Find the LU factorization of

$$A = \begin{bmatrix} -4 & -2 & -2 & 3 \\ -12 & -8 & -4 & 13 \\ 16 & 16 & 1 & -32 \\ -16 & -6 & -13 & 23 \end{bmatrix}.$$

and use it to solve the system

$$\begin{bmatrix} -4 & -2 & -2 & 3 \\ -12 & -8 & -4 & 13 \\ 16 & 16 & 1 & -32 \\ -16 & -6 & -13 & 23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -36 \\ 78 \end{bmatrix}.$$

$A = LU$ where

$$L = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$$U = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix},$$

$x_1 =$ $\square$,
$x_2 =$ $\square$,
$x_3 =$ $\square$,
$x_4 =$ $\square$.

$$A = \begin{bmatrix} -4 & -2 & -2 & 3 \\ -12 & -8 & -4 & 13 \\ 16 & 16 & 1 & -32 \\ -16 & -6 & -13 & 23 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cccc|cccc} -4 & -2 & -2 & 3 & 1 & 0 & 0 & 0 \\ -12 & -8 & -4 & 13 & 0 & 1 & 0 & 0 \\ 16 & 16 & 1 & -32 & 0 & 0 & 1 & 0 \\ 16 & -6 & -13 & 23 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \overset{[E_3E_2E_1A \mid E_3E_2E_1]}{\left[\begin{array}{cccc|cccc} -4 & -2 & -2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 4 & -3 & 1 & 0 & 0 \\ 0 & 8 & -7 & -20 & 4 & 0 & 1 & 0 \\ 0 & 2 & -5 & 11 & -4 & 0 & 0 & 1 \end{array}\right]}$$

$$\longrightarrow \overset{[E_5 \cdots E_1A \mid E_5 \cdots E_1]}{\left[\begin{array}{cccc|cccc} -4 & -2 & -2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 4 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & -8 & 4 & 1 & 0 \\ 0 & 0 & -3 & 15 & -7 & 1 & 0 & 1 \end{array}\right]} \longrightarrow \overset{E_6 \cdots E_1A = U \qquad E_6 \cdots E_1}{\left[\begin{array}{cccc|cccc} -4 & -2 & -2 & 3 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 4 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & -8 & 4 & 1 & 0 \\ 0 & 0 & 0 & 3 & -31 & 13 & 3 & 1 \end{array}\right]} \qquad (*)$$

$$E_6 \cdots E_1 A = U \iff A = (E_6 \cdots E_1)^{-1} U = \underbrace{\left(E_1^{-1} E_2^{-1} \cdots E_6^{-1}\right)}_{L} U$$

As in class: $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -4 & -4 & 1 & 0 \\ 4 & -1 & -3 & 1 \end{bmatrix}$; $U = \begin{bmatrix} -4 & -2 & -2 & 3 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

By (*):

$$L^{-1} = E_6 \cdots E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -8 & 4 & 1 & 0 \\ -31 & 13 & 3 & 1 \end{bmatrix}$$

$$AX = B \iff (LU)X = B \iff L(UX) = B \iff UX = L^{-1}B$$

With $B = \begin{bmatrix} 2 \\ 8 \\ -36 \\ 78 \end{bmatrix}$ $\quad L^{-1}B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -8 & 4 & 1 & 0 \\ -31 & 13 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ -36 \\ 78 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -20 \\ 12 \end{bmatrix}$

Solve $UX = \begin{bmatrix} 2 \\ 2 \\ -20 \\ 12 \end{bmatrix}$:

$$\left[\begin{array}{cccc|c} -4 & -2 & -2 & 3 & 2 \\ 0 & -2 & 2 & 4 & 2 \\ 0 & 0 & 1 & -4 & -20 \\ 0 & 0 & 0 & 3 & 12 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \text{solution} \begin{bmatrix} 3 \\ 3 \\ -4 \\ 4 \end{bmatrix}$$