LINKAGE CLASSES OF GRADE 3 PERFECT IDEALS

LARS WINTHER CHRISTENSEN, OANA VELICHE, AND JERZY WEYMAN

Abstract. While every grade 2 perfect ideal in a regular local ring is linked to a complete intersection ideal, it is known not to be the case for ideals of grade 3. We soften the blow by proving that every grade 3 perfect ideal in a regular local ring is linked to a complete intersection or a Golod ideal. Our proof is indebted to a homological classification of Cohen–Macaulay local rings of codimension 3. That debt is swiftly repaid, as we use linkage to reveal some of the finer structures of this classification.

1. Introduction

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The difference between the *embedding dimension* and *depth* of R,

$$\operatorname{edim} R = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 \qquad \text{and} \qquad \operatorname{depth} R = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(k,R) \neq 0\}\,,$$

i.e. the number $c = \operatorname{edim} R - \operatorname{depth} R$, is called the *embedding codepth* of R. Rings with $c = 0$ are regular, rings with $c = 1$ are hypersurfaces, and for rings with $c = 2$ there are two possibilities: complete intersection or Golod. For $c = 3$ the field of possibilities widens: Such rings can be Gorenstein and not complete intersection, or they may not even belong to any of the classes mentioned thus far. There is, nevertheless, a classification of local rings of embedding codepth 3. It is based on multiplicative structures in homology, and the details are discussed below.

In the 1980s Weyman [15] and Avramov, Kustin, and Miller [4] established a classification scheme which—though it is discrete in the sense that it does not involve moduli—is subtle enough to facilitate a proof of the rationality of Poincaré series of codepth 3 local rings, an open question at the time. For this purpose it was not relevant to know if local rings of every class in the scheme actually exist, and it later turned out that they do not. In 2012 Avramov [3] returned to the classification and tightened it; that is, he limited the range of possible classes. This was necessary to use the classification to answer the codepth 3 case of a question ascribed to Huneke about growth in the minimal injective resolution of a local ring. In the same paper [3, Question 3.8], Avramov formally raised the question about realizability: Which classes of codepth 3 local rings do actually occur? This paper deals with aspects of the realizability question that can be studied by linkage theory and, in turn, provide new insight on linkage of grade 3 perfect ideals.

Date: December 30, 2018.

2010 *Mathematics Subject Classification.* Primary 13C40. Secondary 13D02; 13H10.

Key words and phrases. Complete intersection, Golod, linkage, local ring, Tor algebra.

This work is part of a body of research that started during the authors' visit to MSRI in Spring 2013 and continued during a months-long visit by L.W.C. to Northeastern University; the hospitality of both institutions is acknowledged with gratitude. L.W.C. was partly supported by NSA grant H98230-14-0140 and Simons Foundation collaboration grant 428308, and J.W. was partly supported by NSF DMS grants 1400740 and 1802067.

To discuss our contributions towards an answer to the realizability question, we need to describe the classification in further detail. The $\mathfrak{m}$-adic completion $\widehat{R}$ of R is by Cohen's Structure Theorem a quotient of a regular local ring. To be precise, there is a regular local ring Q with maximal ideal $\mathfrak{M}$ and an ideal $\mathfrak{I} \subseteq \mathfrak{M}^2$ with $Q/\mathfrak{I} \cong \widehat{R}$. The Auslander-Buchsbaum Formula thus yields

$$c = \operatorname{edim} Q - \operatorname{depth} \widehat{R} = \operatorname{depth} Q - \operatorname{depth}_Q \widehat{R} = \operatorname{pd}_Q \widehat{R}\,.$$

For $c = 3$, Buchsbaum and Eisenbud [7] show that the minimal free resolution

$$F_\bullet \;=\; 0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0$$

of $\widehat{R}$ over Q has a structure of a commutative differential graded algebra. The multiplicative structure on $F_\bullet$ induces a graded-commutative algebra structure on $\operatorname{Tor}^Q_\bullet(\widehat{R},k) = \mathrm{H}_\bullet(F_\bullet \otimes_Q k)$; as a k-algebra, $\operatorname{Tor}^Q_\bullet(\widehat{R},k)$ is isomorphic to the Koszul homology algebra $\mathrm{H}(K^R)$, in particular it is independent of the presentation of $\widehat{R}$ and the choice of $F_\bullet$. The multiplicative structure on $\operatorname{Tor}^Q_\bullet(\widehat{R},k)$ is the basis for the classification. In [4] the possible multiplication tables are explicitly described, up to isomorphism; they are labeled $\mathbf{B}$, $\mathbf{C}(3)$, $\mathbf{G}(r)$, $\mathbf{H}(p,q)$, and $\mathbf{T}$, where the parameters p, q, and r are non-negative integers bounded by functions of the invariants $m = \operatorname{rank}_Q F_1$, called the *first derivation* of R, and $n = \operatorname{rank}_Q F_3$, called the *type* of R. Thus, for fixed m and n there are only finitely many possible structures, and [3, Question 3.8] asks which ones actually occur. It is known that a complete answer will have to take into account the Cohen–Macaulay defect of the ring; in this paper we only consider Cohen–Macaulay rings, and our main result pertains to the two-parameter family:

1.1 Theorem. *Let R be a Cohen–Macaulay local ring of codimension 3 and class $\mathbf{H}(p,q)$. Let m and n denote the first derivation and the type of R. The inequalities*

$$p \leqslant m-1 \qquad \text{and} \qquad q \leqslant n$$

hold, and the following conditions are equivalent

$$(i)\;\; p = n+1 \qquad (ii)\;\; q = m-2 \qquad (iii)\;\; p = m-1 \quad \text{and} \quad q = n\,.$$

Otherwise, i.e. when these conditions are not satisfied, there are inequalities

$$p \leqslant n-1 \qquad \text{and} \qquad q \leqslant m-4$$

with

$$p = n-1 \quad \text{only if} \quad q \equiv_2 m-4 \qquad \text{and} \qquad q = m-4 \quad \text{only if} \quad p \equiv_2 n-1\,.$$

The first set of inequalities in this theorem can be read off immediately from the multiplication table (2.1.1), and the equivalence of conditions (i)–(iii) is known from [3, cor. 3.3]. Thus, what is new is the "otherwise" statement. The inequalities $p \leqslant n-1$ and $q \leqslant m-4$ are proved in Section 5, and the last assertions about equalities and congruences are proved in Section 6.

To parse the theorem it may be helpful to visualize an instance of it. For $m = 7$ and $n = 5$ the parameter p can take values between 0 and $m-1 = 6$, and q can take values between 0 and $n = 5$; thus the classes fit naturally in a 7×6 grid; see Table 1.1. It follows from the theorem that at most 17 of the 42 fields in the grid are populated, and in the experiments that informed the statement of Theorem 1.1 we encountered all of them; we discuss this point further in Section 7.

						$\mathbf{H}(6,5)$
$\mathbf{H}(0,3)$		$\mathbf{H}(2,3)$		$\mathbf{H}(4,3)$		
$\mathbf{H}(0,2)$	$\mathbf{H}(1,2)$	$\mathbf{H}(2,2)$	$\mathbf{H}(3,2)$			
$\mathbf{H}(0,1)$	$\mathbf{H}(1,1)$	$\mathbf{H}(2,1)$	$\mathbf{H}(3,1)$	$\mathbf{H}(4,1)$		
$\mathbf{H}(0,0)$	$\mathbf{H}(1,0)$	$\mathbf{H}(2,0)$	$\mathbf{H}(3,0)$			

TABLE 1.1. For $m = 7$ and $n = 5$ the innate bounds $p \leqslant 6$ and $q \leqslant 5$ are optimal. Theorem 1.1 exposes a finer structure to the classification that precludes 25 of the 42 classes $\mathbf{H}(p,q)$ within these bounds.

* * *

If R and, therefore, $\widehat{R}$ is Cohen–Macaulay of codimension 3, then the defining ideal $\mathfrak{I}$ with $\widehat{R} = Q/\mathfrak{I}$ is perfect of grade 3. We obtain Theorem 1.1 as a special case of results—Theorems 5.7 and 6.6—about grade 3 perfect ideals in regular local rings.

Our main tool of investigation is linkage. It is known that every grade 2 perfect ideal in Q is linked to an ideal $\mathfrak{I}$ such that the local ring $Q/\mathfrak{I}$ is complete intersection; such ideals are called *licci*. Not every grade 3 perfect ideal is licci, see for example [9, prop. 3.5], but as a special case of Theorem 3.7 we obtain:

1.2 Theorem. *Every grade* 3 *perfect ideal in a regular local ring Q is linked to a grade* 3 *perfect ideal $\mathfrak{I} \subseteq Q$ such that $Q/\mathfrak{I}$ is complete intersection or Golod.*

The paper is organized as follows. We recall the details of the classification from [4, 15] in Section 2. The motor of the paper is Section 3, it has four statements that track relations between the multiplicative structures on the Tor algebras of linked ideals. The proofs of these statements are deferred to the Appendix, but as a first application Theorem 1.2 is proved in Section 3. The applications to the classification scheme begin in Section 4 and continue in Sections 5 and 6, which provide the proof of Theorem 1.1. In Section 7 we provide a summary of the status of the realizability question.

2. Multiplicative structures in homology

Throughout this paper, Q denotes a commutative noetherian local ring with maximal ideal $\mathfrak{M}$ and residue field $k = Q/\mathfrak{M}$. For an ideal $\mathfrak{I} \subseteq Q$ with $\operatorname{pd}_Q Q/\mathfrak{I} = 3$, let $F_\bullet \to Q/\mathfrak{I}$ be a minimal free resolution over Q and set

$$m_\mathfrak{I} = \operatorname{rank}_Q F_1 \qquad \text{and} \qquad n_\mathfrak{I} = \operatorname{rank}_Q F_3\,;$$

if Q is regular, then these numbers are the first derivation and the type of $Q/\mathfrak{I}$. Notice that one has $\operatorname{rank}_Q F_0 = 1$, which forces $\operatorname{rank}_Q F_2 = m_\mathfrak{I} + n_\mathfrak{I} - 1$ as $F_\bullet$ has Euler characteristic 0.

2.1. By a result of Buchsbaum and Eisenbud [7] the resolution $F_\bullet$ has a structure of a commutative differential graded algebra. This structure is not unique, but the induced graded-commutative algebra structure on $\mathsf{T}_\bullet = \mathrm{H}_\bullet(F_\bullet \otimes_Q k) = \operatorname{Tor}_\bullet^Q(Q/\mathfrak{I}, k)$ is unique. By [4] there exist bases

$$\mathsf{e}_1, \ldots, \mathsf{e}_{m_\mathfrak{I}} \;\; \text{for } \mathsf{T}_1\,, \quad \mathsf{f}_1, \ldots, \mathsf{f}_{m_\mathfrak{I}+n_\mathfrak{I}-1} \;\; \text{for } \mathsf{T}_2\,, \quad \text{and} \quad \mathsf{g}_1, \ldots, \mathsf{g}_{n_\mathfrak{I}} \;\; \text{for } \mathsf{T}_3$$

such that the multiplication on $\mathsf{T}_\bullet$ is one of following:

$$
\begin{array}{lrllll}
 & \mathbf{C}(3): & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 \ \ \mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1 \ \ \mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2 & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } 1 \leqslant i \leqslant 3 \\
 & \mathbf{T}: & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 \ \ \mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1 \ \ \mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2 & & & \\
(2.1.1) & \mathbf{B}: & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } 1 \leqslant i \leqslant 2 \\
 & \mathbf{G}(r): & [r \geqslant 2] & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } 1 \leqslant i \leqslant r \\
 & \mathbf{H}(p,q): & \mathsf{e}_{p+1}\mathsf{e}_i = \mathsf{f}_i \ \ \text{for } 1 \leqslant i \leqslant p & & \mathsf{e}_{p+1}\mathsf{f}_{p+j} = \mathsf{g}_j & \text{for } 1 \leqslant j \leqslant q
\end{array}
$$

Here it is understood that all products that are not mentioned—and not given by those mentioned and the rules of graded commutativity—are zero. We say that $\mathfrak{I}$, or $Q/\mathfrak{I}$, is of class $\mathbf{C}(3)$ if the multiplication on $\mathsf{T}_\bullet$ is given by $\mathbf{C}(3)$ in (2.1.1); similarly for $\mathbf{B}, \mathbf{G}(r), \mathbf{H}(p,q)$, and $\mathbf{T}$.

2.2. To deal with the multiplicative structures on $\mathsf{T}_\bullet$ it is helpful to consider a few additional invariants; set

$$p_{\mathfrak{I}} = \operatorname{rank}_k \mathsf{T}_1 \cdot \mathsf{T}_1\,, \quad q_{\mathfrak{I}} = \operatorname{rank}_k \mathsf{T}_1 \cdot \mathsf{T}_2\,, \quad \text{and} \quad r_{\mathfrak{I}} = \operatorname{rank}_k \delta_2^{\mathsf{T}}$$

where $\delta_2^{\mathsf{T}} : \mathsf{T}_2 \to \operatorname{Hom}_k(\mathsf{T}_1, \mathsf{T}_3)$ is defined by $\delta_2^{\mathsf{T}}(\mathsf{f})(\mathsf{e}) = \mathsf{fe}$ for $\mathsf{e} \in \mathsf{T}_1$ and $\mathsf{f} \in \mathsf{T}_2$. Depending on the class of $\mathfrak{I}$, the values of these invariants are

(2.2.1)

Class of $\mathfrak{I}$	$p_{\mathfrak{I}}$	$q_{\mathfrak{I}}$	$r_{\mathfrak{I}}$
$\mathbf{B}$	1	1	2
$\mathbf{C}(3)$	3	1	3
$\mathbf{G}(r)$ $[r \geqslant 2]$	0	1	r
$\mathbf{H}(p,q)$	p	q	q
$\mathbf{T}$	3	0	0

2.3. Recall that an ideal $\mathfrak{X} \subseteq Q$ is called *complete intersection of grade* g if it is generated by a regular sequence of length g. For such an ideal the minimal free resolution of $Q/\mathfrak{X}$ over Q is the Koszul complex on the regular sequence. In particular, a complete intersection ideal is perfect. A perfect ideal of grade g is called *almost complete intersection* if it is minimally generated by $g+1$ elements.

For a grade 3 perfect ideal $\mathfrak{I} \subseteq Q$ one has $m_{\mathfrak{I}} \geqslant 3$, and the next conditions are equivalent; for the equivalence of (*ii*) and (*iii*) see the remark after [4, def. 2.2].

(*i*) $m_{\mathfrak{I}} \leqslant 3$.
(*ii*) $\mathfrak{I}$ is of class $\mathbf{C}(3)$.
(*iii*) $\mathfrak{I}$ is complete intersection.
(*iv*) $m_{\mathfrak{I}} = 3$ and $n_{\mathfrak{I}} = 1$.

2.4. Recall that an ideal $\mathfrak{X} \subseteq Q$ is called *Gorenstein of grade* g if it is perfect of grade g with $n_{\mathfrak{X}} = 1$, in which case one has $\operatorname{Ext}^g_Q(Q/\mathfrak{X}, Q) \cong Q/\mathfrak{X}$.

If $\mathfrak{I} \subseteq Q$ is Gorenstein of grade 3, then $\mathfrak{I}$ is of class $\mathbf{C}(3)$ or of class $\mathbf{G}(m_{\mathfrak{I}})$ with odd $m_{\mathfrak{I}} \geqslant 5$; see the remark after [4, def. 2.2].

We refer to [3, 1.4] for the following facts and precise references to their origins.

2.5. Assume that Q is regular, and let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal.

(a) The ring $Q/\mathfrak{I}$ is complete intersection if and only if $\mathfrak{I}$ is of class $\mathbf{C}(3)$.
(b) The ring $Q/\mathfrak{I}$ is Gorenstein if and only if $\mathfrak{I}$ is of class $\mathbf{G}(m_{\mathfrak{I}})$ with $n_{\mathfrak{I}} = 1$.
(c) The ring $Q/\mathfrak{I}$ is Golod if and only if $\mathfrak{I}$ is of class $\mathbf{H}(0,0)$.

3. Multiplication in Tor algebras of linked ideals

Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal. Recall that an ideal $\mathfrak{B} \subseteq Q$ is said to be *directly linked* to $\mathfrak{A}$ if there exists a complete intersection ideal $\mathfrak{X} \subseteq \mathfrak{A}$ of grade 3 with $\mathfrak{B} = (\mathfrak{X} : \mathfrak{A})$. The ideal $\mathfrak{B}$ is then also a perfect ideal of grade 3 with $\mathfrak{X} \subseteq \mathfrak{B}$, and one has $\mathfrak{A} = (\mathfrak{X} : \mathfrak{B})$; see Golod [11]. In particular, being directly linked is a reflexive relation. An ideal $\mathfrak{B}$ is said to be *linked* to $\mathfrak{A}$ if there exists a sequence of ideals $\mathfrak{A} = \mathfrak{B}_0, \mathfrak{B}_1, \dots, \mathfrak{B}_n = \mathfrak{B}$ such that $\mathfrak{B}_{i+1}$ is directly linked to $\mathfrak{B}_i$ for each $i = 0, \dots, n-1$. Evidently, being linked is an equivalence relation; the equivalence class of $\mathfrak{A}$ under this relation is called the *linkage class* of $\mathfrak{A}$.

3.1 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal.*

(a) *If $\mathfrak{A}$ is of class* **B***, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 2.$$

Moreover, $\mathfrak{B}$ is of class **H**.

(b) *If $\mathfrak{A}$ is of class* **G***, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant \min\{r_{\mathfrak{A}}, 3\}.$$

Moreover, $\mathfrak{B}$ is of class **H**.

(c) *If $\mathfrak{A}$ is of class* **T** *with $m_{\mathfrak{A}} \geqslant 5$, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3, \quad \text{and} \quad q_{\mathfrak{B}} \geqslant 2.$$

In particular, $\mathfrak{B}$ is of class **H**.

(d) *If $\mathfrak{A}$ is of class* **T***, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3, \quad q_{\mathfrak{B}} = 1, \quad \text{and} \quad r_{\mathfrak{B}} \geqslant 2.$$

In particular, $\mathfrak{B}$ is of class **B** *or* **G**.

(e) *If $\mathfrak{A}$ is of class* **T***, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} \quad \text{and} \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3.$$

Proof. See A.6. □

The next three propositions deal with rings of class **H** the way Proposition 3.1 deals with rings of class **B**, **G**, and **T**.

3.2 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal. If $\mathfrak{A}$ is of class* **H** *with $m_{\mathfrak{A}} - 3 \geqslant p_{\mathfrak{A}}$, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3, \quad p_{\mathfrak{B}} \geqslant q_{\mathfrak{A}}, \quad \text{and} \quad q_{\mathfrak{B}} \geqslant p_{\mathfrak{A}}.$$

Moreover, the following assertions hold:

(a) *If $p_{\mathfrak{A}} \geqslant 1$, then $p_{\mathfrak{B}} = q_{\mathfrak{A}}$.*

(b) *If $p_{\mathfrak{A}} \geqslant 2$, then $\mathfrak{B}$ is of class* $\mathbf{H}(q_{\mathfrak{A}}, \cdot\,)$.

(c) *If $q_{\mathfrak{A}} \geqslant 2$, then $q_{\mathfrak{B}} = p_{\mathfrak{A}}$.*

(d) *If $q_{\mathfrak{A}} \geqslant 3$, then $\mathfrak{B}$ is of class* $\mathbf{H}(\,\cdot\,, p_{\mathfrak{A}})$.

(e) *If $q_{\mathfrak{A}} = 1$, then $\mathfrak{B}$ is of class* **B** *or* **H**.

(f) *If $q_{\mathfrak{A}} = 1$ and $p_{\mathfrak{A}} = 0$, then $\mathfrak{B}$ is of class* **H**.

Proof. See A.7. □

3.3 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal. If $\mathfrak{A}$ is of class* **H** *with $m_{\mathfrak{A}} - 2 \geqslant p_{\mathfrak{A}} \geqslant 1$, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2\,, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3\,, \quad p_{\mathfrak{B}} \geqslant q_{\mathfrak{A}}\,, \quad \text{and} \quad q_{\mathfrak{B}} \geqslant p_{\mathfrak{A}} - 1\,.$$

Moreover, the following assertions hold:

(a) *If $p_{\mathfrak{A}} \geqslant 2$, then $p_{\mathfrak{B}} = q_{\mathfrak{A}}$.*
(b) *If $p_{\mathfrak{A}} \geqslant 3$, then $\mathfrak{B}$ is of class $\mathbf{H}(q_{\mathfrak{A}},\,\cdot\,)$.*
(c) *If $q_{\mathfrak{A}} \geqslant 2$, then $q_{\mathfrak{B}} = p_{\mathfrak{A}} - 1$.*
(d) *If $q_{\mathfrak{A}} \geqslant 3$, then $\mathfrak{B}$ is of class $\mathbf{H}(\,\cdot\,, p_{\mathfrak{A}} - 1)$.*
(e) *If $q_{\mathfrak{A}} = 1$, then $\mathfrak{B}$ is of class* **B** *or* **H**.
(f) *If $q_{\mathfrak{A}} = 1 = p_{\mathfrak{A}}$, then $\mathfrak{B}$ is of class* **H**.

Proof. See A.8. □

3.4 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal. If $\mathfrak{A}$ is of class* **H** *with $m_{\mathfrak{A}} - 1 \geqslant p_{\mathfrak{A}} \geqslant 2$, then it is directly linked to a grade 3 perfect ideal $\mathfrak{B}$ with*

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 1\,, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3\,, \quad p_{\mathfrak{B}} \geqslant q_{\mathfrak{A}}\,, \quad \text{and} \quad q_{\mathfrak{B}} \geqslant p_{\mathfrak{A}} - 2\,.$$

Moreover, if $m_{\mathfrak{A}} \geqslant 5$ or $n_{\mathfrak{A}} \geqslant 3$, then the following assertions hold:

(a) *If $p_{\mathfrak{A}} \geqslant 3$, then $p_{\mathfrak{B}} = q_{\mathfrak{A}}$.*
(b) *If $p_{\mathfrak{A}} \geqslant 4$, then $\mathfrak{B}$ is of class $\mathbf{H}(q_{\mathfrak{A}},\,\cdot\,)$.*
(c) *If $q_{\mathfrak{A}} \geqslant 2$, then $q_{\mathfrak{B}} = p_{\mathfrak{A}} - 2$.*
(d) *If $q_{\mathfrak{A}} \geqslant 3$, then $\mathfrak{B}$ is of class $\mathbf{H}(\,\cdot\,, p_{\mathfrak{A}} - 2)$.*
(e) *If $q_{\mathfrak{A}} = 1$, then $\mathfrak{B}$ is of class* **B** *or* **H**.
(f) *If $q_{\mathfrak{A}} = 1$ and $p_{\mathfrak{A}} = 2$, then $\mathfrak{B}$ is of class* **H**.

Proof. See A.9. □

3.5. For a grade 3 perfect ideal $\mathfrak{I} \subseteq Q$ the quantity $m_{\mathfrak{I}} + n_{\mathfrak{I}}$ is a measure of the size of the minimal free resolution $F_{\bullet}$ of $Q/\mathfrak{I}$ over Q. Indeed, one has

$$\sum_{i=0}^{3} \operatorname{rank}_Q F_i = 2(m_{\mathfrak{I}} + n_{\mathfrak{I}})\,;$$

we refer to this number as the *total Betti number* of $\mathfrak{I}$. As one has $m_{\mathfrak{I}} \geqslant 3$ and $n_{\mathfrak{I}} \geqslant 1$ the least possible total Betti number is 8 and attained if and only if $\mathfrak{I}$ is complete intersection; see 2.3.

The next corollary records the observation, already used in [4], that any grade 3 perfect ideal $\mathfrak{I}$ with $p_{\mathfrak{I}} > 0$ is linked to an ideal with lower total Betti number.

3.6 Corollary. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal not of class $\mathbf{C}(3)$. There exists a grade 3 perfect ideal $\mathfrak{B}$ that is directly linked to $\mathfrak{A}$ and has*

$$m_{\mathfrak{B}} + n_{\mathfrak{B}} \leqslant m_{\mathfrak{A}} + n_{\mathfrak{A}} - \min\{2, p_{\mathfrak{A}}\}\,.$$

Proof. Immediate from Propositions 3.1–3.4. □

Theorem 1.2 from the introduction is a special case of the next result; see 2.5.

3.7 Theorem. *Every grade 3 perfect ideal in Q is linked to a grade 3 perfect ideal of class $\mathbf{C}(3)$ or $\mathbf{H}(0,0)$.*

Proof. Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal, and assume that $\mathfrak{A}$ is not of class $\mathbf{C}(3)$ or $\mathbf{H}(0,0)$. If $p_{\mathfrak{A}} = 0$, then $\mathfrak{A}$ is of class $\mathbf{G}$ or $\mathbf{H}$ per (2.2.1), so $q_{\mathfrak{A}} \geqslant 1$. It now follows from Proposition 3.1(b) and Proposition 3.2 that $\mathfrak{A}$ is linked to a grade 3 perfect ideal $\mathfrak{B}$ with $p_{\mathfrak{B}} \geqslant 1$ and $m_{\mathfrak{B}} + n_{\mathfrak{B}} = m_{\mathfrak{A}} + n_{\mathfrak{A}}$. Thus it follows from Corollary 3.6 that every grade 3 perfect ideal that is not of class $\mathbf{C}(3)$ or $\mathbf{H}(0,0)$ can be linked to a grade 3 perfect ideal with smaller total Betti number, and ideals of class $\mathbf{C}(3)$ have the smallest possible total Betti number; see 3.5. □

4. Ideals of deviation at most 2

The next result is [10, thm. 4.1]; under the assumption that Q is Gorenstein it is proved by Avramov [1].

4.1 Fact. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} = 4$.*

(a) *If $n_{\mathfrak{A}}$ is odd, then $n_{\mathfrak{A}} \geqslant 3$ and $\mathfrak{A}$ is of class $\mathbf{T}$.*
(b) *If $n_{\mathfrak{A}} = 2$, then $\mathfrak{A}$ is of class $\mathbf{H}(3,2)$.*
(c) *If $n_{\mathfrak{A}} \geqslant 4$ is even, then $\mathfrak{A}$ is of class $\mathbf{H}(3,0)$.*

In particular, one has $p_{\mathfrak{A}} = 3$ and $q_{\mathfrak{A}} \in \{0,2\}$.

The results in Section 3 easily yield the codimension 3 case of the fact that every almost complete intersection ideal is linked to a Gorenstein ideal; see [7, prop. 5.2].

4.2 Remark. Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} = 4$. It follows from Fact 4.1, Proposition 3.1(e), Proposition 3.4, and 2.4 that $\mathfrak{A}$ is directly linked to a Gorenstein ideal $\mathfrak{B}$ with

$$m_{\mathfrak{B}} = \begin{cases} n_{\mathfrak{A}} & \text{if } n_{\mathfrak{A}} \text{ is odd;} \\ n_{\mathfrak{A}} + 1 & \text{if } n_{\mathfrak{A}} \text{ is even.} \end{cases}$$

From the proof of the theorem in J. Watanabe's [14] one can deduce that every almost complete intersection ideal is linked to a complete intersection ideal. Here is an explicit statement.

4.3 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} = 4$. There exists a grade 3 perfect ideal of class $\mathbf{C}(3)$ that is linked to $\mathfrak{A}$ in at most $n_{\mathfrak{A}} - 2$ links if $n_{\mathfrak{A}}$ is odd and at most $n_{\mathfrak{A}} - 1$ links if $n_{\mathfrak{A}}$ is even.*

Proof. First assume that $n_{\mathfrak{A}}$ is even. If $n_{\mathfrak{A}} = 2$, then it follows from Remark 4.2 that $\mathfrak{A}$ is directly linked to a Gorenstein ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = 3$, i.e. $\mathfrak{B}$ is of class $\mathbf{C}(3)$; see 2.3. Now let $n \geqslant 2$ be an integer and assume that the statement holds for ideals $\mathfrak{A}'$ with $n_{\mathfrak{A}'} = 2(n-1)$. If $n_{\mathfrak{A}} = 2n$, then it follows from Remark 4.2 that $\mathfrak{A}$ is directly linked to a Gorenstein ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = 2n+1$. By Proposition 3.1(b) there is an ideal $\mathfrak{A}'$ that is directly linked to $\mathfrak{B}$ and has $m_{\mathfrak{A}'} = 4$ and $n_{\mathfrak{A}'} = 2(n-1)$. By assumption $\mathfrak{A}'$ is linked to an ideal of class $\mathbf{C}(3)$ in at most $2(n-1) - 1$ links, so $\mathfrak{A}$ is linked to the same ideal in at most $2 + 2(n-1) - 1 = n_{\mathfrak{A}} - 1$ links.

Now assume that $n_{\mathfrak{A}}$ is odd. If $n_{\mathfrak{A}} = 3$, then it follows from Remark 4.2 that $\mathfrak{A}$ is directly linked to a Gorenstein ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = 3$, i.e. $\mathfrak{B}$ is of class $\mathbf{C}(3)$; see 2.3. If $n_{\mathfrak{A}} = 2n+1$ for some $n \geqslant 2$, then it follows from Remark 4.2 that $\mathfrak{A}$ is directly linked to a Gorenstein ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = 2n+1$. By Proposition 3.1(b) there is an ideal $\mathfrak{A}'$ that is directly linked to $\mathfrak{B}$ and has $m_{\mathfrak{A}'} = 4$ and $n_{\mathfrak{A}'} = 2(n-1)$. By what has already been proved, $\mathfrak{A}'$ is linked to an ideal of class $\mathbf{C}(3)$ in at most $2(n-1)-1$ links, so $\mathfrak{A}$ is linked to the same ideal in at most $2+2(n-1)-1 = n_{\mathfrak{A}} - 2$ links. □

Brown [5, thm. 4.5] proves:

4.4 Fact. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} \geqslant 5$ and $n_{\mathfrak{A}} = 2$. If $p_{\mathfrak{A}} \geqslant 1$ holds, then*

(a) *$\mathfrak{A}$ is of class $\mathbf{B}$ if $m_{\mathfrak{A}}$ is odd.*

(b) *$\mathfrak{A}$ is of class $\mathbf{H}(1,2)$ if $m_{\mathfrak{A}}$ is even.*

In particular, one has $p_{\mathfrak{A}} = 1$; see (2.2.1).

Proposition 3.1 applies to restrict the possible classes of five generated ideals.

4.5 Theorem. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} = 5$. The following assertions hold:*

(a) *$\mathfrak{A}$ is of class $\mathbf{B}$ only if $n_{\mathfrak{A}} = 2$.*

(b) *$\mathfrak{A}$ is of class $\mathbf{G}$ only if $n_{\mathfrak{A}} = 1$.*

(c) *$\mathfrak{A}$ is of class $\mathbf{T}$ only if $n_{\mathfrak{A}} \geqslant 4$.*

Notice that by part (b) a five generated ideal of class $\mathbf{G}$ is Gorenstein and hence of class $\mathbf{G}(5)$; see 2.4.

Proof. (a): If $\mathfrak{A}$ is of class $\mathbf{B}$, then it is by Proposition 3.1(a) linked to a grade 3 perfect ideal $\mathfrak{B}$ with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 2\,.$$

If $n_{\mathfrak{A}} = 1$, then one has $m_{\mathfrak{B}} = 3$ and, therefore, $n_{\mathfrak{B}} = 1$ per 2.3; a contradiction. If $n_{\mathfrak{A}} \geqslant 3$, then one has $m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2 \geqslant 5$ so Fact 4.4 yields $p_{\mathfrak{B}} = 1$; a contradiction.

(b): If $\mathfrak{A}$ is of class $\mathbf{G}$, then it is by Proposition 3.1(b) linked to a grade 3 perfect ideal $\mathfrak{B}$ with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant \min\{r_{\mathfrak{A}}, 3\} \geqslant 2\,.$$

If $n_{\mathfrak{A}} \geqslant 2$ holds, then one has $m_{\mathfrak{B}} \geqslant 5$ and hence $p_{\mathfrak{B}} = 1$ by Fact 4.4; a contradiction.

(c): If $\mathfrak{A}$ is of class $\mathbf{T}$, then it is by Proposition 3.1(e) linked to a grade 3 perfect ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = n_{\mathfrak{A}}$ and $n_{\mathfrak{B}} = 2$; by 2.3 this forces $n_{\mathfrak{A}} \geqslant 4$. □

The next result was first proved by Sánchez [13].

4.6 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal. If $m_{\mathfrak{A}} \geqslant 5$ and $n_{\mathfrak{A}} = 3$, then $p_{\mathfrak{A}} \neq 3$. In particular, $\mathfrak{A}$ is not of class $\mathbf{T}$.*

Proof. Assume towards a contradiction that $m_{\mathfrak{A}} \geqslant 5$ and $n_{\mathfrak{A}} = 3 = p_{\mathfrak{A}}$ hold. Per (2.2.1) the ideal $\mathfrak{A}$ is of class $\mathbf{H}$ or $\mathbf{T}$.

If $\mathfrak{A}$ is of class $\mathbf{H}$, then there exists by Proposition 3.4(a) a grade 3 perfect ideal $\mathfrak{B}$ with $m_{\mathfrak{B}} = 4$ and $p_{\mathfrak{B}} = q_{\mathfrak{A}}$. By Fact 4.1 one has then $p_{\mathfrak{B}} = 3$ and, therefore, $q_{\mathfrak{A}} = 3$. Thus 3.4(c) yields $q_{\mathfrak{B}} = p_{\mathfrak{A}} - 2 = 1$, which contradicts 4.1.

If $\mathfrak{A}$ is of class $\mathbf{T}$, then it follows from Proposition 3.1(d) there exists a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{B}$ or $\mathbf{G}$ with

$$m_{\mathfrak{B}} = 5\,, \quad n_{\mathfrak{B}} = m_{\mathfrak{A}} - 3\,, \quad q_{\mathfrak{B}} = 1\,, \quad \text{and} \quad r_{\mathfrak{B}} \geqslant 2\,.$$

By Theorem 4.5 one has $m_{\mathfrak{A}} \geqslant 6$ and, therefore, $n_{\mathfrak{B}} \geqslant 3$, which contradicts 4.5. □

5. Class H: Bounds on the parameters of multiplication

Throughout this section, the local ring Q is assumed to be regular.

5.1 Lemma. *Let $\mathfrak{I}$ be a grade 3 perfect ideal in Q. If $\mathfrak{I}$ is not contained in $\mathfrak{M}^2$, then one has $m_\mathfrak{I} - n_\mathfrak{I} = 2$.*

Proof. Choose an element $x \in \mathfrak{I} \setminus \mathfrak{M}^2$ and set $P = Q/(x)$; notice that x is a Q-regular element. By the Auslander–Buchsbaum Formula one has $\operatorname{pd}_P Q/\mathfrak{I} = 2$, so the format of the minimal free resolution of $Q/\mathfrak{I}$ as a P-module is

$$0 \longrightarrow P^{a-1} \longrightarrow P^a \longrightarrow P$$

for some $a \geqslant 2$. Now it follows from [2, thm. 2.2.3] that the minimal free resolution of $Q/\mathfrak{I}$ as a Q-module has format $0 \to Q^{a-1} \to Q^{2a-1} \to Q^{a+1} \to Q$. In particular, one has $m_\mathfrak{I} = a+1$ and $n_\mathfrak{I} = a-1$. □

5.2. Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal. If $\mathfrak{I}$ is not contained in $\mathfrak{M}^2$, then $n_\mathfrak{I} = 1$ implies that $\mathfrak{I}$ is of class $\mathbf{C}(3)$ by Lemma 5.1 and 2.3. If $\mathfrak{I}$ is contained in $\mathfrak{M}^2$, then $n_\mathfrak{I} = 1$ holds if and only if $\mathfrak{I}$ is Gorenstein, i.e. of class $\mathbf{C}(3)$ or $\mathbf{G}(m_\mathfrak{I})$; see 2.4. In particular, one has the following lower bounds on $m_\mathfrak{I}$ and $n_\mathfrak{I}$ depending on the class of $\mathfrak{I}$.

(5.2.1)

Class of $\mathfrak{I}$	$m_\mathfrak{I}$	$n_\mathfrak{I}$
$\mathbf{B}$	$\geqslant 4$	$\geqslant 2$
$\mathbf{C}(3)$	3	1
$\mathbf{G}(r)$	$\geqslant 4$	$\geqslant 1$
$\mathbf{H}(p,q)$	$\geqslant 4$	$\geqslant 2$
$\mathbf{T}$	$\geqslant 4$	$\geqslant 2$

Lemma 5.1 together with results from [3] yield bounds on the parameters of multiplication for ideals of class $\mathbf{H}$.

5.3 Proposition. *Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal of class $\mathbf{H}$; one has*

(5.3.1) $\quad p_\mathfrak{I} \leqslant m_\mathfrak{I} - 1 \quad$ *and* $\quad q_\mathfrak{I} \leqslant n_\mathfrak{I}\,;$

(5.3.2) $\quad p_\mathfrak{I} \leqslant n_\mathfrak{I} + 1 \quad$ *and* $\quad q_\mathfrak{I} \leqslant m_\mathfrak{I} - 2\,.$

Moreover, if $\mathfrak{I}$ is contained in $\mathfrak{M}^2$, then the following conditions are equivalent

(i) $p_\mathfrak{I} = n_\mathfrak{I} + 1 \qquad$ *(ii)* $q_\mathfrak{I} = m_\mathfrak{I} - 2 \qquad$ *(iii)* $p_\mathfrak{I} = m_\mathfrak{I} - 1$ *and* $q_\mathfrak{I} = n_\mathfrak{I}\,.$

Notice that when these conditions are satisfied, one has $m_\mathfrak{I} - n_\mathfrak{I} = 2$.

Proof. The inequalities (5.3.1) are immediate from (2.1.1). If $\mathfrak{I}$ is not contained in $\mathfrak{M}^2$, then by Lemma 5.1 the inequalities in (5.3.2) agree with those in (5.3.1). If $\mathfrak{I}$ is contained in $\mathfrak{M}^2$, then the inequalities (5.3.2) hold by [3, thm. 3.1]. For such an ideal the conditions (*i*)–(*iii*) are equivalent by [3, cor. 3.3]. □

The goal of this section is to establish part of Theorem 1.1 by proving (in Corollary 5.8) the following statement:

Proposition. *Let $\mathfrak{I} \subseteq \mathfrak{M}^2$ be a grade 3 perfect ideal of class $\mathbf{H}$. If $p_\mathfrak{I} \neq n_\mathfrak{I} + 1$ or $q_\mathfrak{I} \neq m_\mathfrak{I} - 2$, then one has $p_\mathfrak{I} \leqslant n_\mathfrak{I} - 1$ and $q_\mathfrak{I} \leqslant m_\mathfrak{I} - 4$.*

The proof takes up the balance of the section; it is propelled by Propositions 3.2, 3.3, and 3.4. For example, an ideal $\mathfrak{A}$ with $m_{\mathfrak{A}} = 6$ is by these results linked to an ideal $\mathfrak{B}$ with $n_{\mathfrak{B}} = 3$, and $p_{\mathfrak{B}}$ and $q_{\mathfrak{B}}$ are essentially determined by $q_{\mathfrak{A}}$ and $p_{\mathfrak{A}}$, respectively. Thus restrictions on $q_{\mathfrak{B}}/p_{\mathfrak{B}}$ imply restrictions on $p_{\mathfrak{A}}/q_{\mathfrak{A}}$. Here is an outline: Let $\mathfrak{I} \subseteq \mathfrak{M}^2$ be a grade 3 perfect ideal of class $\mathbf{H}$. By Proposition 5.3 one can assume that $p_{\mathfrak{I}} \neq n_{\mathfrak{I}} + 1$ and $q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 2$ hold, and it suffices to prove

$$(1) \quad p_{\mathfrak{I}} \neq n_{\mathfrak{I}} \qquad \text{and} \qquad (2) \quad q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 3\,.$$

By (5.2.1) one has $m_{\mathfrak{I}} \geqslant 4$ and $n_{\mathfrak{I}} \geqslant 2$. If $m_{\mathfrak{I}} = 4$ or $n_{\mathfrak{I}} = 2$ holds, then both (1) and (2) follow from Facts 4.1 and 4.4. A few other low values of $m_{\mathfrak{I}}$ and $n_{\mathfrak{I}}$ require special attention, and after that the proof proceeds by induction: (2) for $n_{\mathfrak{I}} = 3$ implies (1) for $m_{\mathfrak{I}} = 6$, which in turn implies (2) for $n_{\mathfrak{I}} = 4$ etc.

5.4 Lemma. *For every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ with $m_{\mathfrak{A}} = 5$, one has $q_{\mathfrak{A}} \neq 2$.*

Proof. If $\mathfrak{A}$ is not of class $\mathbf{H}$, then per (2.2.1) one has $q_{\mathfrak{A}} \leqslant 1$, so assume that $\mathfrak{A}$ is of class $\mathbf{H}$. Towards a contradiction, assume that $q_{\mathfrak{A}} = 2$ holds. By (5.3.1) one has

$$(1) \qquad p_{\mathfrak{A}} \leqslant 4 \qquad \text{and} \qquad n_{\mathfrak{A}} \geqslant 2\,.$$

If $p_{\mathfrak{A}} \leqslant 2$ holds, then as $q_{\mathfrak{A}} = 2$ by assumption it follows from Proposition 3.2 that there exists a grade 3 perfect ideal $\mathfrak{B}$ in Q with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 2\,.$$

By (1) one has $m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3 \geqslant 5$, so Fact 4.4 yields $p_{\mathfrak{B}} = 1$; a contradiction.

If $p_{\mathfrak{A}} = 3$ holds, then as $q_{\mathfrak{A}} = 2$ by assumption, Proposition 3.3(b,c) yields the existence a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}(2,2)$ with $n_{\mathfrak{B}} = 2$. As $n_{\mathfrak{B}} \neq 1$ one has $m_{\mathfrak{B}} \geqslant 4$, see (5.2.1), and it follows from Facts 4.1 and 4.4 that no such ideal $\mathfrak{B}$ exists.

If $p_{\mathfrak{A}} = 4$ holds, then as $q_{\mathfrak{A}} = 2$ by assumption, Proposition 3.4(b,c) yields the existence of a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}(2,2)$ with $n_{\mathfrak{B}} = 2$. As $n_{\mathfrak{B}} \neq 1$ one has $m_{\mathfrak{B}} \geqslant 4$, see (5.2.1), and it follows from Facts 4.1 and 4.4 that no such ideal $\mathfrak{B}$ exists. □

5.5 Lemma. *Let $n \geqslant 3$ be an integer and assume that for every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ with $n_{\mathfrak{B}} = n$ one has $p_{\mathfrak{B}} \neq n_{\mathfrak{B}}$. For every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ with $m_{\mathfrak{A}} = n + 3$ one then has $q_{\mathfrak{A}} \neq m_{\mathfrak{A}} - 3$.*

Proof. Let $n \geqslant 3$ be an integer and $\mathfrak{A} \subseteq Q$ a grade 3 perfect ideal with $m_{\mathfrak{A}} = n+3$. Assume towards a contradiction that $q_{\mathfrak{A}} = m_{\mathfrak{A}} - 3 = n$ holds. As $n \geqslant 3$, it follows that $\mathfrak{A}$ is of class $\mathbf{H}$, see (2.2.1). Per (5.3.1) one has

$$(1) \qquad p_{\mathfrak{A}} \leqslant n + 2 \qquad \text{and} \qquad n_{\mathfrak{A}} \geqslant n\,.$$

We treat the cases $p_{\mathfrak{A}} \leqslant n$, $p_{\mathfrak{A}} = n + 1$, and $p_{\mathfrak{A}} = n + 2$ separately.

Case 1. If $p_{\mathfrak{A}} \leqslant n = m_{\mathfrak{A}} - 3$ holds, then as $q_{\mathfrak{A}} = n$ by assumption it follows from Proposition 3.2(d) that there exists a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}$ with

$$(2) \qquad m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} = n\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant n\,.$$

As (2) and (1) yield

$$m_{\mathfrak{B}} - n_{\mathfrak{B}} = n_{\mathfrak{A}} + 3 - n \geqslant 3\,,$$

it follows from Lemma 5.1 that $\mathfrak{B}$ is contained in $\mathfrak{M}^2$, and then Proposition 5.3 yields $p_{\mathfrak{B}} \leqslant n_{\mathfrak{B}}$. By (2) one now has $p_{\mathfrak{B}} = n_{\mathfrak{B}}$, which is a contradiction.

Case 2. If $p_{\mathfrak{A}} = n+1 = m_{\mathfrak{A}} - 2$ holds, then as $q_{\mathfrak{A}} = n$ by assumption it follows from Proposition 3.3(b,c) that there exists a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}(n,n)$ with $n_{\mathfrak{B}} = n$, and that is a contradiction.

Case 3. If $p_{\mathfrak{A}} = n+2 = m_{\mathfrak{A}} - 1$ holds, then as $q_{\mathfrak{A}} = n$ by assumption it follows from Proposition 3.4(b,c) that there exists a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}(n,n)$ with $n_{\mathfrak{B}} = n$, and that is a contradiction. □

5.6 Lemma. *Let $m \geqslant 6$ be an integer and assume that for every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ with $m-1 \leqslant m_{\mathfrak{A}} \leqslant m$ one has $q_{\mathfrak{A}} \neq m_{\mathfrak{A}} - 3$. For every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ with $n_{\mathfrak{B}} = m-2$ one then has $p_{\mathfrak{B}} \neq n_{\mathfrak{B}}$.*

Proof. Let $m \geqslant 6$ be an integer and $\mathfrak{B} \subseteq Q$ a grade 3 perfect ideal with $n_{\mathfrak{B}} = m-2$. Assume towards a contradiction that

$$(0) \qquad p_{\mathfrak{B}} = n_{\mathfrak{B}} = m-2$$

holds. As $m-2 \geqslant 4$, it follows that $\mathfrak{B}$ is of class $\mathbf{H}$, see (2.2.1), and (5.3.1) yields

$$(1) \qquad m-2 = p_{\mathfrak{B}} \leqslant m_{\mathfrak{B}} - 1\,.$$

We treat the cases $m_{\mathfrak{B}} = m$ and $m_{\mathfrak{B}} \neq m$ separately.

Case 1. Assume that $m_{\mathfrak{B}} = m$ holds. Per (0) one has $p_{\mathfrak{B}} = m_{\mathfrak{B}} - 2$, so by Proposition 3.3(b) there exists a grade 3 perfect ideal $\mathfrak{A}$ in Q of class $\mathbf{H}$ with

$$(2) \qquad m_{\mathfrak{A}} = m\,, \quad n_{\mathfrak{A}} = m-3\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant m-3\,.$$

In view of (5.3.2) one now has

$$m_{\mathfrak{A}} - 2 \geqslant q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 3\,.$$

As $m_{\mathfrak{A}} = m$ one has $q_{\mathfrak{A}} \neq m_{\mathfrak{A}} - 3$ by assumption, which forces $q_{\mathfrak{A}} = m_{\mathfrak{A}} - 2 = m-2$. As one has $m_{\mathfrak{A}} - n_{\mathfrak{A}} = 3$ it follows from Lemma 5.1 that $\mathfrak{A}$ is contained in $\mathfrak{M}^2$. Now Proposition 5.3 yields $q_{\mathfrak{A}} = n_{\mathfrak{A}}$, and by (2) that is a contradiction.

Case 2. Assume that $m_{\mathfrak{B}} \neq m$ holds. Per (1) one has $p_{\mathfrak{B}} \leqslant m_{\mathfrak{B}} - 1$, so by (0) and Proposition 3.4(b) there exists a grade 3 perfect ideal $\mathfrak{A}$ in Q of class $\mathbf{H}$ with

$$(3) \qquad m_{\mathfrak{A}} = m-1\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3 \quad \text{and} \quad q_{\mathfrak{A}} \geqslant m-4\,.$$

In view of (5.3.2) one now has

$$m_{\mathfrak{A}} - 2 \geqslant q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 3\,.$$

As $m_{\mathfrak{A}} = m-1$, one has $q_{\mathfrak{A}} \neq m_{\mathfrak{A}} - 3$ by assumption, so that forces $q_{\mathfrak{A}} = m_{\mathfrak{A}} - 2 = m-3$. As one has $m_{\mathfrak{A}} - n_{\mathfrak{A}} = m - m_{\mathfrak{B}} + 2 \neq 2$, it follows from Lemma 5.1 that $\mathfrak{A}$ is contained in $\mathfrak{M}^2$. Now Proposition 5.3 yields $q_{\mathfrak{A}} = n_{\mathfrak{A}}$, and by (3) that is a contradiction, as one has $n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3 \neq m-3$. □

5.7 Theorem. *Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal of class $\mathbf{H}$. If $p_{\mathfrak{I}} \neq n_{\mathfrak{I}} + 1$ and $q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 2$ hold, then one has $p_{\mathfrak{I}} \leqslant n_{\mathfrak{I}} - 1$ and $q_{\mathfrak{I}} \leqslant m_{\mathfrak{I}} - 4$.*

Proof. By (5.3.2) there are inequalities $p_{\mathfrak{I}} \leqslant n_{\mathfrak{I}} + 1$ and $q_{\mathfrak{I}} \leqslant m_{\mathfrak{I}} - 2$, so under the assumptions it is sufficient to show that $p_{\mathfrak{I}} \neq n_{\mathfrak{I}}$ and $q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 3$ hold. Per (5.2.1) one has $m_{\mathfrak{I}} \geqslant 4$ and $n_{\mathfrak{I}} \geqslant 2$; to prove the assertion we show by induction on $l \in \mathbb{N}$ that the following hold:

(a) Every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ of class $\mathbf{H}$ with $m_{\mathfrak{A}} = l+3$ has $q_{\mathfrak{A}} \neq l$.

(b) Every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class $\mathbf{H}$ with $n_{\mathfrak{B}} = l+1$ has $p_{\mathfrak{B}} \neq l+1$.

For $l = 1$ part (a) follows from Fact 4.1 and (b) follows from Facts 4.1 and 4.4.

For $l = 2$ part (a) is Lemma 5.4, and (b) holds by Fact 4.1 and Proposition 4.6.

Let $l \geqslant 3$ and assume that (a) and (b) hold for all lower values of l. By part (b) for $l-1$ the hypothesis in Lemma 5.5 is satisfied for $n = l+1$, and it follows that (a) holds for l. By part (a) for l and $l-1$ the hypothesis in Lemma 5.6 is satisfied for $m = l+3$, and it follows that (b) holds for l. □

5.8 Corollary. *Let $\mathfrak{I} \subseteq \mathfrak{M}^2$ be a grade 3 perfect ideal of class* **H**. *If $p_{\mathfrak{I}} \neq n_{\mathfrak{I}} + 1$ or $q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 2$, then one has $p_{\mathfrak{I}} \leqslant n_{\mathfrak{I}} - 1$ and $q_{\mathfrak{I}} \leqslant m_{\mathfrak{I}} - 4$.*

Proof. By Proposition 5.3 one has $p_{\mathfrak{I}} = n_{\mathfrak{I}} + 1$ if and only if $q_{\mathfrak{I}} = m_{\mathfrak{I}} - 2$. □

5.9 Corollary. *Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal of class* **H**. *If $m_{\mathfrak{I}} - n_{\mathfrak{I}} \neq 2$ holds, then one has $p_{\mathfrak{I}} \leqslant n_{\mathfrak{I}} - 1$ and $q_{\mathfrak{I}} \leqslant m_{\mathfrak{I}} - 4$.*

Proof. It follows from Lemma 5.1 that $\mathfrak{I}$ is contained in $\mathfrak{M}^2$, so by Corollary 5.8 it suffices to show that $p_{\mathfrak{I}} \neq n_{\mathfrak{I}} + 1$ or $q_{\mathfrak{I}} \neq m_{\mathfrak{I}} - 2$ holds. If $m_{\mathfrak{I}} - n_{\mathfrak{I}} < 2$, then (5.3.1) yields $p_{\mathfrak{I}} \leqslant m_{\mathfrak{I}} - 1 < n_{\mathfrak{I}} + 1$, and if $m_{\mathfrak{I}} - n_{\mathfrak{I}} > 2$, then (5.3.1) yields $q_{\mathfrak{I}} \leqslant n_{\mathfrak{I}} < m_{\mathfrak{I}} - 2$. □

6. Class H: Extremal values of the parameters of multiplication

Throughout this section, the local ring Q is assumed to be regular. We establish the remaining part of Theorem 1.1 by proving (as a special case of Theorem 6.6) the following:

Proposition. *For every grade 3 perfect ideal $\mathfrak{I} \subseteq \mathfrak{M}^2$ of class* **H** *the following hold:*

(a) *If $p_{\mathfrak{I}} \geqslant n_{\mathfrak{I}} - 1$, then $q_{\mathfrak{I}} \equiv_2 m_{\mathfrak{I}} - 4$.*
(b) *If $q_{\mathfrak{I}} \geqslant m_{\mathfrak{I}} - 4$, then $p_{\mathfrak{I}} \equiv_2 n_{\mathfrak{I}} - 1$.*

The proof follows the template from Section 5. It is, eventually, an induction argument with the low values of $m_{\mathfrak{I}}$ and $n_{\mathfrak{I}}$ handled separately in Facts 4.1 and 4.4 and Lemmas 6.1–6.3.

6.1 Lemma. *For every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ of class* **H** *with $m_{\mathfrak{A}} = 5$ and $q_{\mathfrak{A}} \geqslant 1$ one has $p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1$.*

Proof. By (5.2.1) and (5.3.1) one has $n_{\mathfrak{A}} \geqslant 2$ and $p_{\mathfrak{A}} \leqslant 4$. The assumption $q_{\mathfrak{A}} \geqslant 1$ together with (5.3.2) and Lemma 5.4 yield $q_{\mathfrak{A}} = 1$ or $q_{\mathfrak{A}} = 3$. We treat odd and even values of $n_{\mathfrak{A}}$ separately.

Assume that $n_{\mathfrak{A}}$ is even. The goal is to show that $p_{\mathfrak{A}}$ is odd, i.e. not 0, 2, or 4.

- If $p_{\mathfrak{A}} = 0$, then there would by Proposition 3.2—part (f) if $q_{\mathfrak{A}} = 1$ and part (d) if $q_{\mathfrak{A}} = 3$—exist a grade 3 perfect ideal $\mathfrak{B}$ in Q of class **H** with
$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 1\,.$$
As $m_{\mathfrak{B}}$ is odd, it follows from Fact 4.4 that no such ideal exists.
- If $p_{\mathfrak{A}} = 2$, then there would by Proposition 3.4—part (f) if $q_{\mathfrak{A}} = 1$ and part (d) if $q_{\mathfrak{A}} = 3$—exist a grade 3 perfect ideal $\mathfrak{B}$ in Q of class **H** with
$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 1\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 1\,.$$
As $m_{\mathfrak{B}}$ is odd, it follows from Fact 4.4 that no such ideal exists.

- If $p_{\mathfrak{A}} = 4$, then there would by Proposition 3.4(b) exist a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}$ with
$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 1\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 1\,.$$
As $m_{\mathfrak{B}}$ is odd, it follows from Fact 4.4 that no such ideal exists.

Assume now that $n_{\mathfrak{A}}$ is odd. The goal is to show that $p_{\mathfrak{A}}$ is even, i.e. not 1 or 3.

- If $p_{\mathfrak{A}} = 1$, then there would by Proposition 3.3—part (f) if $q_{\mathfrak{A}} = 1$ and part (d) if $q_{\mathfrak{A}} = 3$—exist a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}$ with
$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 1\,.$$
As $m_{\mathfrak{B}}$ is odd, it follows from Fact 4.4 that no such ideal exists.
- If $p_{\mathfrak{A}} = 3$, then there would by Proposition 3.3(b) exist a grade 3 perfect ideal $\mathfrak{B}$ in Q of class $\mathbf{H}$ with
$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2\,, \quad n_{\mathfrak{B}} = 2\,, \quad \text{and} \quad p_{\mathfrak{B}} \geqslant 1\,.$$
As $m_{\mathfrak{B}}$ is odd, it follows from Fact 4.4 that no such ideal exists. □

6.2 Lemma. *For every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class $\mathbf{H}$ with $n_{\mathfrak{B}} = 3$ and $p_{\mathfrak{B}} \geqslant 2$ one has $q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4$.*

Proof. By (5.2.1) and (5.3.1) one has $m_{\mathfrak{B}} \geqslant 4$ and $q_{\mathfrak{B}} \leqslant 3$. The assumption $p_{\mathfrak{B}} \geqslant 2$ together with (5.3.2) and Proposition 4.6 yields $p_{\mathfrak{B}} = 2$ or $p_{\mathfrak{B}} = 4$. We treat odd and even values of $m_{\mathfrak{B}}$ separately.

Assume that $m_{\mathfrak{B}}$ is even. The goal is to show that $q_{\mathfrak{B}}$ is even, i.e. not 1 or 3.

- If $q_{\mathfrak{B}} = 1$, then there would by Proposition 3.4—part (f) if $p_{\mathfrak{B}} = 2$ and part (b) if $p_{\mathfrak{B}} = 4$—exist a grade 3 perfect ideal $\mathfrak{A}$ in Q of class $\mathbf{H}$ with $m_{\mathfrak{A}} = 4$ and $n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3$. As $n_{\mathfrak{A}}$ is odd, this contradicts Fact 4.1.
- If $q_{\mathfrak{B}} = 3$, then there would by Proposition 3.4(d) exist a grade 3 perfect ideal $\mathfrak{A}$ in Q of class $\mathbf{H}$ with $m_{\mathfrak{A}} = 4$ and $n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3$. As $n_{\mathfrak{A}}$ is odd, this contradicts Fact 4.1.

Assume now that $m_{\mathfrak{B}}$ is odd. The goal is to show that $q_{\mathfrak{B}}$ is odd, i.e. not 0 or 2.

- If $q_{\mathfrak{B}} = 0$, then there would by Proposition 3.3(a) exist a grade 3 perfect ideal $\mathfrak{A}$ in Q with
$$m_{\mathfrak{A}} = 5\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3\,, \quad p_{\mathfrak{A}} = 0\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant 1\,.$$
It follows from (2.2.1), 2.3, and Theorem 4.5 that $\mathfrak{A}$ is of class $\mathbf{H}$. As $n_{\mathfrak{A}}$ is even, it follows from Lemma 6.1 that no such ideal $\mathfrak{A}$ exists.
- If $q_{\mathfrak{B}} = 2$, then there would by Proposition 3.3(a,c) exist a grade 3 perfect ideal $\mathfrak{A}$ in Q with
$$m_{\mathfrak{A}} = 5\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3\,, \quad p_{\mathfrak{A}} = 2\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant 1\,.$$
It follows from (2.2.1) that $\mathfrak{A}$ is of class $\mathbf{H}$. As $n_{\mathfrak{A}}$ is even, it follows from Lemma 6.1 that no such ideal $\mathfrak{A}$ exists. □

6.3 Lemma. *For every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class $\mathbf{H}$ with $n_{\mathfrak{B}} = 4$ and $p_{\mathfrak{B}} \geqslant 3$ one has $q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4$.*

Proof. By (5.2.1) one has $m_{\mathfrak{B}} \geqslant 4$. Notice that if $m_{\mathfrak{B}} = 4$ holds, then Fact 4.1 yields $q_{\mathfrak{B}} = 0 = m_{\mathfrak{B}} - 4$. If $m_{\mathfrak{B}} \geqslant 5$, then Proposition 3.4(a) yields a grade 3 perfect ideal $\mathfrak{A}$ in Q with
$$m_{\mathfrak{A}} = 5\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3 \geqslant 2\,, \quad p_{\mathfrak{A}} = q_{\mathfrak{B}}\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant 1\,.$$

It follows from Theorem 4.5 and (2.2.1) that $\mathfrak{A}$ is of class **B** or **H**.

If $\mathfrak{B}$ is of class **B**, then 4.5(a) yields $n_{\mathfrak{A}} = 2$, and hence $m_{\mathfrak{B}} = 5$. Per (2.2.1) one thus has $q_{\mathfrak{B}} = p_{\mathfrak{A}} = 1 = m_{\mathfrak{B}} - 4$.

If $\mathfrak{B}$ is of class **H**, then Lemma 6.1 yields $q_{\mathfrak{B}} = p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1 = m_{\mathfrak{B}} - 4$. □

6.4 Lemma. *Let $n \geqslant 3$ be an integer. Assume that for every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class* **H** *with $n_{\mathfrak{B}} = n$ and $p_{\mathfrak{B}} \geqslant n_{\mathfrak{B}} - 1$, one has $q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4$. For every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ of class* **H** *with $m_{\mathfrak{A}} = n+3$ and $q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4$, one then has $p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1$.*

Proof. Let $n \geqslant 3$ be an integer and $\mathfrak{A}$ a grade 3 perfect ideal in Q of class **H** with

(1) $$m_{\mathfrak{A}} = n + 3 \geqslant 6 \quad \text{and} \quad q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4 = n - 1 \geqslant 2\,.$$

It needs to be shown $p_{\mathfrak{A}}$ and $n_{\mathfrak{A}}$ have opposite parity. By (5.3.1) one has $p_{\mathfrak{A}} \leqslant n+2$; we treat the cases $p_{\mathfrak{A}} \leqslant n$, $p_{\mathfrak{A}} = n+1$, and $p_{\mathfrak{A}} = n+2$ separately.

Case 1. Assume that $0 \leqslant p_{\mathfrak{A}} \leqslant n = m_{\mathfrak{A}} - 3$ holds. As $q_{\mathfrak{A}} \geqslant 2$ by (1), there exists by Proposition 3.2(c) and (1) a grade 3 perfect ideal $\mathfrak{B}$ in Q with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} = n\,, \quad p_{\mathfrak{B}} \geqslant n - 1\,, \quad \text{and} \quad q_{\mathfrak{B}} = p_{\mathfrak{A}}\,.$$

For $n \geqslant 4$ one has $q_{\mathfrak{A}} \geqslant 3$ by (1), so it follows from Proposition 3.2(d) that $\mathfrak{B}$ is of class **H**; for $n = 3$ the same conclusion follows from (2.2.1) and Proposition 4.6. The assumptions now yield the congruence

$$p_{\mathfrak{A}} = q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4 = n_{\mathfrak{A}} - 1\,.$$

Case 2. Assume that $p_{\mathfrak{A}} = n + 1 = m_{\mathfrak{A}} - 2$ holds. As $q_{\mathfrak{A}} \geqslant 2$ by (1), there exists by Proposition 3.3(c) and (1) a grade 3 perfect ideal $\mathfrak{B}$ in Q with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 2\,, \quad n_{\mathfrak{B}} = n\,, \quad p_{\mathfrak{B}} \geqslant n - 1\,, \quad \text{and} \quad q_{\mathfrak{B}} = p_{\mathfrak{A}} - 1 = n\,.$$

As $q_{\mathfrak{B}} \geqslant 3$, the ideal $\mathfrak{B}$ is by (2.2.1) of class **H**, so the assumptions yield

$$p_{\mathfrak{A}} = q_{\mathfrak{B}} + 1 \equiv_2 m_{\mathfrak{B}} - 3 = n_{\mathfrak{A}} - 1\,.$$

Case 3. Assume that $p_{\mathfrak{A}} = n + 2 = m_{\mathfrak{A}} - 1$ holds. As $q_{\mathfrak{A}} \geqslant 2$ by (1), there exists by Proposition 3.4(c) and (1) a grade 3 perfect ideal $\mathfrak{B}$ in Q with

$$m_{\mathfrak{B}} = n_{\mathfrak{A}} + 1\,, \quad n_{\mathfrak{B}} = n\,, \quad p_{\mathfrak{B}} \geqslant n - 1\,, \quad \text{and} \quad q_{\mathfrak{B}} = p_{\mathfrak{A}} - 2 = n\,.$$

As $q_{\mathfrak{B}} \geqslant 3$, the ideal $\mathfrak{B}$ is by (2.2.1) of class **H**, so the assumptions yield

$$p_{\mathfrak{A}} = q_{\mathfrak{B}} + 2 \equiv_2 m_{\mathfrak{B}} - 2 = n_{\mathfrak{A}} - 1\,.$$ □

6.5 Lemma. *Let $m \geqslant 7$ be an integer. Assume that for every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ of class* **H** *with $m - 1 \leqslant m_{\mathfrak{A}} \leqslant m$ and $q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4$ one has $p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1$. For every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class* **H** *with $n_{\mathfrak{B}} = m-2$ and $p_{\mathfrak{B}} \geqslant n_{\mathfrak{B}} - 1$, one then has $q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4$.*

Proof. Let $m \geqslant 7$ be an integer and $\mathfrak{B}$ a grade 3 perfect ideal in Q of class **H** with

(1) $$n_{\mathfrak{B}} = m - 2 \geqslant 5 \quad \text{and} \quad p_{\mathfrak{B}} \geqslant n_{\mathfrak{B}} - 1 = m - 3 \geqslant 4\,.$$

It needs to be shown $q_{\mathfrak{B}}$ and $m_{\mathfrak{B}}$ have the same parity. By (5.3.1) one has $p_{\mathfrak{B}} \leqslant m_{\mathfrak{B}} - 1$. We treat inequality and equality separately.

Case 1. Assume that $p_{\mathfrak{B}} \leqslant m_{\mathfrak{B}} - 2$ holds. As $p_{\mathfrak{B}} \geqslant 4$ by (1), there exists by Proposition 3.3(b) a grade 3 perfect ideal $\mathfrak{A}$ in Q of class **H** with

$$m_{\mathfrak{A}} = m\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3\,, \quad p_{\mathfrak{A}} = q_{\mathfrak{B}}\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant p_{\mathfrak{B}} - 1 \geqslant m - 4\,.$$

As $q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4$, the assumptions now yield

$$q_{\mathfrak{B}} = p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1 = m_{\mathfrak{B}} - 4\,.$$

Case 2. Assume that $p_{\mathfrak{B}} = m_{\mathfrak{B}} - 1$ holds. As $p_{\mathfrak{B}} \geqslant 4$ by (1), there exists by Proposition 3.4(b) a grade 3 perfect ideal $\mathfrak{A}$ in Q of class $\mathbf{H}$ with

$$m_{\mathfrak{A}} = m - 1\,, \quad n_{\mathfrak{A}} = m_{\mathfrak{B}} - 3\,, \quad p_{\mathfrak{A}} = q_{\mathfrak{B}}\,, \quad \text{and} \quad q_{\mathfrak{A}} \geqslant p_{\mathfrak{B}} - 2 \geqslant m - 5\,.$$

As $q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4$, the assumptions now yield

$$q_{\mathfrak{B}} = p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1 = m_{\mathfrak{B}} - 4\,.$$ □

6.6 Theorem. *For every grade 3 perfect ideal $\mathfrak{I} \subseteq Q$ of class $\mathbf{H}$ the following hold*

(a) *If $p_{\mathfrak{I}} \geqslant n_{\mathfrak{I}} - 1$, then $q_{\mathfrak{I}} \equiv_2 m_{\mathfrak{I}} - 4$.*

(b) *If $q_{\mathfrak{I}} \geqslant m_{\mathfrak{I}} - 4$, then $p_{\mathfrak{I}} \equiv_2 n_{\mathfrak{I}} - 1$.*

Proof. Per (5.2.1) one has $m_{\mathfrak{I}} \geqslant 4$ and $n_{\mathfrak{I}} \geqslant 2$; to prove the assertion we show by induction on $l \in \mathbb{N}$ that the following hold:

(1) For every grade 3 perfect ideal $\mathfrak{A} \subseteq Q$ of class $\mathbf{H}$ with $m_{\mathfrak{A}} = l + 3$ and $q_{\mathfrak{A}} \geqslant m_{\mathfrak{A}} - 4$ one has $p_{\mathfrak{A}} \equiv_2 n_{\mathfrak{A}} - 1$.

(2) For every grade 3 perfect ideal $\mathfrak{B} \subseteq Q$ of class $\mathbf{H}$ with $n_{\mathfrak{B}} = l + 1$ and $p_{\mathfrak{B}} \geqslant n_{\mathfrak{B}} - 1$ one has $q_{\mathfrak{B}} \equiv_2 m_{\mathfrak{B}} - 4$.

For $l = 1$ the assertion (1) holds by Fact 4.1 and (2) holds by Fact 4.4.

For $l = 2$ the assertion (1) is Lemma 6.1 and (2) is Lemma 6.2.

Let $l = 3$. As (2) holds for $l = 2$, the premise in Lemma 6.4 is satisfied for $n = 3$, and it follows that (1) holds. The assertion (2) is Lemma 6.3.

Let $l \geqslant 4$ and assume that (1) and (2) hold for all lower values of l. By part (2) for $l - 1$ the premise in Lemma 6.4 is satisfied for $n = l$, and it follows that (1) holds for l. By the assertion (1) for l and $l - 1$ the premise in Lemma 6.5 is satisfied for $m = l + 3$, and it follows that (2) holds for l. □

7. The realizability question

We sum up the contributions of the previous sections towards an answer to the realizability question [3, Question 3.8]. The focus is still on restricting the range of potentially realizable classes.

7.1 Class B. Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal of class $\mathbf{B}$. Fact 4.1, 2.3, and 2.4 yield $m_{\mathfrak{I}} \geqslant 5$ and $n_{\mathfrak{I}} \geqslant 2$. Moreover, if $m_{\mathfrak{I}} = 5$ holds, then Theorem 4.5 yields $n_{\mathfrak{I}} = 2$, and if $n_{\mathfrak{I}} = 2$ holds, then $m_{\mathfrak{I}}$ is odd by Fact 4.4.

7.2 Class T. Let $\mathfrak{I} \subseteq Q$ be a grade 3 perfect ideal of class $\mathbf{T}$; by 2.3 one has $m_{\mathfrak{I}} \geqslant 4$. If $m_{\mathfrak{I}} = 4$ holds, then it follows from Fact 4.1 that $n_{\mathfrak{I}}$ is odd and at least 3. If $m_{\mathfrak{I}} \geqslant 5$ holds, then Fact 4.4 and Proposition 4.6 yield $n_{\mathfrak{I}} \geqslant 4$.

7.3 Summary. Table 7.1 illustrates which Cohen–Macaulay local rings of class $\mathbf{H}$ are possible per Theorem 1.1. The table also shows the restrictions imposed by 7.1 and 7.2 on the existence of rings of class $\mathbf{B}$ or $\mathbf{T}$.

7.4 Conjectures. The statement of Theorem 1.1 was informed by experiments conducted using the Macaulay 2 implementation of Christensen and Veliche's classification algorithm [8, 12]. Based on these experiments we conjecture that for

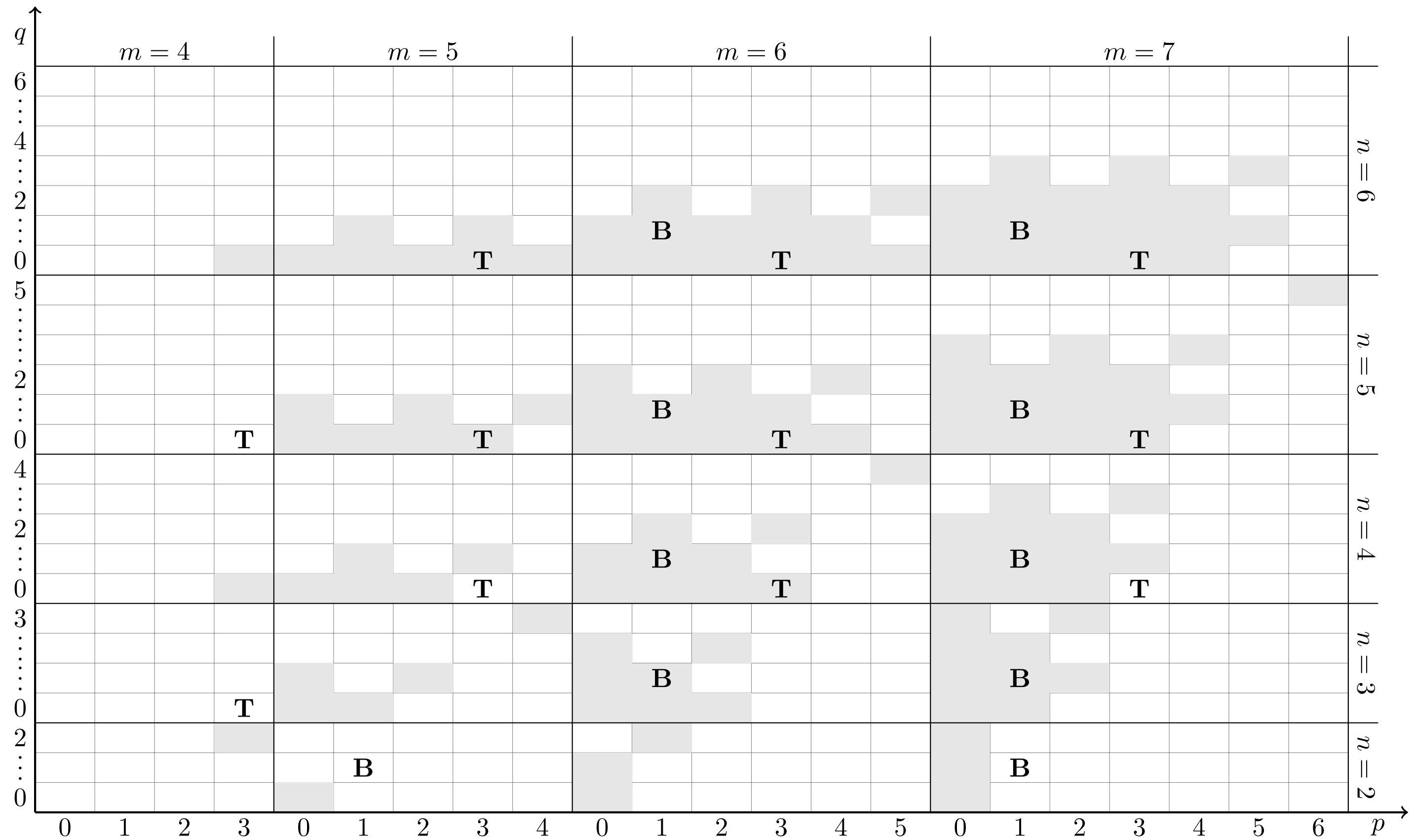

TABLE 7.1. Codimension 3 Cohen–Macaulay local rings that are not Gorenstein have first derivation $m \geqslant 4$ and type $n \geqslant 2$. For such rings with $m \leqslant 7$ and $n \leqslant 6$ the table shows which of the classes **B**, **H**, and **T** may exist according to Theorem 1.1, 7.1, and 7.2. To avoid overloading the table, the possible classes $\mathbf{H}(p, q)$ are only indicated by shading the corresponding fields in the pq-grid.

$m \geqslant 5$ and $n \geqslant 3$ there are no further restrictions on realizability of codimension 3 Cohen–Macaulay local rings of class $\mathbf{B}$, $\mathbf{H}$, or $\mathbf{T}$ than those captured by 1.1, 7.1, and 7.2 and illustrated by Table 7.1.

Absent from Table 7.1 are rings of classes $\mathbf{C}(3)$ and $\mathbf{G}$. A Cohen–Macaulay local ring of codimension 3, first derivation m, and type n is Gorenstein if and only if $n = 1$ holds, and they exist for odd $m \geqslant 3$; see 2.4 and 2.5. For $m = 3$ they are complete intersections, precisely the rings of class $\mathbf{C}(3)$, and for $m \geqslant 5$ they are of class $\mathbf{G}(m)$. Per [3, thm. 3.1] one has $2 \leqslant r \leqslant m-2$ for rings of class $\mathbf{G}(r)$ that are not Gorenstein. Our experiments suggests that there are further restrictions.

Let $\mathfrak{J} \subseteq Q$ be a grade 3 perfect ideal of class $\mathbf{G}$ and not Gorenstein. By 2.3, Fact 4.1, and Theorem 4.5 one has $m_{\mathfrak{J}} \geqslant 6$. We conjecture that the following hold:

(a) If $n_{\mathfrak{J}} = 2$, then one has $2 \leqslant r_{\mathfrak{J}} \leqslant m_{\mathfrak{J}} - 5$ or $r_{\mathfrak{J}} = m_{\mathfrak{J}} - 3$.

(b) If $n_{\mathfrak{J}} \geqslant 3$, then one has $2 \leqslant r_{\mathfrak{J}} \leqslant m_{\mathfrak{J}} - 4$.

The next proposition proves part (b) of this conjecture for six generated ideals.

7.5 Proposition. *Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal with $m_{\mathfrak{A}} = 6$ and $n_{\mathfrak{A}} \geqslant 3$. If $\mathfrak{A}$ is of class $\mathbf{G}$, then $r_{\mathfrak{A}} = 2$.*

Proof. If $\mathfrak{A}$ is of class $\mathbf{G}$, then there exists by Proposition 3.1(b) a grade 3 perfect ideal $\mathfrak{B}$ in Q with

$$m_{\mathfrak{B}} \;=\; n_{\mathfrak{A}} + 3\,, \quad n_{\mathfrak{B}} \;=\; 3\,, \quad \text{and} \quad p_{\mathfrak{B}} \;\geqslant\; \min\{r_{\mathfrak{A}}, 3\} \geqslant 2\,.$$

As $m_{\mathfrak{B}} - n_{\mathfrak{B}} = n_{\mathfrak{A}} \geqslant 3$, Corollary 5.9 yields $p_{\mathfrak{B}} \leqslant 2$, which forces $r_{\mathfrak{A}} = 2$. □

Appendix. Multiplication in Tor algebras of linked ideals—proofs

A.1 Setup. Let Q be a local ring with maximal ideal $\mathfrak{M}$. Let $\mathfrak{A} \subseteq Q$ be a grade 3 perfect ideal; set

$$m = m_{\mathfrak{A}}\,, \quad n = n_{\mathfrak{A}}\,, \quad \text{and} \quad l = m + n - 1\,.$$

Let $A_\bullet \to Q/\mathfrak{A}$ be a minimal free resolution over Q and set

$$\mathsf{A}_\bullet \;=\; \operatorname{Tor}^Q_\bullet(Q/\mathfrak{A}, k) \;=\; \mathrm{H}_\bullet(A_\bullet \otimes_Q k)\,.$$

Let $\{e_i\}_{1\leqslant i\leqslant m}$, $\{f_j\}_{1\leqslant j\leqslant l}$, and $\{g_h\}_{1\leqslant h\leqslant n}$ denote bases for A_1, A_2, and A_3.

A.2 Multiplicative structures. Adopt Setup A.1. The homology classes of the bases for A_1, A_2, and A_3 yield bases

$$\text{(A.2.1)} \qquad \mathsf{e}_1, \dots, \mathsf{e}_m \ \text{ for } \mathsf{A}_1\,, \quad \mathsf{f}_1, \dots, \mathsf{f}_l \ \text{ for } \mathsf{A}_2\,, \quad \text{and} \quad \mathsf{g}_1, \dots, \mathsf{g}_n \ \text{ for } \mathsf{A}_3\,.$$

As recalled in 2.1, bases can be chosen such that the multiplication on $\mathsf{A}_\bullet$ is one of:

$$\text{(A.2.2)} \qquad \begin{array}{rllll}
\mathbf{C}(3): & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 \ \ \mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1 \ \ \mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2 & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } \ 1 \leqslant i \leqslant 3 \\
\mathbf{T}: & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 \ \ \mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1 \ \ \mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2 & & & \\
\mathbf{B}: & \mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } \ 1 \leqslant i \leqslant 2 \\
\mathbf{G}(r): & [r \geqslant 2] & & \mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 & \text{for } \ 1 \leqslant i \leqslant r \\
\mathbf{H}(p,q): & \mathsf{e}_{p+1}\mathsf{e}_i = \mathsf{f}_i \ \ \text{for } \ 1 \leqslant i \leqslant p & & \mathsf{e}_{p+1}\mathsf{f}_{p+j} = \mathsf{g}_j & \text{for } \ 1 \leqslant j \leqslant q
\end{array}$$

A.3 Linkage. Adopt Setup A.1. Let $\mathfrak{X} \subset \mathfrak{A}$ be a complete intersection ideal generated by a regular sequence x_1, x_2, x_3 and set $\mathfrak{B} = (\mathfrak{X} : \mathfrak{A})$. Let $K_\bullet = \mathrm{K}_\bullet(x_1, x_2, x_3)$ be the Koszul complex and $\varphi_\bullet \colon K_\bullet \to A_\bullet$ be a lift of the canonical surjection $Q/\mathfrak{X} \to Q/\mathfrak{A}$ to a morphism of DG algebras.

$$\begin{array}{cccccccccccc} K_\bullet & & 0 \longrightarrow & K_3 & \xrightarrow{\partial_3^{K_\bullet}} & K_2 & \xrightarrow{\partial_2^{K_\bullet}} & K_1 & \xrightarrow{\partial_1^{K_\bullet}} & K_0 & \longrightarrow 0 \\ \downarrow{\scriptstyle\varphi_\bullet} & & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_1} & & \|{\scriptstyle 1_Q} & \\ A_\bullet & & 0 \longrightarrow & A_3 & \xrightarrow{\partial_3^{A_\bullet}} & A_2 & \xrightarrow{\partial_2^{A_\bullet}} & A_1 & \xrightarrow{\partial_1^{A_\bullet}} & A_0 & \longrightarrow 0 \end{array}$$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote the generators of K_1. From the mapping cone of this morphism one gets, see e.g. [4, prop. 1.6], a free resolution of $Q/\mathfrak{B}$ over Q:

$$D_\bullet \;=\; 0 \longrightarrow A_1^* \longrightarrow A_2^* \oplus K_2 \longrightarrow A_3^* \oplus K_1 \longrightarrow Q\,.$$

The complex $D_\bullet$ carries a multiplicative structure given by [4, thm. 1.13] in terms of the basis

(A.3.1) $$\begin{pmatrix} g_1^* \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} g_n^* \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_3 \end{pmatrix} \quad \text{for } D_1\,,$$

(A.3.2) $$\begin{pmatrix} f_1^* \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} f_l^* \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_2\varepsilon_3 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_1\varepsilon_3 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon_1\varepsilon_2 \end{pmatrix} \quad \text{for } D_2\,, \text{ and}$$

(A.3.3) $$e_1^*, \dots, e_m^* \quad \text{for } D_3\,.$$

Here we recall from [4, thm. 1.13] the products that involve the last three vectors in the basis for D_1.

(A.3.4) $$\begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix}\begin{pmatrix} 0 \\ \varepsilon_j \end{pmatrix} = -\begin{pmatrix} 0 \\ \varepsilon_i\varepsilon_j \end{pmatrix} \quad \text{for } i,j \in \{1,2,3\}\,.$$

(A.3.5) $$\begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix}\begin{pmatrix} g_j^* \\ 0 \end{pmatrix} = \sum_{h=1}^{l} g_j^*(\varphi_1(\varepsilon_i) f_h)\begin{pmatrix} f_h^* \\ 0 \end{pmatrix}.$$

(A.3.6) $$\begin{aligned} \begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix}\begin{pmatrix} 0 \\ \varepsilon_i\varepsilon_j \end{pmatrix} &= 0 \quad \text{and} \\ \begin{pmatrix} 0 \\ \varepsilon_h \end{pmatrix}\begin{pmatrix} 0 \\ \varepsilon_i\varepsilon_j \end{pmatrix} &= \pm \partial_1^{A_\bullet} \in A_1^* \quad \text{for } \{h,i,j\} = \{1,2,3\}\,. \end{aligned}$$

(A.3.7) $$\begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix}\begin{pmatrix} f_j^* \\ 0 \end{pmatrix} = \sum_{h=1}^{n} f_j^*(\varphi_1(\varepsilon_i) e_h) e_h^*\,.$$

From the free resolution $D_\bullet$ one gets a minimal free resolution of $Q/\mathfrak{B}$ that we denote $B_\bullet$. From [4, (1.80)] one has :

$$\begin{aligned} \operatorname{rank}_Q B_1 &= n + 3 - \operatorname{rank}_k(\varphi_3 \otimes k) - \operatorname{rank}_k(\varphi_2 \otimes k)\,, \\ \operatorname{rank}_Q B_2 &= l + 3 - \operatorname{rank}_k(\varphi_2 \otimes k) - \operatorname{rank}_k(\varphi_1 \otimes k)\,, \text{ and} \\ \operatorname{rank}_Q B_3 &= m - \operatorname{rank}_k(\varphi_1 \otimes k)\,. \end{aligned}$$

The identity $1 - \operatorname{rank}_Q B_1 + \operatorname{rank}_Q B_2 - \operatorname{rank}_Q B_3 = 0$ yields $\operatorname{rank}_k(\varphi_3 \otimes k) = 0$.

Set

$$\mathsf{B}_\bullet \;=\; \operatorname{Tor}_\bullet^Q(Q/\mathfrak{B}, k) \;=\; \mathrm{H}_\bullet(B_\bullet \otimes_Q k) \;=\; \mathrm{H}_\bullet(D_\bullet \otimes_Q k)$$

and denote the homology classes of the basis vectors as follows:

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n,\mathsf{E}_{n+1},\mathsf{E}_{n+2},\mathsf{E}_{n+3} &\quad \text{for the vectors in (A.3.1),}\\ \mathsf{F}_1,\dots,\mathsf{F}_l,\mathsf{F}_{l+1},\mathsf{F}_{l+2},\mathsf{F}_{l+3} &\quad \text{for the vectors in (A.3.2), and}\\ \mathsf{G}_1,\dots,\mathsf{G}_m &\quad \text{for the vectors in (A.3.3).}\end{aligned}$$

Notice that (A.3.6) yields

$$\text{(A.3.8)} \qquad \mathsf{E}_{n+i}\mathsf{F}_{l+j} \;=\; 0 \quad \text{for } i,j \in \{1,2,3\}\,.$$

A.4 Linkage via minimal generators. Adopt the setup established in A.1–A.3. If x_1, x_2, x_3 are part of a minimal system of generators for $\mathfrak{A}$, then the homomorphism $\varphi_1 \colon K_1 \to A_1$ is given by $\varphi_1(\varepsilon_i) = e_i$ for $1 \le i \le 3$.

One has $\operatorname{rank}_k(\varphi_1 \otimes k) = 3$. More precisely, let h, i, and j denote the three elements in $\{1,2,3\}$, now [4, prop. 1.6] yields

$$\partial_3^{D_\bullet}(e_i^*) \;=\; \pm\begin{pmatrix} 0 \\ \varepsilon_h\varepsilon_j \end{pmatrix} \quad \operatorname{mod} \mathfrak{M}D_2\,,$$

so in the algebra $\mathsf{B}_\bullet$ one has

$$\text{(A.4.1)} \qquad \mathsf{F}_{l+1} = \mathsf{F}_{l+2} = \mathsf{F}_{l+3} = 0 \qquad \text{and} \qquad \mathsf{G}_1 = \mathsf{G}_2 = \mathsf{G}_3 = 0\,.$$

Moreover, the rank of $\varphi_2 \otimes k$ is the number of linearly independent products e_ie_j mod $\mathfrak{M}A_2$ for $i,j \in \{1,2,3\}$. More precisely, if $e_ie_j = \pm f_k \mod \mathfrak{M}A_2$ holds for some $k \in \{1,\dots,l\}$, then [4, prop. 1.6] yields

$$\partial_2^{D_\bullet}\begin{pmatrix} f_k^* \\ 0 \end{pmatrix} \;=\; \pm\begin{pmatrix} 0 \\ \varepsilon_h \end{pmatrix} \quad \operatorname{mod} \mathfrak{M}D_1\,,$$

so in $\mathsf{B}_\bullet$ one has

$$\text{(A.4.2)} \qquad \mathsf{E}_{n+h} = 0 \quad \text{and} \quad \mathsf{F}_k = 0\,.$$

In particular, it follows from (A.4.1) that the only possible non-zero products among the basis vectors for $\mathsf{B}_\bullet$ that involve E_{n+1}, E_{n+2}, or E_{n+3} are:

$$\text{(A.4.3)} \qquad \mathsf{E}_{n+i}\mathsf{E}_j \;=\; \sum_{h=1}^{l} \mathsf{g}_j^*(\mathsf{e}_i\mathsf{f}_h)\mathsf{F}_h \quad \text{for } 1 \leqslant i \leqslant 3 \text{ and } 1 \leqslant j \leqslant n\,.$$

$$\text{(A.4.4)} \qquad \mathsf{E}_{n+i}\mathsf{F}_j \;=\; \sum_{h=4}^{n} \mathsf{f}_j^*(\mathsf{e}_i\mathsf{e}_h)\mathsf{G}_h \quad \text{for } 1 \leqslant i \leqslant 3 \text{ and } 1 \leqslant j \leqslant l\,.$$

A.5 Regular seqeunces. Adopt Setup A.1. Let $x_1,\dots,x_m$ be a minimal set of generators for $\mathfrak{A}$; the homomorphism $\partial_1^{A_\bullet} \colon A_1 \to A_0 = Q$ is given by $\partial_1^A(e_i) = x_i$. By a standard argument, see for example the proofs of [9, prop. 2.2] or [6, lem. 8.2], one can add elements from $\mathfrak{M}\mathfrak{A}$ to modify the sequence of generators such that x_1, x_2, x_3 form a regular sequence. This corresponds to a change of basis on A_1, where e_i for $1 \leqslant i \leqslant 3$ is replaced by $e_i + \sum_{j=1}^m a_{ij}e_j$ with coefficients $a_{ij} \in \mathfrak{M}$. Notice that the homology classes in $\mathsf{A}_\bullet$ of the basis vectors do not change: one has $[e_i + \sum_{j=1}^m a_{ij}e_j] = [e_i] = \mathsf{e}_i$. This means that given any basis $\mathsf{e}_1,\dots,\mathsf{e}_m$ for A_1 one can without loss of generality assume that the generators x_1, x_2, x_3 of $\mathfrak{A}$ corresponding to $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3$ form a regular sequence. We tacitly do so in A.6–A.9.

For ease of reference, still in A.6–A.9, we recall from (2.2.1) the parameters that describe the multiplicative structures:

(A.5.1)

Class of $\mathfrak{A}$	$p_{\mathfrak{A}}$	$q_{\mathfrak{A}}$	$r_{\mathfrak{A}}$
$\mathbf{B}$	1	1	2
$\mathbf{C}(3)$	3	1	3
$\mathbf{G}(r)$ $[r \geqslant 2]$	0	1	r
$\mathbf{H}(p,q)$	p	q	q
$\mathbf{T}$	3	0	0

The rest of this appendix is taken up by the arguments for Propositions 3.1–3.4.

A.6 Proof of Proposition 3.1. Adopt Setup A.1.

(a): One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3 \qquad \text{and} \qquad \mathsf{e}_1\mathsf{f}_1 = \mathsf{g}_1 = \mathsf{e}_2\mathsf{f}_2\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\ldots,\mathsf{E}_n,\mathsf{E}_{n+1},\mathsf{E}_{n+2} &\quad \text{for } \mathsf{B}_1\,,\\ \mathsf{F}_1,\mathsf{F}_2,\mathsf{F}_4,\ldots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and}\\ \mathsf{G}_4,\ldots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned}$$

In particular, one has $m_{\mathfrak{B}} = n+2$ and $n_{\mathfrak{B}} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1} and E_{n+2} are precisely

(1) $$\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1 \qquad \text{and} \qquad \mathsf{E}_{n+2}\mathsf{E}_1 = \mathsf{F}_2\,.$$

This means that $p_{\mathfrak{B}}$ is at least 2, so $\mathfrak{B}$ is of class $\mathbf{H}$ or $\mathbf{T}$; see (A.5.1).

Assume towards a contradiction that $\mathfrak{B}$ is of class $\mathbf{T}$. Per (A.2.2) there are bases $\{\mathsf{E}'_i\}$ for B_1 and $\{\mathsf{F}'_j\}$ for B_2 with nonzero products

$$\mathsf{E}'_1\mathsf{E}'_2 = \mathsf{F}'_3\,, \quad \mathsf{E}'_2\mathsf{E}'_3 = \mathsf{F}'_1\,, \quad \text{and} \quad \mathsf{E}'_3\mathsf{E}'_1 = \mathsf{F}'_2\,.$$

Write $\mathsf{E}_{n+1} = \sum_{i=1}^{n+2}\alpha_i\mathsf{E}'_i$ and $\mathsf{E}_{n+2} = \sum_{i=1}^{n+2}\beta_i\mathsf{E}'_i$; now (1) yields

$$\mathsf{F}_1 \;=\; \sum_{i=1}^{3}\alpha_i\mathsf{E}'_i\mathsf{E}_1 \qquad \text{and} \qquad \mathsf{F}_2 \;=\; \sum_{i=1}^{3}\beta_i\mathsf{E}'_i\mathsf{E}_1\,.$$

As the vectors F_1 and F_2 are linearly independent, so are the vectors $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$. That is, the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$$

has rank 2, whence it follows that the product

$$\mathsf{E}_{n+1}\mathsf{E}_{n+2} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathsf{F}'_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathsf{F}'_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathsf{F}'_3$$

is nonzero, and that contradicts (1).

(b): One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1 \quad \text{for } 1 \leqslant i \leqslant r_{\mathfrak{A}}\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n,\mathsf{E}_{n+1},\mathsf{E}_{n+2},\mathsf{E}_{n+3} &\quad \text{for } \mathsf{B}_1\,, \\ \mathsf{F}_1,\dots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and} \\ \mathsf{G}_4,\dots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned} \tag{2}$$

In particular, one has $m_\mathfrak{B} = n+3$ and $n_\mathfrak{B} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1}, E_{n+2}, and E_{n+3} are precisely

$$\mathsf{E}_{n+i}\mathsf{E}_1 = \mathsf{F}_i \quad \text{for } 1 \leqslant i \leqslant \min\{r_\mathfrak{A}, 3\}\,.$$

As $r_\mathfrak{A} \geqslant 2$ the ideal $\mathfrak{B}$ is per (A.5.1) of class **H** or **T**, and the argument from the proof of part (a) applies to show that $\mathfrak{B}$ is not of class **T**.

(c): One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_3\mathsf{e}_4 = \mathsf{f}_5\,, \quad \mathsf{e}_4\mathsf{e}_5 = \mathsf{f}_3\,, \quad \text{and} \quad \mathsf{e}_5\mathsf{e}_3 = \mathsf{f}_4\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has basis (2). In particular, one has $m_\mathfrak{B} = n+3$ and $n_\mathfrak{B} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1}, E_{n+2}, and E_{n+3} are precisely

$$\mathsf{E}_{n+3}\mathsf{F}_5 = \mathsf{G}_4 \qquad \text{and} \qquad \mathsf{E}_{n+3}\mathsf{F}_4 = -\mathsf{G}_5\,.$$

Thus one has $q_\mathfrak{B} \geqslant 2$, in particular $\mathfrak{B}$ is of class **H**; see (A.5.1).

(d): One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_4\,, \quad \mathsf{e}_3\mathsf{e}_4 = \mathsf{f}_3, \quad \text{and} \quad \mathsf{e}_4\mathsf{e}_2 = \mathsf{f}_3\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n,\mathsf{E}_{n+2},\mathsf{E}_{n+3} &\quad \text{for } \mathsf{B}_1\,, \\ \mathsf{F}_1,\mathsf{F}_2,\mathsf{F}_3,\mathsf{F}_5,\dots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and} \\ \mathsf{G}_4,\dots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned}$$

In particular, one has $m_\mathfrak{B} = n+2$ and $n_\mathfrak{B} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+2} and E_{n+3} are precisely

$$\mathsf{E}_{n+3}\mathsf{F}_3 = \mathsf{G}_4 = -\mathsf{E}_{n+2}\mathsf{F}_2\,. \tag{3}$$

As $n_\mathfrak{A} \geqslant 2$ by 2.4, one has $m_\mathfrak{B} \geqslant 4$, $q_\mathfrak{B} \geqslant 1$, and $r_\mathfrak{B} \geqslant 2$, so per (A.5.1) the ideal $\mathfrak{B}$ is of class **B**, **G**, or **H**, and if $\mathfrak{B}$ is of class **B** or **G**, then $q_\mathfrak{B} = 1$ holds. If $\mathfrak{B}$ is of class **H**, then per (A.2.2) there are bases $\{\mathsf{E}'_i\}$ for B_1, $\{\mathsf{F}'_j\}$ for B_2, and $\{\mathsf{G}'_k\}$ for B_3 with nonzero products

$$\begin{aligned} \mathsf{E}'_{p'+1}\mathsf{E}'_j = \mathsf{F}'_j &\quad \text{for } 1 \leqslant j \leqslant p' \quad \text{and} \\ \mathsf{E}'_{p'+1}\mathsf{F}'_{p'+i} = \mathsf{G}'_i &\quad \text{for } 1 \leqslant i \leqslant q'\,. \end{aligned}$$

Write $\mathsf{E}_{n+3} = \sum_{i=1}^{n+2} \alpha_i \mathsf{E}'_i$. It follows from the nonzero product $\mathsf{E}_{n+3}\mathsf{F}_3 = \mathsf{G}_4$ that $\alpha_{p'+1}$ is nonzero. As $q' = r_\mathfrak{B} \geqslant 2$ per (A.5.1) it follows that there are two linearly independent products of the form $\mathsf{E}_{n+3}\mathsf{F}_j$ which contradicts (3).

(e): One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3\,, \quad \mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1\,, \quad \text{and} \quad \mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n &\quad \text{for } \mathsf{B}_1\,, \\ \mathsf{F}_4,\dots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and} \\ \mathsf{G}_4,\dots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned}$$

with $m_{\mathfrak{B}} = n$ and $n_{\mathfrak{B}} = m$.

A.7 Proof of Proposition 3.2. Adopt Setup A.1 and set $p = p_{\mathfrak{A}}$ and $q = q_{\mathfrak{A}}$. One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_1\mathsf{e}_{3+i} = \mathsf{f}_{q+i} \quad \text{for } 1 \leqslant i \leqslant p \qquad \text{and} \qquad \mathsf{e}_1\mathsf{f}_j = \mathsf{g}_j \quad \text{for } 1 \leqslant j \leqslant q\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n,\mathsf{E}_{n+1},\mathsf{E}_{n+2},\mathsf{E}_{n+3} &\quad \text{for } \mathsf{B}_1\,, \\ \mathsf{F}_1,\dots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and} \\ \mathsf{G}_4,\dots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned}$$

In particular, one has $m_{\mathfrak{B}} = n+3$ and $n_{\mathfrak{B}} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1} are precisely

$$(1)\qquad \begin{aligned} \mathsf{E}_{n+1}\mathsf{E}_j = \mathsf{F}_j &\qquad \text{for } 1 \leqslant j \leqslant q \quad \text{and} \\ \mathsf{E}_{n+1}\mathsf{F}_{q+i} = \mathsf{G}_{3+i} &\qquad \text{for } 1 \leqslant i \leqslant p\,. \end{aligned}$$

Set $p' = p_{\mathfrak{B}}$ and $q' = q_{\mathfrak{B}}$; evidently one has

$$(2)\qquad p' \geqslant q \qquad \text{and} \qquad q' \geqslant p\,.$$

Notice that $\mathfrak{B}$ is not complete intersection, as one has $m_{\mathfrak{B}} \geqslant 4$; cf. 2.3. That is, $\mathfrak{B}$ is of class **B**, **G**, **H**, or **T**.

(a): If $p \geqslant 1$ holds, then (2) yields $q' \geqslant 1$ and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **B**, **G**, or **H**.

If $\mathfrak{B}$ is of class **B**, then there are per (A.2.2) bases $\{\mathsf{E}'_i\}$ for B_1, $\{\mathsf{F}'_j\}$ for B_2, and $\{\mathsf{G}'_k\}$ for B_3 with nonzero products

$$(3)\qquad \mathsf{E}'_1\mathsf{E}'_2 = \mathsf{F}'_3 \qquad \text{and} \qquad \mathsf{E}'_1\mathsf{F}'_1 = \mathsf{G}'_1 = \mathsf{E}'_2\mathsf{F}'_2\,.$$

By (A.5.1) and (2) one has $1 = p' \geqslant q$, so assume towards a contraction that $q = 0$ holds. Write $\mathsf{E}_{n+1} = \sum_{i=1}^{n+3} \alpha_i \mathsf{E}'_i$ and $\mathsf{F}_{q+1} = \sum_{j=1}^{l} \beta_j \mathsf{F}'_j$. By (1) and (3) one has

$$0 \neq \mathsf{G}_4 = \mathsf{E}_{n+1}\mathsf{F}_{q+1} = (\alpha_1\beta_1 + \alpha_2\beta_2)\mathsf{G}'_1$$

and, therefore, $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. From the equalities

$$\mathsf{E}_{n+1}\mathsf{E}'_1 = \alpha_2\mathsf{E}'_2\mathsf{E}'_1 = -\alpha_2\mathsf{F}'_3 \qquad \text{and} \qquad \mathsf{E}_{n+1}\mathsf{E}'_2 = \alpha_1\mathsf{E}'_1\mathsf{E}'_2 = \alpha_1\mathsf{F}'_3$$

it follows that $\mathsf{E}_{n+1}\mathsf{E}_j$ is nonzero for some j, which contradicts the assumption $q = 0$; see (1). Thus $q = 1 = p'$ holds.

If $\mathfrak{B}$ is of class **G**, then (A.5.1) and (2) yield $0 = p' \geqslant q$, so $p' = 0 = q$ holds.

If $\mathfrak{B}$ is of class **H**, then there are per (A.2.2) bases $\{\mathsf{E}'_i\}$ for B_1, $\{\mathsf{F}'_j\}$ for B_2, and $\{\mathsf{G}'_k\}$ for B_3 with nonzero products

$$(4)\qquad \begin{aligned} \mathsf{E}'_{p'+1}\mathsf{E}'_j = \mathsf{F}'_j &\quad \text{for } 1 \leqslant j \leqslant p' \quad \text{and} \\ \mathsf{E}'_{p'+1}\mathsf{F}'_{p'+i} = \mathsf{G}'_i &\quad \text{for } 1 \leqslant i \leqslant q'\,. \end{aligned}$$

Write $\mathsf{E}_{n+1} = \sum_{i=1}^{n+3} \alpha_i \mathsf{E}'_i$. The product $\mathsf{E}_{n+1}\mathsf{F}_{q+1} = \mathsf{G}_4$ is by (1) nonzero, so (4) yields $\alpha_{p'+1} \neq 0$. From

$$\mathsf{E}_{n+1}\mathsf{E}'_j = \alpha_{p'+1}\mathsf{E}'_{p'+1}\mathsf{E}'_j = \alpha_{p'+1}\mathsf{F}'_j \neq 0 \quad \text{for } 1 \leqslant j \leqslant p'$$

it follows that there are p' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$, which forces $p' \leqslant q$; see (1). Thus $p' = q$ holds by (2).

(b): If $p \geqslant 2$ holds, then (2) yields $q' \geqslant 2$, and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **H**. Moreover, $p' = q$ holds by (a).

(c): Assume that $q \geqslant 2$ holds. As one has $p' \geqslant q$, see (2), it follows per (A.5.1) that $\mathfrak{B}$ is of class **H** or **T**. By (2) it suffices to prove that $q' \leqslant p$ holds.

If $\mathfrak{B}$ is of class **T**, then per (A.5.1) one has $0 = q'$ so $q' \leqslant p$ trivially holds.

If $\mathfrak{B}$ is of class **H**, then as in the proof of part (a) there exist bases with the nonzero products given in (4). Write

$$\mathsf{E}_{n+1} = \sum_{i=1}^{n+3} \alpha_i \mathsf{E}'_i\,, \quad \mathsf{E}_1 = \sum_{i=1}^{n+3} \beta_i \mathsf{E}'_i\,, \quad \text{and} \quad \mathsf{E}_2 = \sum_{i=1}^{n+3} \gamma_i \mathsf{E}'_i\,.$$

Now (1) and (4) yield

$$0 \neq \mathsf{F}_1 = \mathsf{E}_{n+1}\mathsf{E}_1 = \sum_{j=1}^{p'} (\alpha_{p'+1}\beta_j - \alpha_j\beta_{p'+1})\mathsf{F}'_j \quad \text{and}$$
$$0 \neq \mathsf{F}_2 = \mathsf{E}_{n+1}\mathsf{E}_2 = \sum_{j=1}^{p'} (\alpha_{p'+1}\gamma_j - \alpha_j\gamma_{p'+1})\mathsf{F}'_j\,.$$

The vectors F_1 and F_2 are linearly independent, while one has

$$\gamma_{p'+1}\sum_{j=1}^{p'} \alpha_j\beta_{p'+1}\mathsf{F}'_j = \beta_{p'+1}\sum_{i=j}^{p'} \alpha_j\gamma_{p'+1}\mathsf{F}'_j\,;$$

it follows that $\alpha_{p'+1}$ is nonzero. Thus, one has

$$\mathsf{E}_{n+1}\mathsf{F}'_{p'+i} = \alpha_{p'+1}\mathsf{E}'_{p'+1}\mathsf{F}'_{p'+i} = \alpha_{p'+1}\mathsf{G}'_i \quad \text{for } 1 \leqslant i \leqslant q'\,.$$

It follows that there are q' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{F}_j$, which forces $q' \leqslant p$; see (1).

(d): If $q \geqslant 3$ holds, then (2) yields $p' \geqslant 3$, and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **H** or **T**. If $\mathfrak{B}$ is of class **T**, then there are per (A.2.2) bases $\{\mathsf{E}'_i\}$ for B_1 and $\{\mathsf{F}'_j\}$ for B_2 with nonzero products

(5) $$\mathsf{E}'_1\mathsf{E}'_2 = \mathsf{F}'_3\,, \quad \mathsf{E}'_2\mathsf{E}'_3 = \mathsf{F}'_1\,, \quad \text{and} \quad \mathsf{E}'_3\mathsf{E}'_1 = \mathsf{F}'_2\,.$$

It follow that for any element E' in B_1, the map $\mathsf{B}_1 \to \mathsf{B}_2$ given by multiplication by E' has rank at most 2. However, multiplication by E_{n+1} has rank $q \geqslant 3$; see (1). Thus $\mathfrak{B}$ is not of class **T** and hence of class **H**; finally (c) yields $q' = p$.

(e): Assume that $q = 1$ holds. By (2) one has $p' \geqslant 1$, so the ideal $\mathfrak{B}$ is per (A.5.1) of class **B**, **H**, or **T**. If $\mathfrak{B}$ is of class **T** then, as in the proof of part (d), there are bases with nonzero products as in (5). Write $\mathsf{E}_{n+1} = \sum_{i=1}^{n+3} \alpha_i \mathsf{E}'_i$ and $\mathsf{E}_1 = \sum_{i=1}^{n+3} \beta_i \mathsf{E}'_i$. By (1) and (5) one has

$$0 \neq \mathsf{F}_1 = \mathsf{E}_{n+1}\mathsf{E}_1 = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathsf{F}'_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathsf{F}'_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathsf{F}'_3\,.$$

It follows that at least one of α_1, α_2, or α_3 is nonzero. From the expressions

$$\begin{aligned}
\mathsf{E}_{n+1}\mathsf{E}'_1 &= \alpha_2\mathsf{E}'_2\mathsf{E}'_1 + \alpha_3\mathsf{E}'_3\mathsf{E}'_1 = \alpha_3\mathsf{F}'_2 - \alpha_2\mathsf{F}'_3\,,\\
\mathsf{E}_{n+1}\mathsf{E}'_2 &= \alpha_1\mathsf{E}'_1\mathsf{E}'_2 + \alpha_3\mathsf{E}'_3\mathsf{E}'_2 = \alpha_1\mathsf{F}'_3 - \alpha_3\mathsf{F}'_1\,, \quad \text{and}\\
\mathsf{E}_{n+1}\mathsf{E}'_3 &= \alpha_1\mathsf{E}'_1\mathsf{E}'_3 + \alpha_2\mathsf{E}'_2\mathsf{E}'_3 = \alpha_2\mathsf{F}'_1 - \alpha_1\mathsf{F}'_2
\end{aligned}$$

it now follows that there are two linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$, which contradicts the assumption $q = 1$; see (1). Thus $\mathfrak{B}$ is not of class $\mathbf{T}$.

(f): It follows from (e) that the ideal $\mathfrak{B}$ is of class $\mathbf{B}$ or $\mathbf{H}$. If $\mathfrak{B}$ is of class $\mathbf{B}$, then as in the proof of part (a) there exist bases with the nonzero products given in (3). Write $\mathsf{E}_{n+1} = \sum_{i=1}^{n+3} \alpha_i\mathsf{E}'_i$ and $\mathsf{E}_1 = \sum_{i=1}^{n+3} \beta_i\mathsf{E}'_i$. Now (1) and (3) yield

$$0 \neq \mathsf{F}_1 = \mathsf{E}_{n+1}\mathsf{E}_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)\mathsf{F}'_3$$

and, therefore, $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. As one has

$$\mathsf{E}_{n+1}\mathsf{F}'_1 = \alpha_1\mathsf{E}'_1\mathsf{F}'_1 = \alpha_1\mathsf{G}'_1 \qquad \text{and} \qquad \mathsf{E}_{n+1}\mathsf{F}'_2 = \alpha_2\mathsf{E}'_2\mathsf{F}'_2 = \alpha_2\mathsf{G}'_1$$

it follows that $\mathsf{E}_{n+1}\mathsf{F}_j$ is nonzero for some j, which contradicts the assumption $p = 0$; see (1). Thus $\mathfrak{B}$ is not of class $\mathbf{B}$.

A.8 Proof of Proposition 3.3. Adopt Setup A.1 and set $p = p_{\mathfrak{A}}$ and $q = q_{\mathfrak{A}}$. One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_1\mathsf{e}_{2+i} = \mathsf{f}_{q+i} \quad \text{for } 1 \leqslant i \leqslant p \qquad \text{and} \qquad \mathsf{e}_1\mathsf{f}_j = \mathsf{g}_j \quad \text{for } 1 \leqslant j \leqslant q\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned}
\mathsf{E}_1, \ldots, \mathsf{E}_n, \mathsf{E}_{n+1}, \mathsf{E}_{n+3} &\quad \text{for } \mathsf{B}_1\,,\\
\mathsf{F}_1, \ldots, \mathsf{F}_q, \mathsf{F}_{q+2}, \ldots, \mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and}\\
\mathsf{G}_4, \ldots, \mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,.
\end{aligned}$$

In particular, one has $m_{\mathfrak{B}} = n+2$ and $n_{\mathfrak{B}} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1} are precisely

$$\begin{aligned}
\mathsf{E}_{n+1}\mathsf{E}_j &= \mathsf{F}_j \qquad \text{for } 1 \leqslant j \leqslant q \quad \text{and}\\
\mathsf{E}_{n+1}\mathsf{F}_{q+i} &= \mathsf{G}_{2+i} \quad \text{for } 2 \leqslant i \leqslant p\,.
\end{aligned} \tag{1}$$

Set $p' = p_{\mathfrak{B}}$ and $q' = q_{\mathfrak{B}}$; evidently one has

$$p' \geqslant q \qquad \text{and} \qquad q' \geqslant p - 1\,. \tag{2}$$

Notice that $\mathfrak{B}$ is not complete intersection, as one has $n \geqslant 2$ by assumption and hence $m_{\mathfrak{B}} \geqslant 4$; see 2.3 and 2.4. That is, $\mathfrak{B}$ is of class $\mathbf{B}$, $\mathbf{G}$, $\mathbf{H}$, or $\mathbf{T}$.

(a): If $p \geqslant 2$ holds, then (2) yields $q' \geqslant 1$, and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class $\mathbf{B}$, $\mathbf{G}$, or $\mathbf{H}$.

If $\mathfrak{B}$ is of class $\mathbf{B}$, then one has $1 = p' \geqslant q$, see (A.5.1) and (2), so assume towards a contraction that $q = 0$ holds. By an argument parallel to the one given in the proof of Proposition 3.2(a), the nonzero product $\mathsf{E}_{n+1}\mathsf{F}_{q+2} = \mathsf{G}_4$ from (1) forces $\mathsf{E}_{n+1}\mathsf{E}_j \neq 0$ for some j, which contradicts the assumption $q = 0$; see (1). Thus $q = 1 = p'$ holds.

If $\mathfrak{B}$ is of class $\mathbf{G}$, then (A.5.1) and (2) yield $0 = p' \geqslant q$, so $p' = 0 = q$ holds.

If $\mathfrak{B}$ is of class $\mathbf{H}$, then an argument parallel to the one given in the proof of Proposition 3.2(a) shows that the nonzero product $\mathsf{E}_{n+1}\mathsf{F}_{q+2} = \mathsf{G}_4$ from (1) forces

p' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$. This implies $p' \leqslant q$, see (1), whence $p' = q$ holds by (2).

(b): If $p \geqslant 3$ holds, then (2) yields $q' \geqslant 2$, and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **H**. Moreover, $p' = q$ holds by (a).

(c): Assume that $q \geqslant 2$ holds. As one has $p' \geqslant 2$, see (2), it follows per (A.5.1) that $\mathfrak{B}$ is of class **H** or **T**. By (2) it is sufficient to prove that $q' \leqslant p - 1$ holds.

If $\mathfrak{B}$ is of class **T**, then per (A.5.1) one has $0 = q'$, so $q' \leqslant p - 1$ trivially holds.

If $\mathfrak{B}$ is of class **H**, then the argument given in the proof of Proposition 3.2(c) applies to show that the nontrivial products $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ and $\mathsf{E}_{n+1}\mathsf{E}_2 = \mathsf{F}_2$ from (1) force q' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{F}_j$. This implies $q' \leqslant p-1$; see (1).

(d): The proof of Proposition 3.2(d) applies.

(e): Assume that $q = 1$ holds. By (2) one has $p' \geqslant 1$, so the ideal $\mathfrak{B}$ is per (A.5.1) of class **B**, **H**, or **T**. If $\mathfrak{B}$ is of class **T**, then the argument given in the proof of Proposition 3.2(e) applies to show that the nonzero product $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ from (1) forces two linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$, which contradicts the assumption $q = 1$; see (1).

(f): It follows from (e) that the ideal $\mathfrak{B}$ is of class **B** or **H**. If $\mathfrak{B}$ is of class **B**, then the argument in the proof of Proposition 3.2(f) applies to show that the nonzero product $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ from (1) forces $\mathsf{E}_{n+1}\mathsf{F}_j \neq 0$ for some j, and that contradicts the assumption $p = 1$; see (1).

A.9 Proof of Proposition 3.4. Adopt Setup A.1 and set $p = p_{\mathfrak{A}}$ and $q = q_{\mathfrak{A}}$. One may assume that the nonzero products of elements from (A.2.1) are

$$\mathsf{e}_1\mathsf{e}_{1+i} = \mathsf{f}_{q+i} \quad \text{for } 1 \leqslant i \leqslant p \qquad \text{and} \qquad \mathsf{e}_1\mathsf{f}_j = \mathsf{g}_j \quad \text{for } 1 \leqslant j \leqslant q\,.$$

Proceeding as in A.3, it follows from A.4 that $\mathfrak{A}$ is directly linked to an ideal $\mathfrak{B}$ whose Tor algebra $\mathsf{B}_\bullet$ has bases

$$\begin{aligned} \mathsf{E}_1,\dots,\mathsf{E}_n,\mathsf{E}_{n+1} &\quad \text{for } \mathsf{B}_1\,,\\ \mathsf{F}_1,\dots,\mathsf{F}_q,\mathsf{F}_{q+3},\dots,\mathsf{F}_l &\quad \text{for } \mathsf{B}_2\,, \text{ and}\\ \mathsf{G}_4,\dots,\mathsf{G}_m &\quad \text{for } \mathsf{B}_3\,. \end{aligned}$$

In particular, one has $m_{\mathfrak{B}} = n+1$ and $n_{\mathfrak{B}} = m-3$. Further it follows from (A.4.3) and (A.4.4) that the nonzero products in $\mathsf{B}_\bullet$ that involve E_{n+1} are precisely

$$(1) \qquad \begin{aligned} \mathsf{E}_{n+1}\mathsf{E}_j = \mathsf{F}_j &\quad \text{for } 1 \leqslant j \leqslant q \quad \text{and}\\ \mathsf{E}_{n+1}\mathsf{F}_{q+i} = \mathsf{G}_{1+i} &\quad \text{for } 3 \leqslant i \leqslant p\,. \end{aligned}$$

Set $p' = p_{\mathfrak{B}}$ and $q' = q_{\mathfrak{B}}$; evidently one has

$$(2) \qquad p' \geqslant q \qquad \text{and} \qquad q' \geqslant p-2\,.$$

Notice from 2.3 that $\mathfrak{B}$ is not complete intersection as one has $m \geqslant 5$ or $n \geqslant 3$ by assumption and hence $m_{\mathfrak{B}} \geqslant 4$ or $n_{\mathfrak{B}} \geqslant 2$. That is, $\mathfrak{B}$ is of class **B**, **G**, **H**, or **T**.

(a): If $p \geqslant 3$ holds, then (2) yields $q' \geqslant 1$ and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **B**, **G**, or **H**.

If $\mathfrak{B}$ is of class **B**, then one has $1 = p' \geqslant q$, see (A.5.1) and (2), so assume towards a contraction that $q = 0$ holds. By an argument parallel to the one given in the proof of Proposition 3.2(a), the nonzero product $\mathsf{E}_{n+1}\mathsf{F}_{q+3} = \mathsf{G}_4$ from (1) forces $\mathsf{E}_{n+1}\mathsf{E}_j \neq 0$ for some j, which contradicts the assumption $q = 0$; see (1). Thus $q = 1 = p'$ holds.

If $\mathfrak{B}$ is of class **G**, then (A.5.1) and (2) yield $0 = p' \geqslant q$, so $p' = 0 = q$ holds.

If $\mathfrak{B}$ is of class **H**, then an argument parallel to the one given in the proof of Proposition 3.2(a) shows that the nonzero product $\mathsf{E}_{n+1}\mathsf{F}_{q+3} = \mathsf{G}_4$ from (1) forces p' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$. This implies $p' \leqslant q$, see (1), whence $p' = q$ holds by (2).

(b): If $p \geqslant 4$ holds, then (2) yields $q' \geqslant 2$, and per (A.5.1) it follows that the ideal $\mathfrak{B}$ is of class **H**. Moreover, $p' = q$ holds by (a).

(c): Assume that $q \geqslant 2$ holds. As one has $p' \geqslant 2$, see (2), it follows per (A.5.1) that $\mathfrak{B}$ is of class **H** or **T**. By (2) it is sufficient to prove that $q' \leqslant p - 2$ holds.

If $\mathfrak{B}$ is of class **T**, then per (A.5.1) one has $0 = q'$, so $q' \leqslant p - 2$ trivially holds.

If $\mathfrak{B}$ is of class **H**, then the argument given in the proof of Proposition 3.2(c) applies to show that the nontrivial products $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ and $\mathsf{E}_{n+1}\mathsf{E}_2 = \mathsf{F}_2$ from (1) force q' linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{F}_j$. This implies $q' \leqslant p - 2$; see (1).

(d): The proof of Proposition 3.2(d) applies.

(e): Assume that $q = 1$ holds. By (2) one has $p' \geqslant 1$, so the ideal $\mathfrak{B}$ is per (A.5.1) of class **B**, **H**, or **T**. If $\mathfrak{B}$ is of class **T**, then the argument given in the proof of Proposition 3.2(e) applies to show that the nonzero product $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ from (1) forces two linearly independent products of the form $\mathsf{E}_{n+1}\mathsf{E}_j$, which contradicts the assumption $q = 1$; see (1).

(f): It follows from (e) that the ideal $\mathfrak{B}$ is of class **B** or **H**. If $\mathfrak{B}$ is of class **B**, then the argument in the proof of Proposition 3.2(f) applies to show that the nonzero product $\mathsf{E}_{n+1}\mathsf{E}_1 = \mathsf{F}_1$ from (1) forces $\mathsf{E}_{n+1}\mathsf{F}_j \neq 0$ for some j, and that contradicts the assumption $p = 2$; see (1).

References

[1] Luchezar L. Avramov, *Poincaré series of almost complete intersections of embedding dimension three*, PLISKA Stud. Math. Bulgar. **2** (1981), 167–172. MR0633871

[2] Luchezar L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR1648664

[3] Luchezar L. Avramov, *A cohomological study of local rings of embedding codepth 3*, J. Pure Appl. Algebra **216** (2012), no. 11, 2489–2506. MR2927181

[4] Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334

[5] Anne E. Brown, *A structure theorem for a class of grade three perfect ideals*, J. Algebra **105** (1987), no. 2, 308–327. MR0873666

[6] David A. Buchsbaum and David Eisenbud, *Some structure theorems for finite free resolutions*, Advances in Math. **12** (1974), 84–139. MR0340240

[7] David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723

[8] Lars Winther Christensen and Oana Veliche, *Local rings of embedding codepth 3: a classification algorithm*, J. Softw. Algebra Geom. **6** (2014), 1–8. MR3338667

[9] Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Free resolutions of Dynkin format and the licci property of grade* 3 *perfect ideals*, preprint arXiv:1712.04016 [math.AC]; 13 pp.

[10] Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *The structure of almost complete intersections of codimension* 3, in preparation.

[11] Evgeniy S. Golod, *A note on perfect ideals*, Algebra (A.I. Kostrikin, Ed.), Moscow State Universit Press, 1980, pp. 37–39.

[12] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `https://faculty.math.illinois.edu/Macaulay2/`.

[13] Rafael Sánchez, *A structure theorem for type* 3*, grade* 3 *perfect ideals*, J. Algebra **123** (1989), no. 2, 263–288. MR1000488

[14] Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. **50** (1973), 227–232. MR0319985

[15] Jerzy Weyman, *On the structure of free resolutions of length* 3, J. Algebra **126** (1989), no. 1, 1–33. MR1023284

Texas Tech University, Lubbock, TX 79409, U.S.A.
Email address: lars.w.christensen@ttu.edu
URL: http://www.math.ttu.edu/~lchriste

Northeastern University, Boston, MA 02115, U.S.A.
Email address: o.veliche@northeastern.edu
URL: https://web.northeastern.edu/oveliche

University of Connecticut, Storrs, CT 06269, U.S.A.
Email address: jerzy.weyman@uconn.edu
URL: http://www.math.uconn.edu/~weyman