Name: ________________________________

Math 3310
Exam 2
Fall 2018

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Part I

Solve problems 1–5 below; they count for 8% each. Your answers must be written on this sheet of paper. No aids are allowed on this part of the test. When you have turned in this part, you may use books and notes to solve the problems on Part II.

1. Give counterexamples to the following statements:
   (a) $\forall x \in \mathbb{R}, x = |x^2|$.
   (b) For irrational numbers p and q, the sum $p+q$ is also irrational.

2. Mark with a ✓ each statement that is proved by the following argument.

   Let l, m, n be integers with $l^3 = m^3 + n^3$. Multiplication by 8 yields $8l^3 = 8m^3 + 8n^3$ and hence $(2l)^3 = (2m)^3 + (2n)^3$.

   (a) There are infinitely many triples (x, y, z) of integers with $x^3 = y^3 + z^3$.
   (b) If there exists a triple (x, y, z) of integers with $x^3 = y^3 + z^3$ then there are infinitely many such triples.
   (c) None of the above.

3. Consider the relation on $\mathbb{Z}$ given by

$$R = \{(1,1), (2,2), (5,5), (2,5), (1,2)\}.$$

   Is it
   (a) Reflexive?
   (b) Symmetric?
   (c) Transitive?

4. For each of the following sets decide if it has a least element, and if it has one identify it.

   (a) $\mathbb{Z}$.

   (b) $\{5n + 11 \mid n \in \mathbb{N}\}$.

   (c) $[-4, \infty)$.

   (d) $\mathbb{N}$.

5. Here is an induction argument.

   One has

   $$2^1 = 2 > 1\,.$$

   Let $k \in \mathbb{N}$ and assume that $2^k > k$ holds. One then has

   $$2^{k+1} = 2(2^k) > 2k = k + k \geq k + 1\,.$$

   Identify the following elements of the argument:

   (a) induction start,

   (b) induction hypothesis,

   (c) induction step,

   and decide which statement is being proved.

Part II

Solve five (5) of the problems 6–11; they count for 12% each. Your solutions to these problems must be written on blank pages or in a blue book. Turn in your solutions to five problems only!

Books, notes, and old homework are allowed aids on this part of the test. Calculators are also allowed, but phones, PDAs, Apple watches etc. are not. For full credit, you must show complete, correct, legible work. Read carefully before you start working.

6. Let a, b, c, d be real numbers. Show that at most four of the numbers ab,ac,ad,bc,bd, and cd are negative.

7. Let $n \geq 2$ be an integer and $A_1, A_2, \ldots, A_n$ be sets. Use mathematical induction to prove the equality
$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}\,.$$

8. Show that
$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$$
holds for all $n \in \mathbb{N}$.

9. Recall that for a natural number n the factorial $n!$ is the product $1 \cdot 2 \cdots n$. Show that
$$2^n < n!$$
holds for all integers $n \geq 4$.

10. Show that the equation
$$x^3 + 2x + -8 = 0$$
has a unique solution in the interval $[1, 2]$.

11. On the set $A = \{a, b, c, d\}$ consider the relation
$$R = \{(a,a), (a,b), (b,b), (b,a), (a,c), (c,c), (c,d), (d,c)\}$$
   (a) Which element can be added to R to make it a reflexive relation?
   (b) Which element can be removed from R to make it symmetric?
   (c) Which element can be added to R to make it symmetric?
   (d) Which two elements can be removed from R to make it transitive?