holds for all $x \in S$. A lower bound $g$ is called a *greatest lower bound* for $S$ if $l \leq g$ holds for every lower bound of $S$.

   (a) Show that not every subset of $\mathbb{R}$ has a lower bound.

   (b) Give an example of a subset of $\mathbb{R}$ that has a lower bound and identify one.

   (c) Show that if a set has a greatest lower bound, then it is unique.