MATH 3310 PROBLEM 6.44

KEY

(a) $F_1 = 1 = F_2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
(b) One has $2|F_n$ if and only if $3|n$.

Proof. By induction. As one has $F_1 = 1 = F_2$ and $F_3 = 1 + 1 = 2$, the statement is true for natural numbers $n \leq 3$.

Let $k \geq 3$ and assume that the statement is true for all natural numbers $n \leq k$.

First assume that $2|F_n$, that is, F_n is even. It follows from the recursion formula $F_n = F_{n-1} + F_{n-2}$ that F_{n-1} and F_{n-2} have the same parity. It is by Ecercise 5.14 not possible that $3|n-1$ and $3|n-2$, so by the induction hypothesis both F_{n-1} and F_{n-2} are odd and 3 divides neither $n-1$ nor $n-2$, whence 3 divides n.

Next assume that F_n is odd. It follows from the recursion formula $F_n = F_{n-1} + F_{n-2}$ that F_{n-1} and F_{n-2} have opposite parity. Thus F_{n-1} or F_{n-2} is even, and it follows by the induction hypothesis that 3 divides $n-1$ or $n-2$. In particular, 3 does not divide n.

By the Strong Principle of Mathematical Induction, one has $2|F_n$ if and only if $3|n$ for all $n \in \mathbb{N}$. □