Local rings of embedding codepth 3:
a classification algorithm

Lars Winther Christensen∗
Texas Tech University,
Lubbock, TX 79409, U.S.A.
lars.w.christensen@ttu.edu

Oana Veliche
Northeastern University,
Boston, MA 02115, U.S.A.
o.veliche@neu.edu

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Abstract

Let \( I \) be an ideal of a regular local ring \( Q \) with residue field \( k \). The length of the minimal free resolution of \( R = Q/I \) is called the codepth of \( R \). If it is at most 3, then the resolution carries a structure of a differential graded algebra, and the induced algebra structure on \( \text{Tor}_Q^*(R, k) \) provides for a classification of such local rings.

We describe the Macaulay 2 package \textit{CodepthThree} that implements an algorithm for classifying a local ring as above by computation of a few cohomological invariants.

1 Introduction and notation

Let \( R \) be a commutative noetherian local ring with residue field \( k \). Assume that \( R \) has the form \( Q/I \) where \( Q \) is a regular local ring with maximal ideal \( n \) and \( I \subseteq n^2 \). The embedding dimension of \( R \) (and of \( Q \)) is denoted \( e \). Let

\[
F = 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0
\]

be a minimal free resolution of \( R \) over \( Q \). Set \( d = \text{depth} \, R \); the length \( c \) of the resolution \( F \) is by the Auslander–Buchsbaum formula

\[
c = \text{proj.dim}_Q \, R = \text{depth} \, Q - \text{depth}_Q \, R = e - d,
\]

and one refers to this invariant as the \textit{codepth} of \( R \). In the following we assume that \( c \) is at most 3. By a theorem of Buchsbaum and Eisenbud [3, 3.4.3] the resolution \( F \) carries a differential graded algebra structure, which induces a unique graded-commutative algebra structure on \( A = \text{Tor}_Q^*(R, k) \). The possible structures were identified by Weyman [5] and by Avramov, Kustin, and Miller [2]. According to the multiplicative structure on \( A \), the ring \( R \)

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belongs to exactly one of the classes designated \( B, C(c), G(r), H(p,q), S, \) and \( T. \) Here the parameters \( p, q, \) and \( r \) are given by
\[
\begin{align*}
p &= \text{rank}_k(A_1 \cdot A_1), \\
q &= \text{rank}_k(A_1 \cdot A_2), \quad \text{and} \\
r &= \text{rank}_k(\delta: A_2 \to \text{Hom}_k(A_1, A_3)),
\end{align*}
\]
where \( \delta \) is the canonical map. See [1, 2, 5] for further background and details.

When, in the following, we talk about classification of a local ring \( R, \) we mean the classification according to the multiplicative structure on \( A. \) To describe the classification algorithm, we need a few more invariants of \( R. \) Set
\[
\begin{align*}
l &= \text{rank}_Q F_1 - 1 \quad \text{and} \\
n &= \text{rank}_Q F_c;
\end{align*}
\]
the latter invariant is called the type of \( R. \) The Cohen–Macaulay defect of \( R \) is \( h = \dim R - d. \) The Betti numbers \( \beta_i \) and the Bass numbers \( \mu_i \) record ranks of cohomology groups,
\[
\begin{align*}
\beta_i &= \beta^R_i(k) = \text{rank}_k \text{Ext}_R^i(k,k) \\
\mu_i &= \mu_i(R) = \text{rank}_k \text{Ext}_R^i(k,R).
\end{align*}
\]
The generating functions \( \sum_{i=0}^{\infty} \beta_i t^i \) and \( \sum_{i=0}^{\infty} \mu_i t^i \) are called the Poincaré series and the Bass series of \( R. \)

2 The algorithm

For a local ring of codepth \( c \leq 3, \) the class together with the invariants \( e, c, l, \) and \( n \) completely determine the Poincaré series and the Bass series of \( R; \) see [1]. Conversely, one can determine the class of \( R \) based on \( e, c, l, n, \) and a few Betti and Bass numbers; in the following we describe how.

**Lemma 1.** For a local ring \( R \) of codepth 3 the invariants \( p, q, \) and \( r \) are determined by \( e, l, n, \beta_2, \beta_3, \beta_4, \) and \( \mu_{e-2} \) through the formulas
\[
\begin{align*}
p &= n + le + \beta_2 - \beta_3 + \binom{e-1}{3}, \\
q &= (n-p)e + l\beta_2 + \beta_3 - \beta_4 + \binom{e-1}{4}, \quad \text{and} \\
r &= l + n - \mu_{e-2}.
\end{align*}
\]

*Proof.* The Poincaré series of \( R \) has by \([1, 2.1]\) the form
\[
\sum_{i=0}^{\infty} \beta_i t^i = \frac{(1 + t)^{e-1}}{1 - t - lt^2 - (n-p)t^3 + qt^4 + \cdots},
\]
and expansion of the rational function yields the expressions for \( p \) and \( q. \)

One has \( d = e - 3 \) and the Bass series of \( R \) has, also by \([1, 2.1]\), the form
\[
\sum_{i=0}^{\infty} \mu_i t^i = \frac{t^dn + (l-r)t + \cdots}{1 - t + \cdots};
\]
expansion of the rational function now yields the expression for \( r. \)

\[\square\]
**Proposition 2.** A local ring \( R \) of codepth 3 can be classified based on the invariants \( e, h, l, n, \beta_2, \beta_3, \beta_4, \mu_{e-2}, \) and \( \mu_{e-1}. \)

**Proof.** First recall that one has \( h = 0 \) and \( n = 1 \) if and only if \( R \) is Gorenstein; see [3, 3.2.10]. In this case \( R \) is in class \( C(3) \) if \( l = 2 \) and otherwise in class \( G(l + 1). \)

Assume now that \( R \) is not Gorenstein. The invariants \( p, q, \) and \( r \) can be computed from the formulas in Lemma 1. It remains to determine the class, which can be done by case analysis. Recall from [1, 1.3 and 3.1] that one has

\[
\begin{array}{c|ccc}
Class & p & q & r \\
\hline
T & 3 & 0 & 0 \\
B & 1 & 1 & 2 \\
G(r) [r \geq 2] & 0 & 1 & r \\
H(p,q) & p & q & q \\
\end{array}
\]

In case \( q \geq 2 \) the ring \( R \) is in class \( H(p,q); \) for \( q \leq 1 \) the case analysis shifts to \( p. \)

In case \( p = 0 \) the distinction between the classes \( G(r) \) and \( H(0,q) \) is made by comparing \( q \) and \( r; \) they are equal if and only if \( R \) is in class \( H(0,q). \)

In case \( p = 1 \) the distinction between the classes \( B \) and \( H(1,q) \) is made by comparing \( q \) and \( r; \) they are equal if and only if \( R \) is in class \( H(1,q). \)

In case \( p = 3 \) the distinction between the classes \( T \) and \( H(3,q) \) is drawn by the invariant \( \mu_{e-1}. \) Recall the relation \( d = e - 3; \) expansion of the expressions from [1, 2.1] yields \( \mu_{e-1} = \mu_{e-2} + ln - 2 \) if \( R \) is in \( T \) and \( \mu_{e-1} = \mu_{e-2} + ln - 3 \) if \( R \) is in \( H(3,q). \)

In all other cases, i.e. \( p = 2 \) or \( p \geq 4, \) the ring \( R \) is in class \( H(p,q). \) \( \square \)

**Remark 3.** One can also classify a local ring \( R \) of codepth 3 based on the invariants \( e, h, l, n, \beta_2, \ldots \beta_5, \) and \( \mu_{e-2}. \) In the case \( p = 3 \) one then discriminates between the classes by looking at \( \beta_5, \) which is \( \beta_4 + l\beta_3 + (n-3)\beta_2 + \tau \) with \( \tau = 0 \) if \( R \) is in class \( H(3,q) \) and \( \tau = 1 \) if \( R \) is in class \( T. \) However, it is not possible to classify \( R \) based on Betti numbers alone. Indeed, rings in the classes \( B \) and \( H(1,1) \) have identical Poincaré series and so do rings in the classes \( G(r) \) and \( H(0,1). \)

**Remark 4.** A local ring \( R \) of codepth \( c \leq 2 \) can be classified based on the invariants \( c, h, \) and \( n. \) Indeed, if \( c \leq 1 \) then \( R \) is a hypersurface; i.e. it belongs to class \( C(c). \) If \( c = 2 \) then \( R \) belongs to class \( C(2) \) if and only if it is Gorenstein \( (h = 0 \) and \( n = 1); \) otherwise it belongs to class \( S. \)

**Algorithm 5.** From Remark 4 and the proof of Proposition 2 one gets the following algorithm that takes as input invariants of a local ring of codepth \( c \leq 3 \) and outputs its class.

**INPUT:** \( c, e, h, l, n, \beta_2, \beta_3, \beta_4, \mu_{e-2}, \mu_{e-1} \)

- In case \( c \leq 1 \) set \( Class = C(c) \)
- In case \( c = 2 \)
  - \( \diamond \) if \( (h = 0 \) and \( n = 1) \) then set \( Class = C(2) \)
  - \( \diamond \) else set \( Class = S \)
In case $c = 3$
  - if ($h = 0$ and $n = 1$) then set $r = l + 1$
    - if $r = 3$ then set $\text{Class} = C(3)$
    - else set $\text{Class} = G(r)$
  - else compute $p$ and $q$
    - if ($q \geq 2$ or $p = 2$ or $p \geq 4$) then set $\text{Class} = H(p, q)$
    - else compute $r$
      - In case $p = 0$
        - if $q = r$ then set $\text{Class} = H(0, q)$
        - else set $\text{Class} = G(r)$
      - In case $p = 1$
        - if $q = r$ then set $\text{Class} = H(1, q)$
        - else set $\text{Class} = B$
      - In case $p = 3$
        - if $\mu_{e-1} = \mu_{e-2} + ln - 2$ then set $\text{Class} = T$
        - else set $\text{Class} = H(3, q)$

**OUTPUT:** $\text{Class}$

**Remark 6.** Given a local ring $R = Q/I$ the invariants $e$ and $h$ can be computed from $R$, and $c$, $l$, and $n$ can be determined by computing a minimal free resolution of $R$ over $Q$. The Betti numbers $\beta_2, \beta_3, \beta_4$ one can get by computing the first five steps of a minimal free resolution $F$ of $k$ over $R$. Recall the relation $d = e - c$; the Bass numbers $\mu_{e-2}$ and $\mu_{e-1}$ one can get by computing the cohomology in degrees $d+1$ and $d+2$ of the dual complex $F^* = \text{Hom}_R(F, R)$. For large values of $d$, this may not be feasible, but one can reduce $R$ modulo a regular sequence $x = x_1, \ldots, x_d$ and obtain the Bass numbers as $\mu_{d+i}(R) = \mu_i(R/(x))$; cf. [3, 3.1.16].

## 3 The implementation

The *Macaulay 2* package *CodepthThree* implements Algorithm 5. The function *torAlgClass* takes as input a quotient $Q/I$ of a polynomial algebra, where $I$ is contained in the irrelevant maximal ideal $\mathfrak{N}$ of $Q$. It returns the class of the local ring $R$ obtained by localization of $Q/I$ at $\mathfrak{N}$. For example, the local ring obtained by localizing the quotient

$$Q[x, y, z]/(xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)$$

is in class $G(2)$; see [4]. Here is how it looks when one calls the function *torAlgClass*.

Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "CodepthThree";
i2 : Q = QQ[x,y,z];
i3 : I = ideal (x*y^2,x*y*z,y*z^2,x^4-y^3*z,x*z^3-y^4);
o3 : Ideal of Q
i4 : torAlgClass (Q/I)
o4 = G(2)

Underlying *torAlgClass* is the workhorse function *torAlgData* which returns a hash table with the following data:

<table>
<thead>
<tr>
<th>Key</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;c&quot;</td>
<td>codepth of R</td>
</tr>
<tr>
<td>&quot;e&quot;</td>
<td>embedding dimension of R</td>
</tr>
<tr>
<td>&quot;h&quot;</td>
<td>Cohen–Macaulay defect of R</td>
</tr>
<tr>
<td>&quot;m&quot;</td>
<td>minimal number of generators of defining ideal of R</td>
</tr>
<tr>
<td>&quot;n&quot;</td>
<td>type of R</td>
</tr>
<tr>
<td>&quot;Class&quot;</td>
<td>(non-parametrized) class of R</td>
</tr>
<tr>
<td>&quot;p&quot;</td>
<td>rank of $A_1 \cdot A_1$</td>
</tr>
<tr>
<td>&quot;q&quot;</td>
<td>rank of $A_1 \cdot A_2$</td>
</tr>
<tr>
<td>&quot;r&quot;</td>
<td>rank of $\delta: A_2 \to \text{Hom}_k(A_1, A_3)$</td>
</tr>
<tr>
<td>&quot;PoincareSeries&quot;</td>
<td>Poincaré series of $R$</td>
</tr>
<tr>
<td>&quot;BassSeries&quot;</td>
<td>Bass series of $R$</td>
</tr>
</tbody>
</table>

In the example from above one gets:

i5 : torAlgData(Q/I)

\[
2 + 2T - T - T + T
\]

o5 = HashTable{BassSeries => ---------------------- }

\[
1 - T - 4T - 2T + T
\]

c => 3
Class => G
e => 3
h => 1
m => 5
n => 2
p => 0

\[
(1 + T)
\]

PoincareSeries => ----------------------

\[
1 - T - 4T - 2T + T
\]

q => 1
r => 2

To facilitate extraction of data from the hash table, the package offers two functions *torAlgDataList* and *torAlgDataPrint* that take as input a quotient ring and a list of keys. In the example from above one gets:
As discussed in Remark 6, the computation of Bass numbers may require a reduction modulo a regular sequence. In our implementation such a reduction is attempted if the embedding dimension of the local ring $R$ is more than 3. The procedure involves random choices of ring elements, and hence it may fail. By default, up to 625 attempts are made, and with the function `setAttemptsAtGenericReduction`, one can change the number of attempts. If none of the attempts are successful, then an error message is displayed:

```
i8 : Q = ZZ/2[u,v,w,x,y,z];
i9 : R = Q/ideal(x*y^2,x*y*z,y*z^2,x^4-y^3*z,x*z^3-y^4);
i10 : setAttemptsAtGenericReduction(R,1)
o10 = 1 attempt(s) will be made to compute the Bass numbers via a generic reduction
i11 : torAlgClass R
stdio:11:1:(3): error: Failed to compute Bass numbers. You may raise the number of attempts to compute Bass numbers via a generic reduction with the function `setAttemptsAtGenericReduction` and try again.
i12 : setAttemptsAtGenericReduction(R,25)
o12 = 625 attempt(s) will be made to compute the Bass numbers via a generic reduction
i13 : torAlgClass R
o13 = G(2)
```

Notice that the maximal number of attempts is $n^2$ where $n$ is the value set with the function `setAttemptsAtGenericReduction`. 
Notes. Given $Q/I$ our implementation of Algorithm 5 in `torAlgData` proceeds as follows.

1. Check if a value is set for `attemptsAtBassNumbers`; if not use the default value 25.

2. Initialize the invariants of $R$ (the localization of $Q/I$ at the irrelevant maximal ideal) that are to be returned; see the table in Section 3.

3. Handle the special case where the defining ideal $I$ or $Q/I$ is 0. In all other cases compute the invariants $c, e, h, m (= l + 1)$, and $n$.

4. If possible, classify $R$ based on $c, e, h, m,$ and $n$. At this point the implementation deviates slightly from Algorithm 5, as it uses that all rings with $c = 3$ and $h = 2$ are of class $H(0, 0)$; see [1, 3.5].

5. For rings not classified in step 3 or 4 one has $c = 3$; cf. Remark 4. Compute the Betti numbers $\beta_2, \beta_3,$ and $\beta_4,$ and with the formula from Lemma 1 compute $p$ and $q$. If possible classify $R$ based on these two invariants.

6. For rings not classified in steps 3–5, compute the Bass numbers $\mu_{e-2}$ and $\mu_{e-1}$. If $d = e - 3$ is positive, then the Bass numbers are computed via a reduction modulo a regular sequence of length $d$ as discussed above. Now compute $r$ with the formula from Lemma 1 and classify $R$.

7. The class of $R$ together with the invariants $c, l = m - 1,$ and $n$ determine its Bass and Poincaré series; cf. [1, 2.1].

If $I$ is homogeneous, then various invariants of $R$ can be determined directly from the graded ring $Q/I$. If $I$ is not homogeneous, and $R$ hence not graded, then functions from the package `LocalRings` are used.

References


