GORENSTEIN DIMENSION OF MODULES OVER HOMOMORPHISMS

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Abstract. Given a homomorphism of commutative noetherian rings $R \to S$ and an S–module N, it is proved that the Gorenstein flat dimension of N over R, when finite, may be computed locally over S. When, in addition, the homomorphism is local and N is finitely generated over S, the Gorenstein flat dimension equals $\sup\{\, m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(E,N) \neq 0\}$, where E is the injective hull of the residue field of R. This result is analogous to a theorem of André on flat dimension.

Introduction

Let R be a commutative noetherian ring and let N be an R–module. We say that N is *finite over a homomorphism* if there exists a homomorphism of rings $R \to S$ such that S is noetherian, N is a finite (that is, finitely generated) S–module, and the S–action is compatible with the action of R.

In the case where $R \to S$ is a local homomorphism, this class of modules has been studied by Apassov [2], who called them almost finite modules, and by Avramov, Foxby, Miller, Sather-Wagstaff and others, cf. [5,23,7]. The work of these and other authors show that modules finite over (local) homomorphisms have homological properties extending those of finite modules (over local rings).

An important property of many invariants of R–modules is that they can be computed locally over R. A basic question is whether the same property holds for modules over a homomorphism; that is, whether an invariant of the R–module N can be computed locally over S. It is easy to see that this is the case for flat dimension; this paper focuses on the Gorenstein flat dimension. Introduced by Enochs, Jenda and Torrecillas [16], this invariant is one generalization to non-finite modules of the notion of G–dimension, due to Auslander and Bridger [3, 4]. In Theorem (2.1) we prove that if $\operatorname{Gfd}_R N$, the Gorenstein flat dimension of N, is finite, then

$$\operatorname{Gfd}_R N = \sup\{\, \operatorname{Gfd}_{R_\mathfrak{p}} N_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} S \text{ and } \mathfrak{p} = R \cap \mathfrak{q}\}.$$

This extends a well-known result [12, 21] for the absolute case $R \xrightarrow{=} S$.

The result above focuses attention on modules over local homomorphisms. In this situation, a theorem of André [1] says that if N is finite, then the flat dimension over R equals $\sup\{\, m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(k,N) \neq 0\}$, where k is the residue field of R.

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Theorem (4.1) gives an analogous result in the context of Gorenstein flat dimension: If N is finite over a local homomorphism, and $\operatorname{Gfd}_R N$ is finite, then

$$\operatorname{Gfd}_R N = \sup\{\, m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(E,N) \neq 0\},$$

where E is the injective hull of the residue field k. The absolute case appears in [9].

A crucial difference between this result and André's is that it must be assumed *a priori* that $\operatorname{Gfd}_R N$ is finite: Vanishing of $\operatorname{Tor}_{\gg 0}^R(E,N)$ does not detect finite Gorenstein flat dimension, see Example (4.3). This example also suggests that André's proof, which relies on the fact that finite flat dimension of N is detected by vanishing of $\operatorname{Tor}_{\gg 0}^R(-,N)$, is not likely to carry over to our context. And, indeed, our arguments have a different flavor.

As a corollary we obtain the following result about completions: If N is finite over a local homomorphism, and $\operatorname{Gfd}_R N$ is finite, then

$$\operatorname{Gfd}_R N = \operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N).$$

The corresponding result for flat dimension is elementary; for Gorenstein flat dimension we are not aware of any other proof.

1. Basic notions

Throughout the paper R and S denote rings; unless stated otherwise, they are assumed to be commutative and noetherian. Given a homomorphism $\varphi\colon R \to S$, any S–module becomes an R–module with the action determined by φ. We say that φ is *local*, if R and S are local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

We work with complexes, which we grade homologically:

$$M = \cdots \to M_{\ell+1} \to M_\ell \to M_{\ell-1} \to \cdots$$

The homological size of a complex is captured by the numbers $\sup M$ and $\inf M$, defined as the supremum and infimum of the set $\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(M) \neq 0\}$. We say that M is *homologically finite* if the R–module $\mathrm{H}(M)$ is *finite*, that is, finitely generated.

We use the notation $\mathsf{D}(R)$ for the derived category of R, and $\mathsf{D}^{\mathrm{f}}(R)$ for its subcategory of homologically finite complexes. We use the symbol $\simeq$ to denote isomorphisms in derived categories.

Let L and M be R–complexes, that is to say, complexes of R–modules. The derived tensor product and Hom functors are denoted $L \otimes_R^{\mathbf{L}} M$ and $\mathbf{R}\mathrm{Hom}_R(L,M)$. We write $\operatorname{pd}_R M$ for the projective dimension, and $\operatorname{fd}_R M$ for the flat dimension, of M over R, cf. [5].

When $(R,\mathfrak{m},k)$ is local, the *depth* of an R–complex M is defined by

(1.0.1) $$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k,M).$$

Thus, $\operatorname{depth}_R M = \infty$ if and only if $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,M)) = 0$.

(1.1) **Supports.** The *support* of an R–complex M is a subset of $\operatorname{Spec} R$:

$$\operatorname{Supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(M_\mathfrak{p}) \neq 0\},$$

and $\operatorname{Max}_R M$ is the subset of maximal ideals in $\operatorname{Supp}_R M$.

Foxby [18] has introduced the *small support* of M as

$$\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(k(\mathfrak{p}) \otimes_R^{\mathbf{L}} M) \neq 0\},$$

where $k(\mathfrak{p})$ denotes the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of R at $\mathfrak{p}$. For convenience we set

$$\max_R M = \{\mathfrak{p} \in \operatorname{supp}_R M \mid \mathfrak{p} \text{ is maximal in } \operatorname{supp}_R M\}.$$

Note that $\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M$ and equality holds when M is homologically finite. Elements in $\operatorname{Max}_R M$ are maximal ideals, while those in $\max_R M$ need not be; see property (c) below.

We recall some properties of these subsets. Let L and M be R–complexes. For $\mathfrak{p}$ in $\operatorname{Spec} R$ we write $\mathrm{E}_R(R/\mathfrak{p})$ for the injective hull of the R–module $R/\mathfrak{p}$.

(a) $\operatorname{supp}_R M = \emptyset$ if and only if $\mathrm{H}(M) = 0$.
(b) $\operatorname{supp}_R(L \otimes^{\mathbf{L}}_R M) = \operatorname{supp}_R L \cap \operatorname{supp}_R M$, for any R–complex L.
(c) $\operatorname{supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$, for any $\mathfrak{p}$ in $\operatorname{Spec} R$.
(d) A prime ideal $\mathfrak{p}$ is in $\operatorname{supp}_R M$ if and only if $\mathrm{H}(\mathrm{E}_R(R/\mathfrak{p}) \otimes^{\mathbf{L}}_R M) \neq 0$.
(e) When $\sup M = s$ is finite, the associated primes of the top homology module belong to the small support: $\operatorname{Ass}_R \mathrm{H}_s(M) \subseteq \operatorname{supp}_R M$.
(f) When $(R, \mathfrak{m}, k)$ is local, $\mathfrak{m}$ is in $\operatorname{supp}_R M$ if and only if $\operatorname{depth}_R M$ is finite.

Indeed, parts (a), (b), (c), and (f) are proved in [18, sec. 2]; part (d) follows immediately from (a), (b) and (c); part (e) is [11, prop. 2.6 and (2.4.1)].

Next we recall the notion of G–dimension; see [4, 8, 9] for details.

(1.2) **G–dimension.** A finite R–module G is said to be *totally reflexive* if there exists an exact complex L of finite free R–modules such that $G \cong \operatorname{Coker}(L_1 \to L_0)$ and $\mathrm{H}(\operatorname{Hom}_R(L, R)) = 0$. Any finite free module is totally reflexive, so each homologically finite R–complex N with $\mathrm{H}_\ell(N) = 0$ for $\ell \ll 0$ admits a resolution by totally reflexive modules. The *G–dimension* is the number

$$\operatorname{G-dim}_R N = \inf \left\{ d \in \mathbb{Z} \;\middle|\; \begin{array}{c} N \text{ is isomorphic in } \mathsf{D}(R) \text{ to a complex of totally} \\ \text{reflexive modules: } 0 \to G_d \to G_{d-1} \to \cdots \to G_i \to 0 \end{array} \right\}$$

Enochs, Jenda and Torrecillas [16, 14] have studied extensions of G–dimension to complexes whose homology may not be finite. One such extension is the Gorenstein flat dimension; see [16, 9].

(1.3) **Gorenstein flat dimension.** An R–module A is *Gorenstein flat* if there exists an exact complex F of flat modules such that $A \cong \operatorname{Coker}(F_1 \to F_0)$ and $\mathrm{H}(J \otimes_R F) = 0$ for any injective R–module J. Any free module is Gorenstein flat, so each complex M with $\mathrm{H}_\ell(M) = 0$ for $\ell \ll 0$ admits a resolution by Gorenstein flat modules. The *Gorenstein flat dimension* is the number

$$\operatorname{Gfd}_R M = \inf \left\{ d \in \mathbb{Z} \;\middle|\; \begin{array}{c} M \text{ is isomorphic in } \mathsf{D}(R) \text{ to a complex of Gorenstein} \\ \text{flat modules: } 0 \to A_d \to A_{d-1} \to \cdots \to A_i \to 0 \end{array} \right\}$$

When M is homologically finite $\operatorname{Gfd}_R M = \operatorname{G-dim}_R M$; see [9, thm. (5.1.11)].

(1.4) **Remark.** By [13, thm. (3.5) and cor. (3.6)], if M is an R–complex of finite Gorenstein flat dimension, then:

$$\begin{aligned} \operatorname{Gfd}_R M &= \sup \{ \sup (J \otimes^{\mathbf{L}}_R M) \mid J \text{ is injective} \} \\ &= \sup \{ \sup (\mathrm{E}_R(R/\mathfrak{p}) \otimes^{\mathbf{L}}_R M) \mid \mathfrak{p} \in \operatorname{Spec} R \}. \end{aligned}$$

2. Localization

The gist of this section is that for complexes over homomorphisms the Gorenstein flat dimension, when it is finite, may be computed locally. We should like to note that the analogue for flat dimensions is elementary to verify, for the finiteness of that invariant is detected by vanishing of Tor functors. The absolute case, $R \xrightarrow{=} S$, is easily deduced from [23, thm. 8.8] and [12, thm. (2.4)].

(2.1) **Theorem.** *Let $\varphi\colon R \to S$ be a homomorphism of rings and let X be an S–complex. If $\operatorname{Gfd}_R X$ is finite, then*

$$\operatorname{Gfd}_R X = \begin{cases} \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} S \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\} \\ \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Max}_S X \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\} \\ \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \max_S X \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\} \end{cases}$$

The proof is given towards the end of this section. In preparation we recall a result about colimits of Gorenstein flat modules:

(2.2) **Remark.** If $(M_i)_{i\in I}$ is a filtered system of Gorenstein flat modules over a coherent ring, then the colimit $\varinjlim M_i$ is Gorenstein flat. This follows from work of Enochs et. al. [17,15] and Holm [21]: By [17, thm. 2.4 (and remarks before sec. 2)] a filtered colimit $M = \varinjlim M_i$ of Gorenstein flat modules has a co-proper right resolution by flat modules. Because colimits commute with tensor products, (1.4) provides an equality

$$\sup\{\sup(J \otimes^{\mathbf{L}}_R M) \mid J \text{ is injective}\} = 0.$$

Therefore, by [21, thm. 3.6], the colimit M is Gorenstein flat.

For the next result note that any R–module M has a natural structure of a module over its endomorphism ring $\operatorname{Hom}_R(M,M)$.

(2.3) **Lemma.** *Let R be a coherent ring and M a Gorenstein flat R–module. Let Z be a multiplicatively closed set in the center of the ring $\operatorname{Hom}_R(M,M)$. Then the R–module $Z^{-1}M$ is Gorenstein flat.*

Proof. Let $\mathcal{V}$ denote the set of finitely generated (as semigroups) multiplicatively closed subsets of Z. The modules $V^{-1}M$, for $V \in \mathcal{V}$, with natural maps

$$\rho^{UV}\colon U^{-1}M \to V^{-1}M \quad \text{for } U \subseteq V$$

form a filtered system. It is straightforward to verify that the colimit $\varinjlim V^{-1}M$ is isomorphic to $Z^{-1}M$ as $\operatorname{Hom}_R(M,M)$–module and, therefore, as an R–module.

By Remark (2.2), a filtered colimit of Gorenstein flat modules is Gorenstein flat, so it remains to see that the modules $V^{-1}M$ are Gorenstein flat. For any $V \in \mathcal{V}$ the module $V^{-1}M$ can be constructed by successively inverting the finitely many generators of V. Thus, it suffices to prove that M_z is Gorenstein flat for any $z \in Z$. Again, M_z is the colimit of the linear system $(M \xrightarrow{z} M \xrightarrow{z} M \xrightarrow{z} \cdots)$ and hence Gorenstein flat by (2.2). □

We should like to stress that in the next result the ring S need not be noetherian.

(2.4) **Proposition.** *Let R be a noetherian ring. Let $\varphi\colon R \to S$ be a homomorphism of rings and X an S–complex. For each $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$, one has*

$$\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} = \operatorname{Gfd}_R X_\mathfrak{q} \leq \operatorname{Gfd}_R X.$$

Proof. The equality in the statement is evident: a Gorenstein flat $R_\mathfrak{p}$–module is Gorenstein flat over R and any Gorenstein flat R–module localizes to give a Gorenstein flat $R_\mathfrak{p}$–module.

In verifying the inequality one may assume that $\operatorname{Gfd}_R X$ is finite. Pick a surjective homomorphism $\tilde{S} \to S$ where $\tilde{S}$ is an R–algebra, free as an R–module, and let $\tilde{\mathfrak{q}}$ be the preimage of $\mathfrak{q}$ in $\tilde{S}$. Evidently, $X_{\tilde{\mathfrak{q}}} \simeq X_\mathfrak{q}$ as $\tilde{S}$–complexes, and hence also as R–complexes, so replacing S with $\tilde{S}$, we assume henceforth that the R–module S is free.

Let U be a free resolution of X over S and set $\Omega = \operatorname{Ker}(\partial^U_{d-1})$ for $d = \operatorname{Gfd}_R X$. Since S is a free R–module, U is also an R–free resolution of X, and since $\operatorname{Gfd}_R X$ is finite, Ω viewed as an R–module is Gorenstein flat. Note that one has isomorphisms

$$U_\mathfrak{q} \simeq X_\mathfrak{q} \quad \text{and} \quad \operatorname{Ker}(\partial^{U_\mathfrak{q}}_{d-1}) \cong \Omega_\mathfrak{q}.$$

The complex $U_\mathfrak{q}$ consists of flat R–modules, so to settle the claim it suffices to prove that the R–module $\Omega_\mathfrak{q}$ is Gorenstein flat. Therefore, it suffices to verify the result in the case where the S–module X is Gorenstein flat over R.

Homothety provides a homomorphism of rings $S \to \operatorname{Hom}_R(X,X)$. Let Z be the image of $S \setminus \mathfrak{q}$ under this map; it is a multiplicatively closed subset in the center of $\operatorname{Hom}_R(X,X)$, and $Z^{-1}X \cong X_\mathfrak{q}$ as S–modules. It now remains to invoke Lemma (2.3). □

Proof of Theorem (2.1). Proposition (2.4) implies the first inequality below

$$\begin{aligned}\operatorname{Gfd}_R X &\geq \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} S \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\}\\ &\geq \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Max}_S X \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\}\\ &\geq \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \max_S X \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\}.\end{aligned}$$

The second inequality holds because of the inclusion $\operatorname{Max}_S X \subseteq \operatorname{Supp}_S X$, and the third follows also by Proposition (2.4) as any ideal in $\max_S X$ is contained in an ideal from $\operatorname{Max}_S X$. This leaves us one inequality to verify:

$$\operatorname{Gfd}_R X \leq \sup\{\operatorname{Gfd}_{R_\mathfrak{p}} X_\mathfrak{q} \mid \mathfrak{q} \in \max_S X \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\}.$$

Set $d = \operatorname{Gfd}_R X$ and pick a $\tilde{\mathfrak{p}}$ in $\operatorname{Spec} R$ for which $\sup(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X) = d$. Pick a prime ideal $\mathfrak{q}'$ associated to the S–module $\mathrm{H}_d(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X)$ and set $\mathfrak{p}' = \mathfrak{q}' \cap R$. In the (in)equalities below:

$$(\dagger) \qquad \begin{aligned} d &= \sup(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X)\\ &= \sup(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X_{\mathfrak{q}'})\\ &= \sup(\mathrm{E}_{R_{\mathfrak{p}'}}(R_{\mathfrak{p}'}/\tilde{\mathfrak{p}}R_{\mathfrak{p}'}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}'}} X_{\mathfrak{q}'})\\ &\leq \operatorname{Gfd}_{R_{\mathfrak{p}'}} X_{\mathfrak{q}'}\end{aligned}$$

the second one holds by choice of $\mathfrak{q}'$, while the third holds because $\mathrm{E}_R(R/\tilde{\mathfrak{p}})$ is an $R_{\mathfrak{p}'}$–module, as $\tilde{\mathfrak{p}} \subseteq \mathfrak{p}'$. By (1.1)(e) the ideal $\mathfrak{q}'$ is in the small support of the

S–complex $\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X$. The first equality below is due to the associativity of the tensor product

$$\begin{aligned}\operatorname{supp}_S(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R X) &= \operatorname{supp}_S((\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S X) \\ &= \operatorname{supp}_S(\mathrm{E}_R(R/\tilde{\mathfrak{p}}) \otimes^{\mathbf{L}}_R S) \cap \operatorname{supp}_S X\end{aligned}$$

while the second one is by (1.1)(b). These show that $\mathfrak{q}'$ is in $\operatorname{supp}_S X$. Finally, choose $\mathfrak{q} \in \max_S X$ containing $\mathfrak{q}'$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. It follows by (†) and Proposition (2.4) that $d \leq \operatorname{Gfd}_{R_{\mathfrak{p}'}} X_{\mathfrak{q}'} \leq \operatorname{Gfd}_{R_{\mathfrak{p}}} X_{\mathfrak{q}}$. □

3. Approximations

In this section we establish an approximation theorem for complexes of finite G–dimension; this is an important ingredient in the proof of Theorem (4.1). It is a common generalization to complexes of [21, thm. 2.10] and [13, lem. (2.17)], which deal with modules. Similar extensions have been obtained by Holm et. al. [19,22]; see (3.5) and the remarks following the statement of the theorem for further relations to earlier work.

(3.1) **Theorem.** *Let S be a ring and N a homologically finite S–complex with finite G–dimension. For each integer $n \leq \text{G–dim}_S N$ there exists an exact triangle*

$$N \to P \to H \to \Sigma N$$

in $\mathsf{D}^{\mathrm{f}}(S)$ with the following properties:

(a) $\operatorname{pd}_S P = \text{G–dim}_S N$ *and* $\text{G–dim}_S H \leq n$.

(b) *There are inequalities:* $\inf P \geq n \geq \sup H$, *and*

$$\max\{n, \sup N\} \geq \sup P \quad \textit{and} \quad \inf H \geq \min\{n, \inf N + 1\}.$$

Moreover, the following induced sequence of S–modules is exact:

$$0 \to \mathrm{H}_n(N) \to \mathrm{H}_n(P) \to \mathrm{H}_n(H) \to \mathrm{H}_{n-1}(N) \to 0.$$

We precede the proof with a couple of remarks and a lemma.

(3.2) **Remark.** As above, let N be a homologically finite S–complex of finite G–dimension. By rotating the exact triangle in (3.1), we see that for each integer $n \leq \text{G–dim}_S N$ there exists an exact triangle

$$P' \to H' \to N \to \Sigma P'$$

in $\mathsf{D}^{\mathrm{f}}(S)$ where $\operatorname{pd}_S P' = \text{G–dim}_S N - 1$ and $\text{G–dim}_S H' \leq n - 1$.

(3.3) **Remark.** Let N be a finite S–module with finite G–dimension. Applying Theorem (3.1) with $n = 0$ we get from part (b) an exact sequence of finite modules

$$0 \to N \to \mathrm{H}_0(P) \to \mathrm{H}_0(H) \to 0.$$

Moreover, $\mathrm{H}_\ell(H) = 0 = \mathrm{H}_\ell(P)$ for $\ell \neq 0$, so from part (a) it follows that $\mathrm{H}_0(H)$ is totally reflexive and $\operatorname{pd}_R \mathrm{H}_0(P) = \text{G–dim}_S N$. Thus we recover [13, lem. (2.17)].

Analogously, if $\text{G–dim}_S N \geq 1$, applying Theorem (3.1) with $n = 1$ yields an exact sequence of finite modules:

$$0 \to \mathrm{H}_1(P) \to \mathrm{H}_1(H) \to N \to 0,$$

where $\mathrm{H}_1(H)$ is totally reflexive and $\operatorname{pd}_R \mathrm{H}_1(P) = \text{G–dim}_S N - 1$. In this way we also recover [21, thm. 2.10].

(3.4) **Lemma.** *Let X be an S–complex. For any injective homomorphism $\iota\colon X_n \to Y_n$ of S–modules there is a commutative diagram*

$$\begin{tikzcd}
X = \cdots \arrow[r] & X_{n+1} \arrow[r, "\alpha"] \arrow[d, equal] & X_n \arrow[r, "\beta"] \arrow[d, hook, "\iota"'] & X_{n-1} \arrow[r, "\gamma"] \arrow[d, hook, "\iota'"'] & X_{n-2} \arrow[r] \arrow[d, equal] & \cdots \\
Y = \cdots \arrow[r] & X_{n+1} \arrow[r, "\alpha'"] & Y_n \arrow[r, "\beta'"] & Y_{n-1} \arrow[r, "\gamma'"] & X_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

such that Y is a complex, $\operatorname{Coker}\iota' \cong \operatorname{Coker}\iota$*, and the induced map* $\mathrm{H}(X) \to \mathrm{H}(Y)$ *is an isomorphism. When X_{n-1} and Y_n are finite, Y_{n-1} can be chosen finite.*

Proof. Set $\alpha' = \iota\alpha$ and let $\beta'\colon Y_n \to Y_{n-1}$ be the pushout of β along ι; thus

$$Y_{n-1} = \frac{Y_n \oplus X_{n-1}}{\{(\iota(x), \beta(x)) \mid x \in X_n\}}.$$

Let $\iota'\colon X_{n-1} \to Y_{n-1}$ be the induced map, which sends x to $(0, x)$; it is injective because ι is. Define $\gamma'\colon Y_{n-1} \to X_{n-2}$ by $(y, x) \mapsto \gamma(x)$. By construction the diagram is commutative. It is elementary to check that Y is a complex, and the induced map $\operatorname{Coker}\iota \to \operatorname{Coker}\iota'$ an isomorphism. Thus, the cokernel of the inclusion of complexes $X \hookrightarrow Y$ is exact, and hence the induced map $\mathrm{H}(X) \to \mathrm{H}(Y)$ is bijective. By construction, Y_{n-1} is finite when X_{n-1} and Y_n are so. □

Proof of Theorem (3.1). The hypothesis is that N is a homologically finite S–complex with finite G–dimension; set $d = \text{G–dim}_R N$ and $i = \inf N$. Let

$$\cdots \to P_\ell \to P_{\ell-1} \to \cdots \to P_i \to 0$$

be a projective resolution of N by finite modules. For integers $n \leq d+1$ we construct, by descending induction on n, complexes $C(n)$ isomorphic to N in $\mathsf{D}(S)$ and of the form

$$C(n) = 0 \to Q_d \to \cdots \to Q_n \to G_{n-1} \to P_{n-2} \to \cdots \to P_i \to 0,$$

where the modules Q_ℓ are also finite projective and G_{n-1} is totally reflexive. For the first step, set $G_d = \operatorname{Ker}(P_d \to P_{d-1})$; this module is totally reflexive, the complex

$$C(d+1) = 0 \to G_d \to P_{d-1} \to \cdots \to P_i \to 0$$

is isomorphic to N in $\mathsf{D}(S)$ and has the desired form. Next we construct $C(n)$ from $C(n+1)$. The totally reflexive module G_n in $C(n+1)$ embeds into a finite free module $\iota\colon G_n \to Q_n$ such that $\operatorname{Coker}\iota$ is totally reflexive. By Lemma (3.4) we have a commutative diagram

$$\begin{tikzcd}
C(n+1) = \cdots \arrow[r] & Q_{n+1} \arrow[r] \arrow[d, equal] & G_n \arrow[r] \arrow[d, hook, "\iota"'] & P_{n-1} \arrow[r] \arrow[d, hook, "\iota'"'] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \\
C(n) = \cdots \arrow[r] & Q_{n+1} \arrow[r] & Q_n \arrow[r] & G_{n-1} \arrow[r] & P_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

The module $\operatorname{Coker}\iota'$ is isomorphic to $\operatorname{Coker}\iota$ and hence totally reflexive; therefore G_{n-1} is totally reflexive. The complex $C(n)$ has the desired form, by construction, and is isomorphic to $C(n+1) \simeq N$, again by (3.4).

Now, fix an integer $n \leq d$ and replace N by $C(n)$. Let P be the truncation $N_{\geqslant n}$ of N and $H = \Sigma(N_{\leqslant n-1})$; the canonical surjection $N \to P$ yields an exact triangle

$$N \to P \to H \to \Sigma N. \tag{Δ}$$

We now verify that this triangle has the desired properties:

(a): It is evident from the construction that $\text{G–dim}_S\, H \leq n$ and $\text{pd}_S\, P \leq d$. To see that $\text{pd}_S\, P = d$, apply $\mathbf{R}\text{Hom}_S(-, S)$ to (Δ) and take homology to get the exact sequence

$$\text{Ext}_S^d(P, S) \to \text{Ext}_S^d(N, S) \to \text{Ext}_S^{d+1}(H, S).$$

Recall that $\text{G–dim}_S\, X = \sup\{\, m \in \mathbb{Z} \mid \text{Ext}_S^m(X, S) \neq 0\}$ for any homologically finite S–complex of finite G–dimension, cf. [9, cor. (2.3.8)]. Therefore, in the exact sequence above, the module on the right is zero as $d \geq n \geq \text{G–dim}_S\, H$, while the middle one is non-zero as $\text{G–dim}_S\, N = d$. Thus, $\text{Ext}_S^d(P, S) \neq 0$.

(b): By construction $\inf P \geq n \geq \sup H$, so the homology exact sequence

$$\cdots \to \text{H}_\ell(N) \to \text{H}_\ell(P) \to \text{H}_\ell(H) \to \text{H}_{\ell-1}(N) \to \cdots$$

associated with (Δ) gives the desired exact sequence and isomorphisms $\text{H}_\ell(P) \cong \text{H}_\ell(N)$ for $\ell \geq n+1$ and $\text{H}_\ell(H) \cong \text{H}_{\ell-1}(N)$ for $\ell \leq n-1$. In particular, $\max\{n, \sup N\} \geq \sup P$ and $\inf H \geq \min\{n, \inf N + 1\}$. $\square$

(3.5) **Remark.** We note that with G–dimension replaced by Gorenstein projective dimension, or by Gorenstein flat dimension, the arguments in the preceding proof carry over to the case where the homology modules of N are not necessarily finite. In this paper we only need the version stated in Theorem (3.1).

4. Local homomorphisms

The main result of this section is:

(4.1) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring, and let N be an R–complex, finite over a local homomorphism. If $\text{Gfd}_R\, N$ is finite, then*

$$\text{Gfd}_R\, N = \sup(\text{E}_R(k) \otimes_R^{\mathbf{L}} N) = \text{depth}\, R - \text{depth}_R\, N.$$

The second equality was proved in [23, thm. 8.7]; the theorem is motivated by the following considerations:

(4.2) **Remarks.** The flat dimension of N can be tested by cyclic modules, $R/\mathfrak{p}$, and if $\text{fd}_R\, N$ is finite, then

$$\sup(k \otimes_R^{\mathbf{L}} N) = \text{depth}_R\, R - \text{depth}_R\, N.$$

This is the Auslander–Buchsbaum formula for N, cf. [18, p. 153]. Analogously, the Gorenstein flat dimension is tested by modules $\text{E}_R(R/\mathfrak{p})$, cf. (1.4), and if $\text{Gfd}_R\, N$ is finite, an analogue of the Auslander–Buchsbaum is provided by [23, thm. 8.7]:

$$\sup(\text{E}_R(k) \otimes_R^{\mathbf{L}} N) = \text{depth}\, R - \text{depth}_R\, N.$$

Assume that N is finite over a local homomorphism, then

$$\text{fd}_R\, N = \sup(k \otimes_R^{\mathbf{L}} N)$$

by [5, prop. 5.5]. By the three displayed equations it follows that

$$\text{(4.2.1)} \qquad \text{fd}_R\, N = \sup(\text{E}_R(k) \otimes_R^{\mathbf{L}} N) \text{ when } \text{fd}_R\, N \text{ is finite.}$$

However, an elementary argument is also available: Set $f = \operatorname{fd}_R N$; associated to the exact sequence $0 \to k \to \mathrm{E}_R(k) \to C \to 0$ is an exact sequence of homology modules
$$0 \to \mathrm{H}_f(k \otimes^{\mathbf{L}}_R N) \to \mathrm{H}_f(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \to \cdots,$$
which shows that also $\mathrm{H}_f(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \neq 0$.

Theorem (4.1) is an analogue of (4.2.1) for Gorenstein flat dimension. When N is finite over R itself, the first equality in (4.1) recovers [9, thm. (2.4.5)(b)]:
$$\text{G–dim}_R N = \sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N).$$
Even in this case one has to assume a priori that the dimension is finite:

(4.3) **Example.** Jorgensen and Şega [24, thm. 1.7] construct an artinian ring R and a finite R–module L with
$$\text{G–dim}_R L = \infty \quad \text{and} \quad \inf \mathbf{R}\mathrm{Hom}_R(L, R) = 0.$$
The last equality translates to $\sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R L) = 0$ by Matlis duality.

It is implicit in Theorem (4.1) that both $\sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N)$ and $\operatorname{depth}_R N$ are finite. This holds in general for complexes finite over local homomorphisms:

(4.4) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and let N be an R–complex, finite over a local homomorphism. If $\mathrm{H}(N) \neq 0$, then*
$$\operatorname{depth}_R N \textit{ is finite} \quad \textit{and} \quad \mathrm{H}(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \neq 0.$$

Proof. By assumption there is a local homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$, such that N is homologically finite over S. With $i = \inf N$ one has
$$\mathrm{H}_i(k \otimes^{\mathbf{L}}_R N) \cong k \otimes_R \mathrm{H}_i(N) \cong \mathrm{H}_i(N)/\mathfrak{m}\,\mathrm{H}_i(N).$$
Since φ is local, and the S–module $\mathrm{H}_i(N)$ is finite and non-zero, Nakayama's lemma implies $\mathrm{H}_i(N)/\mathfrak{m}\,\mathrm{H}_i(N)$ is non-zero. Thus $\mathfrak{m}$ is in $\operatorname{supp}_R N$, in particular, $\operatorname{depth}_R N$ is finite, cf. (1.1)(f). Moreover, $\mathfrak{m}$ is in $\operatorname{supp}_R \mathrm{E}_R(k)$, by (1.1)(c), and thus also in $\operatorname{supp}_R(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N)$, whence $\mathrm{H}(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \neq 0$ by (1.1)(a). □

For Theorem (4.1) it is important that the homology of $\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N$ is non-zero. However, that condition alone is not sufficient for the first equality, not even for (4.2.1); one needs the finiteness of $\mathrm{H}(N)$:

(4.5) **Example.** Let $(R, \mathfrak{m}, k)$ be a regular local ring. For a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ set $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $N = k(\mathfrak{p}) \oplus R$. Then
$$\operatorname{fd}_R N = \operatorname{fd}_{R_\mathfrak{p}} k(\mathfrak{p}) = \dim R_\mathfrak{p} \quad \text{and}$$
$$\sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) = 0 = \sup(k \otimes^{\mathbf{L}}_R N).$$

For the proof of the theorem we need the following lemmas. The first one deals with the restricted flat dimension, introduced by Foxby in [12]. Its relevance for our purpose comes from [23, thm. 8.8], see also [21, thm. 3.19].

As usual, for any local ring $(R, \mathfrak{m}, k)$ its $\mathfrak{m}$-adic completion is denoted $\widehat{R}$.

(4.6) **Lemma.** *Let $\varphi\colon R \to S$ be a homomorphism of rings and X an S–complex.*

(a) *If φ is flat, then $\mathsf{Rfd}_R\, X \leq \mathsf{Rfd}_S\, X$*

(b) *If φ is local, then $\mathsf{Rfd}_R\, X = \mathsf{Rfd}_R(\widehat{S} \otimes_S X) \leq \mathsf{Rfd}_{\widehat{R}}(\widehat{S} \otimes_S X)$*

Proof. Let $\mathsf{F}_0(R)$ be the class of R–modules of finite flat dimension.

(a): For each $T \in \mathsf{F}_0(R)$ the module $T \otimes_R S$ has finite flat dimension over S. With this, the desired inequality follows from:

$$\begin{aligned}
\mathsf{Rfd}_R\, X &= \sup\{\, \sup\,(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathsf{F}_0(R)\}\\
&= \sup\{\, \sup\,(T \otimes^{\mathbf{L}}_R (S \otimes_S X)) \mid T \in \mathsf{F}_0(R)\}\\
&= \sup\{\, \sup\,((T \otimes_R S) \otimes^{\mathbf{L}}_S X) \mid T \in \mathsf{F}_0(R)\}\\
&\leq \mathsf{Rfd}_S\, X,
\end{aligned}$$

where the first equality is the definition.

(b): The inequality follows from (a), and the equality is an easy calculation:

$$\begin{aligned}
\mathsf{Rfd}_R\, X &= \sup\{\, \sup\,(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathsf{F}_0(R)\}\\
&= \sup\{\, \sup\,((T \otimes^{\mathbf{L}}_R X) \otimes_S \widehat{S}) \mid T \in \mathsf{F}_0(R)\}\\
&= \sup\{\, \sup\,(T \otimes^{\mathbf{L}}_R (X \otimes_S \widehat{S})) \mid T \in \mathsf{F}_0(R)\}\\
&= \mathsf{Rfd}_R(X \otimes_S \widehat{S}) \qquad \square
\end{aligned}$$

(4.7) **Lemma.** *Let $\varphi\colon R \to S$ be a local homomorphism and N a homologically finite S–complex. If $\operatorname{Gfd}_R N$ is finite, then $\operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N)$ is finite as well, and there is an inequality: $\operatorname{Gfd}_R N \leq \operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N)$.*

In Corollary (4.8) we strengthen the inequality to an equality.

Proof. By [23, prop. 8.13] the G–dimension of N along φ, introduced in that paper and denoted $\text{G–dim}_\varphi\, N$, is finite. By [23, 3.4.1] also $\text{G–dim}_{\hat{\varphi}}(\widehat{S} \otimes_S N)$ is finite, where $\hat{\varphi}\colon \widehat{R} \to \widehat{S}$ is the completion of φ, and hence $\operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N)$ is finite, by [23, thm. 8.2]. Moreover, we have

$$\operatorname{Gfd}_R N = \mathsf{Rfd}_R\, N \leq \mathsf{Rfd}_{\widehat{R}}(\widehat{S} \otimes_S N) = \operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N),$$

where the equalities are by [23, thm. 8.8] and the inequality is Lemma (4.6)(b). $\square$

Proof of Theorem (4.1). By hypothesis N is an R–complex and there exists a local homomorphism $\varphi\colon R \to S$ such that N is a homologically finite S–complex. It suffices to prove

$$(\dagger) \qquad \operatorname{Gfd}_R N = \sup\,(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N),$$

since the second equality of the claim is [23, thm. 8.7].

1° First we reduce the problem to the case where R and S are complete (in the topologies induced by the respective maximal ideals). The right hand side in (†) is unchanged on tensoring with $\widehat{S}$: Indeed there are isomorphisms of complexes

$$\begin{aligned}(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \otimes_S \widehat{S} &\simeq \mathrm{E}_R(k) \otimes^{\mathbf{L}}_R (N \otimes_S \widehat{S}) \\ &\simeq \mathrm{E}_R(k) \otimes^{\mathbf{L}}_R (\widehat{S} \otimes_S N) \\ &\simeq \mathrm{E}_R(k) \otimes^{\mathbf{L}}_R (\widehat{R} \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes_S N)) \\ &\simeq (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes_S N) \\ &\simeq \mathrm{E}_{\widehat{R}}(k) \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes_S N)\end{aligned}$$

where the first and penultimate ones hold by associativity of tensor products. The second isomorphism holds as $N \otimes_S \widehat{S}$ and $\widehat{S} \otimes_S N$ are isomorphic as S–complexes and hence also as R–complexes. The third isomorphism holds because the composite map $R \to S \to \widehat{S}$ factors through $\widehat{R}$. Being $\mathfrak{m}$-torsion, $\mathrm{E}_R(k)$ is naturally isomorphic to $\mathrm{E}_R(k) \otimes_R \widehat{R}$, and as $\widehat{R}$–modules $\mathrm{E}_R(k) \cong \mathrm{E}_{\widehat{R}}(k)$; this accounts for the last isomorphism. The faithful flatness of $\widehat{S}$ over S and the isomorphisms above yield:

$$\begin{aligned}\sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) &= \sup((\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \otimes_S \widehat{S}) \\ &= \sup(\mathrm{E}_{\widehat{R}}(k) \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes_S N))\end{aligned}$$

The preceding equality and Remark (1.4) yield the first two (in)equalities below, while Lemma (4.7) gives the third one:

$$\begin{aligned}\sup(\mathrm{E}_{\widehat{R}}(k) \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes_S N)) &= \sup(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \\ &\le \mathrm{Gfd}_R N \\ &\le \mathrm{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N).\end{aligned} \tag{‡}$$

Moreover, $\mathrm{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N)$ is finite, again by (4.7), and the complex $\widehat{S} \otimes_S N$ is homologically finite over $\widehat{S}$. Thus, if (†) holds when R and S are complete, then equalities must hold all way through in (‡).

We assume henceforth that R and S are complete.

2° Next we reduce to the case where φ is flat and the closed fiber $S/\mathfrak{m}S$ is regular. Since R and S are complete, the homomorphism φ admits a regular factorization: a commutative diagram of local homomorphisms

$$\begin{tikzcd} & R' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\varphi"'] & & S \end{tikzcd}$$

where φ' is surjective and $\dot{\varphi}$ is flat with $R'/\mathfrak{m}R'$ regular, cf. [6, thm. (1.1)]. Since N is homologically finite over S it is also finite over R', and so it suffices to prove the result for $\dot{\varphi}$; this achieves the desired reduction.

3° Since R is complete, it has a dualizing complex D; since φ is flat with regular closed fiber, the complex $S \otimes_R D$ is dualizing for S, cf. [20]. Now, from [10, prop. (5.3)] it follows that an S–complex X is in the Auslander category $\mathsf{A}(S)$ if and only if it is in $\mathsf{A}(R)$. By [13, thm. (4.1)] complexes in the Auslander category are exactly those of finite Gorenstein flat dimension, that is,

$$(*)\qquad \begin{aligned} \operatorname{Gfd}_R X < \infty &\iff X \in \mathsf{A}(R) \\ &\iff X \in \mathsf{A}(S) \\ &\iff \operatorname{Gfd}_S X < \infty. \end{aligned}$$

Therefore, when $\operatorname{Gfd}_R X$ is finite, so is $\operatorname{Gfd}_S X$, and hence

$$(**)\qquad \operatorname{Gfd}_R X = \mathsf{R}\mathrm{fd}_R X \leq \mathsf{R}\mathrm{fd}_S X = \operatorname{Gfd}_S X,$$

where the inequality is Lemma (4.6)(a) and the equalities are by [23, thm. 8.8].

We may assume that $\mathrm{H}(N) \neq 0$ and set $i = \inf N$. By $(*)$ the complex N has finite Gorenstein flat dimension over S; since it is homologically finite, it thus has finite G–dimension over S, cf. [9, thm. (5.1.11)]. By Theorem (3.1) there is an exact triangle in $\mathsf{D}^{\mathrm{f}}(S)$:

$$N \to P \to H \to \Sigma N,$$

where $\operatorname{pd}_S P = \operatorname{G-dim}_S N$ and $\operatorname{G-dim}_S H \leq i$; in particular $\operatorname{Gfd}_S H \leq i$, again by [9, thm. (5.1.11)]. By $(**)$ it follows that $\operatorname{Gfd}_R H \leq \operatorname{Gfd}_S H \leq i$. For any injective R–module J one therefore has $\sup (J \otimes^{\mathbf{L}}_R H) \leq i$ by (1.4), and hence the exact triangle above yields the following isomorphisms and exact sequence

$$(\dagger\dagger)\qquad \begin{gathered} \mathrm{H}_\ell(J \otimes^{\mathbf{L}}_R N) \cong \mathrm{H}_\ell(J \otimes^{\mathbf{L}}_R P) \quad \text{for} \quad \ell \geq i+1, \\ 0 \to \mathrm{H}_i(J \otimes^{\mathbf{L}}_R N) \to \mathrm{H}_i(J \otimes^{\mathbf{L}}_R P). \end{gathered}$$

Since $\inf P \geq i$ we deduce that $\sup (J \otimes^{\mathbf{L}}_R N) \leq \sup (J \otimes^{\mathbf{L}}_R P)$. Combined with (1.4) this implies the second inequality below

$$\begin{aligned} \sup (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) &\leq \operatorname{Gfd}_R N \\ &\leq \operatorname{Gfd}_R P \\ &\leq \operatorname{fd}_R P \\ &= \sup (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R P); \end{aligned}$$

the first inequality is also by (1.4), the third inequality is trivial, while the equality is by (4.2.1), since φ flat and $\operatorname{pd}_S P$ finite implies $\operatorname{fd}_R P$ finite.

Finally, $\mathrm{H}_\ell(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) \neq 0$ for some $\ell \geq i = \inf N$, cf. Lemma (4.4), and so $(\dagger\dagger)$ shows that $\sup (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) = \sup (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R P)$. Thus, from the preceding display, we conclude that $\sup (\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R N) = \operatorname{Gfd}_R N$. □

(4.8) **Corollary.** *Let $\varphi\colon R \to S$ be a local homomorphism and N a homologically finite S–complex. If $\operatorname{Gfd}_R N$ is finite, then*

$$\operatorname{Gfd}_R N = \operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N).$$

Proof. From Lemma (4.7) one obtains that $\operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N)$ is finite. Since the $\widehat{R}$–complex $\widehat{S} \otimes_S N$ is finite over the completion $\hat{\varphi}\colon \widehat{R} \to \widehat{S}$, Theorem (4.1) gives the first and the last equalities below:

$$\begin{aligned} \operatorname{Gfd}_R N &= \operatorname{depth} R - \operatorname{depth}_R N \\ &= \operatorname{depth} \widehat{R} - \operatorname{depth}_{\widehat{R}}(\widehat{S} \otimes_S N) \\ &= \operatorname{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N); \end{aligned}$$

the second equality is a standard property of depth. □

We conclude with a global version of Theorem (4.1):

(4.9) **Theorem.** *Let $\varphi\colon R \to S$ be a homomorphism of rings and let N be a homologically finite S–complex. If* $\operatorname{Gfd}_R N$ *is finite, then*

$$\operatorname{Gfd}_R X = \sup\{\sup(\mathrm{E}_R(k(\mathfrak{p})) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{Max}_S N \text{ and } \mathfrak{p} = \mathfrak{q} \cap R\}.$$

Note that $\max_S N = \operatorname{Max}_S N$ as N is homologically finite.

Proof. For each $\mathfrak{q} \in \operatorname{Max}_S N$ the $R_\mathfrak{p}$–complex $N_\mathfrak{q}$ is finite over the local homomorphism $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \to S_\mathfrak{q}$, and $\operatorname{Gfd}_{R_\mathfrak{p}} N_\mathfrak{q}$ is finite by Theorem (2.1), so (4.1) yields

$$\operatorname{Gfd}_{R_\mathfrak{p}} N_\mathfrak{q} = \sup(\mathrm{E}_R(k(\mathfrak{p})) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{q}).$$

Combining this equality with that in Theorem (2.1) gives the desired result. □

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