Name: ______________________________

Math 3310
Final Exam
Fall 2014


Department of Math and Stat
Texas Tech University

*Instructions:* Solve nine of the problems 1–10 and five of the problems A–F. If you solve more than nive/five problems, then you must clearly mark which ones you want graded.

The problems 1–10 are worth 10 points each, and your solutions must be written on this piece of paper. The problems A–F count for 15 points each, and your solutions must be written in a blue book. For full credit, you must show complete, correct, legible work. Read carefully before you start working. No books or notes are allowed. Calculators are allowed, phones and PDAs are not.

1. For each of the following functions from $\mathbb{R}$ to $\mathbb{R}$, decide if it is 1-to-1, onto, bijective, or none of the above.

   (a) $f(x) = \cos x$.

   (b) $f(x) = |x|$.

   (c) $f(x) = x^2$.

   (d) $f(x) = e^x$.

   (e) $f(x) = \sqrt{x^2}$.

   (f) $f(x) = \tan x$.

   (g) $f(x) = x^3$.

   (h) $f(x) = \sqrt{x^2}$.

   (i) $f(x) = x|x|$.

   (j) $f(x) = x^2 \sin x$.

2. Identify each of the following sets as finite or infinite.

   (a) $\mathbb{N} \times \mathbb{N}$.

   (b) $\mathbb{N} \times \{1, 2\}$.

   (c) $\mathbb{Z} \times \emptyset$.

   (d) $\{n \in \mathbb{N} \mid n^2 < 100\}$.

   (e) $\mathbb{Z} - \mathbb{N}$.

3. For $n \in \mathbb{Z}$ consider the statement $P$ : If $n$ is odd, then $n^2 + n(n-1)$ is odd.

   (a) State the negation of $P$.

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   (b) State the contrapositive of $P$.

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4. Give a counterexample to each of the following statements.

   (a) Let $n \in \mathbb{Z}$. If $7|(n^2-1)$, then $7|(n+1)$.

   (b) $\forall n \in \mathbb{Z}, n = \sqrt{n^2}$.

   (c) For all prime numbers $p$ and $q$, at least one of the numbers $pq+1$ and $pq-1$ is prime.

   (d) Every denumerable subset $D$ of $\mathbb{Q}$ contains $\mathbb{Z}$.

   (e) $\forall n \in \mathbb{N}, 2^n+1$ is prime.

5. Complete the multiplication table for $\mathbb{Z}_7$

| | [0] | [1] | [2] | [3] | [4] | [5] | [6] |
|---|---|---|---|---|---|---|---|
| [0] | [0] | [0] | [0] | [0] | [0] | [0] | [0] |
| [1] | | | | | | | |
| [2] | | | | | | | |
| [3] | | | | [2] | | | |
| [4] | | | | | | | |
| [5] | | | | | | | |
| [6] | | | | | | | |

6. Mark each statement that is proved by the following argument.

   Let $A$ be a proper subset of $B$ and assume that there exists a bijective function $f\colon A \to B$. Suppose $B$ is finite, then $A$ is finite as well, and since $A$ is a proper subset of $B$ one has $|A| < |B|$. The function $f$ is 1-to-1, so one has $|f(A)| = |A|$. On the other hand, $f$ is also onto, so one has $f(A)| = |B|$. Contradiction!

   (a) A set cannot be numerically equivalent to a proper subset of itself.

   (b) If a set is numerically equivalent to a proper subset of itself, then it is infinite.

   (c) The cardinality of a proper subset $A$ of $B$ is less than the cardinality of $B$.

   (d) If $A$ is a proper subset of $B$, then there cannot exist a bijective function $f\colon A \to B$.

   (e) None of the above.

7. Consider the set $A = \{a, b, c, d, e, f\}$. The partition

$$\{\,\{a, d, e\}, \{b, f\}, \{c\}\,\}$$

of $A$ determines an equivalence relation R on $A$. Describe R as a subset of $A \times A$.

8. Of the relations R on $\mathbb{Q}$ below, mark those that are equivalence relations.

   (a) $x \,\mathrm{R}\, y$ if $y - x \in \mathbb{N}$.

   (b) $x \,\mathrm{R}\, y$ if $y - x \in \mathbb{Z}$.

   (c) $x \,\mathrm{R}\, y$ if $xy \geq 0$.

   (d) $x \,\mathrm{R}\, y$ if $|x| = |y|$.

   (e) $x \,\mathrm{R}\, y$ if $x \geq y$.

9. Complete the truth table

| $P$ | $Q$ | $R$ | $\sim Q$ | $P \Rightarrow \sim Q$ | $P \Rightarrow R$ | $(P \Rightarrow \sim Q) \vee (P \Rightarrow R)$ | $Q \Rightarrow R$ | $P \Rightarrow (Q \Rightarrow R)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | | | | | | |
| T | T | F | | | | | | |
| T | F | T | | | | | | |
| T | F | F | | | | | | |
| F | T | T | | | | | | |
| F | T | F | | | | | | |
| F | F | T | | | | | | |
| F | F | F | | | | | | |

   and use it to decide if the statements

$$(P \Rightarrow \sim Q) \vee (P \Rightarrow R) \qquad \text{and} \qquad (P \Rightarrow (Q \Rightarrow R))$$

   are logically equivalent.

10. Given the following subsets of $\mathbb{R}$:

$$A = (-\infty, 5], \quad B = \mathbb{Q} \cup (20, 28), \quad C = \{n^2 \mid n \in \mathbb{N}\}, \quad \text{and} \quad D = (-\infty, 20]$$

   which of the following sets is $(A \cap B) \cap (C \cup \overline{D})$?

   (a) $\emptyset$.

   (b) $\{0, 1\}$

   (c) $\{1, 4\}$

   (d) $\{0, 1, 4, 5\} \cup (20, 28)$

   (e) $\{1, 4, 5\} \cup (20, \infty)$

A. Let $A$, $B$, and $C$ be sets and consider functions $f\colon A \to B$ and $g\colon B \to C$.

(a) Show that if the composite $g \circ f\colon A \to C$ is 1-to-1, then $f$ is 1-to-1.

(b) Show that if the composite $g \circ f\colon A \to C$ is onto, then $g$ is onto.

B. Let $X$ and $Y$ be non-empty sets and let $f\colon X \to Y$ be a function.

(a) Show that the relation R on $X$ given by $x_1 \,\mathrm{R}\, x_2$ if $f(x_1) = f(x_2)$ is an equivalence relation.

(b) Describe the equivalence classes for R in the special case $X = \mathbb{R}$, $Y = \{0, 1\}$, and $f$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

C. Prove the following result: For all $n \in \mathbb{Z}$, the number $n^3 + n(n-3) + n$ is even.

D. Prove the following result: Let $A$ and $B$ be subsets of some a universal set $U$. If $\overline{A \cup B} = \overline{A}$, then one has $B \subseteq A$.

E. Prove that for all $n \in \mathbb{N}$ one has

$$\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}\,.$$

F. Let $U$ be a universal set and $A_1, \ldots, A_n$ be subsets of $U$. Prove that for every $n \geq 2$ one has

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}$$