Local rings of embedding codepth 3: a classification algorithm

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Abstract

Let I be an ideal of a regular local ring Q with residue field k. The length of the minimal free resolution of $R = Q/I$ is called the codepth of R. If it is at most 3, then the resolution carries a structure of a differential graded algebra, and the induced algebra structure on $\operatorname{Tor}_*^Q(R,k)$ provides for a classification of such local rings.

We describe the *Macaulay 2* package *CodepthThree* that implements an algorithm for classifying a local ring as above by computation of a few cohomological invariants.

1 Introduction and notation

Let R be a commutative noetherian local ring with residue field k. Assume that R has the form Q/I where Q is a regular local ring with maximal ideal $\mathfrak{n}$ and $I \subseteq \mathfrak{n}^2$. The embedding dimension of R (and of Q) is denoted e. Let

$$F = 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a minimal free resolution of R over Q. Set $d = \operatorname{depth} R$; the length c of the resolution F is by the Auslander–Buchsbaum formula

$$c = \operatorname{proj.dim}_Q R = \operatorname{depth} Q - \operatorname{depth}_Q R = e - d,$$

and one refers to this invariant as the *codepth* of R. In the following we assume that c is at most 3. By a theorem of Buchsbaum and Eisenbud [3, 3.4.3] the resolution F carries a differential graded algebra structure, which induces a unique graded-commutative algebra structure on $A = \operatorname{Tor}_*^Q(R,k)$. The possible structures were identified by Weyman [5] and by Avramov, Kustin, and Miller [2]. According to the multiplicative structure on A, the ring R

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belongs to exactly one of the classes designated $\mathbf{B}$, $\mathbf{C}(c)$, $\mathbf{G}(r)$, $\mathbf{H}(p,q)$, $\mathbf{S}$, and $\mathbf{T}$. Here the parameters p, q, and r are given by

$$p = \operatorname{rank}_k(A_1 \cdot A_1), \quad q = \operatorname{rank}_k(A_1 \cdot A_2), \quad \text{and} \quad r = \operatorname{rank}_k(\delta\colon A_2 \to \operatorname{Hom}_k(A_1, A_3)),$$

where δ is the canonical map. See [1, 2, 5] for further background and details.

When, in the following, we talk about classification of a local ring R, we mean the classification according to the multiplicative structure on A. To describe the classification algorithm, we need a few more invariants of R. Set

$$l = \operatorname{rank}_Q F_1 - 1 \qquad \text{and} \qquad n = \operatorname{rank}_Q F_c;$$

the latter invariant is called the *type* of R. The *Cohen–Macaulay defect* of R is $h = \dim R - d$. The Betti numbers β_i and the Bass numbers μ_i record ranks of cohomology groups,

$$\beta_i = \beta_i^R(k) = \operatorname{rank}_k \operatorname{Ext}_R^i(k,k) \qquad \text{and} \qquad \mu_i = \mu_i(R) = \operatorname{rank}_k \operatorname{Ext}_R^i(k,R).$$

The generating functions $\sum_{i=0}^{\infty} \beta_i t^i$ and $\sum_{i=0}^{\infty} \mu_i t^i$ are called the *Poincaré series* and the *Bass series* of R.

2 The algorithm

For a local ring of codepth $c \leq 3$, the class together with the invariants e, c, l, and n completely determine the Poincaré series and the Bass series of R; see [1]. Conversely, one can determine the class of R based on e, c, l, n, and a few Betti and Bass numbers; in the following we describe how.

Lemma 1. *For a local ring R of codepth 3 the invariants p, q, and r are determined by e, l, n, β_2, β_3, β_4, and μ_{e-2} through the formulas*

$$\begin{aligned} p &= n + le + \beta_2 - \beta_3 + \tbinom{e-1}{3}, \\ q &= (n-p)e + l\beta_2 + \beta_3 - \beta_4 + \tbinom{e-1}{4}, \quad \textit{and} \\ r &= l + n - \mu_{e-2}. \end{aligned}$$

Proof. The Poincaré series of R has by [1, 2.1] the form

$$\sum_{i=0}^{\infty} \beta_i t^i = \frac{(1+t)^{e-1}}{1 - t - lt^2 - (n-p)t^3 + qt^4 + \cdots}, \tag{1}$$

and expansion of the rational function yields the expressions for p and q.

One has $d = e - 3$ and the Bass series of R has, also by [1, 2.1], the form

$$\sum_{i=0}^{\infty} \mu_i t^i = t^d \frac{n + (l-r)t + \cdots}{1 - t + \cdots}; \tag{2}$$

expansion of the rational function now yields the expression for r. ∎

Proposition 2. *A local ring R of codepth 3 can be classified based on the invariants e, h, l, n, $\beta_2, \beta_3, \beta_4$, μ_{e-2}, and μ_{e-1}.*

Proof. First recall that one has $h = 0$ and $n = 1$ if and only if R is Gorenstein; see [3, 3.2.10]. In this case R is in class $\mathbf{C}(3)$ if $l = 2$ and otherwise in class $\mathbf{G}(l+1)$.

Assume now that R is not Gorenstein. The invariants p, q, and r can be computed from the formulas in Lemma 1. It remains to determine the class, which can be done by case analysis. Recall from [1, 1.3 and 3.1] that one has

Class	p	q	r
$\mathbf{T}$	3	0	0
$\mathbf{B}$	1	1	2
$\mathbf{G}(r)$ $[r \geq 2]$	0	1	r
$\mathbf{H}(p,q)$	p	q	q

In case $q \geq 2$ the ring R is in class $\mathbf{H}(p,q)$; for $q \leq 1$ the case analysis shifts to p.

In case $p = 0$ the distinction between the classes $\mathbf{G}(r)$ and $\mathbf{H}(0,q)$ is made by comparing q and r; they are equal if and only if R is in class $\mathbf{H}(0,q)$.

In case $p = 1$ the distinction between the classes $\mathbf{B}$ and $\mathbf{H}(1,q)$ is made by comparing q and r; they are equal if and only if R is in class $\mathbf{H}(1,q)$.

In case $p = 3$ the distinction between the classes $\mathbf{T}$ and $\mathbf{H}(3,q)$ is drawn by the invariant μ_{e-1}. Recall the relation $d = e - 3$; expansion of the expressions from [1, 2.1] yields $\mu_{e-1} = \mu_{e-2} + ln - 2$ if R is in $\mathbf{T}$ and $\mu_{e-1} = \mu_{e-2} + ln - 3$ if R is in $\mathbf{H}(3,q)$.

In all other cases, i.e. $p = 2$ or $p \geq 4$, the ring R is in class $\mathbf{H}(p,q)$. □

Remark 3. One can also classify a local ring R of codepth 3 based on the invariants e, h, l, n, $\beta_2, \ldots \beta_5$, and μ_{e-2}. In the case $p = 3$ one then discriminates between the classes by looking at β_5, which is $\beta_4 + l\beta_3 + (n-3)\beta_2 + \tau$ with $\tau = 0$ if R is in class $\mathbf{H}(3,q)$ and $\tau = 1$ if R is in class $\mathbf{T}$. However, it is not possible to classify R based on Betti numbers alone. Indeed, rings in the classes $\mathbf{B}$ and $\mathbf{H}(1,1)$ have identical Poincaré series and so do rings in the classes $\mathbf{G}(r)$ and $\mathbf{H}(0,1)$.

Remark 4. A local ring R of codepth $c \leq 2$ can be classified based on the invariants c, h, and n. Indeed, if $c \leq 1$ then R is a hypersurface; i.e. it belongs to class $\mathbf{C}(c)$. If $c = 2$ then R belongs to class $\mathbf{C}(2)$ if and only if it is Gorenstein ($h = 0$ and $n = 1$); otherwise it belongs to class $\mathbf{S}$.

Algorithm 5. From Remark 4 and the proof of Proposition 2 one gets the following algorithm that takes as input invariants of a local ring of codepth $c \leq 3$ and outputs its class.

INPUT: c, e, h, l, n, β_2, β_3, β_4, μ_{e-2}, μ_{e-1}

- In case $c \leq 1$ set *Class* $= \mathbf{C}(c)$
- In case $c = 2$
  - ◇ if ($h = 0$ and $n = 1$) then set *Class* $= \mathbf{C}(2)$
  - ◇ else set *Class* $= \mathbf{S}$

- In case $c = 3$
  - ◇ if ($h = 0$ and $n = 1$) then set $r = l + 1$
    - if $r = 3$ then set *Class* $= \mathbf{C}(3)$
    - else set *Class* $= \mathbf{G}(r)$
  - ◇ else compute p and q
    - if ($q \geq 2$ or $p = 2$ or $p \geq 4$) then set *Class* $= \mathbf{H}(p, q)$
    - else compute r
      - In case $p = 0$
        - if $q = r$ then set *Class* $= \mathbf{H}(0, q)$
        - else set *Class* $= \mathbf{G}(r)$
      - In case $p = 1$
        - if $q = r$ then set *Class* $= \mathbf{H}(1, q)$
        - else set *Class* $= \mathbf{B}$
      - In case $p = 3$
        - if $\mu_{e-1} = \mu_{e-2} + ln - 2$ then set *Class* $= \mathbf{T}$
        - else set *Class* $= \mathbf{H}(3, q)$

OUTPUT: *Class*

Remark 6. Given a local ring $R = Q/I$ the invariants e and h can be computed from R, and c, l, and n can be determined by computing a minimal free resolution of R over Q. The Betti numbers $\beta_2, \beta_3, \beta_4$ one can get by computing the first five steps of a minimal free resolution F of k over R. Recall the relation $d = e - c$; the Bass numbers μ_{e-2} and μ_{e-1} one can get by computing the cohomology in degrees $d+1$ and $d+2$ of the dual complex $F^* = \mathrm{Hom}_R(F, R)$. For large values of d, this may not be feasible, but one can reduce R modulo a regular sequence $\mathbf{x} = x_1, \ldots, x_d$ and obtain the Bass numbers as $\mu_{d+i}(R) = \mu_i(R/(\mathbf{x}))$; cf. [3, 3.1.16].

3 The implementation

The *Macaulay2* package *CodepthThree* implements Algorithm 5. The function *torAlgClass* takes as input a quotient Q/I of a polynomial algebra, where I is contained in the irrelevant maximal ideal $\mathfrak{N}$ of Q. It returns the class of the local ring R obtained by localization of Q/I at $\mathfrak{N}$. For example, the local ring obtained by localizing the quotient

$$\mathbb{Q}[x, y, z]/(xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)$$

is in class $\mathbf{G}(2)$; see [4]. Here is how it looks when one calls the function *torAlgClass*.

```
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure,
LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "CodepthThree";
i2 : Q = QQ[x,y,z];
```

```
i3 : I = ideal (x*y^2,x*y*z,y*z^2,x^4-y^3*z,x*z^3-y^4);
o3 : Ideal of Q
i4 : torAlgClass (Q/I)
o4 = G(2)
```

Underlying *torAlgClass* is the workhorse function *torAlgData* which returns a hash table with the following data:

Key	Value
`"c"`	codepth of R
`"e"`	embedding dimension of R
`"h"`	Cohen–Macaulay defect of R
`"m"`	minimal number of generators of defining ideal of R
`"n"`	type of R
`"Class"`	(non-parametrized) class of R
	(‘B’, ‘C’, ‘G’, ‘H’, ‘S’, ‘T’, ‘codepth > 3’, or ‘zero ring’)
`"p"`	rank of $A_1 \cdot A_1$
`"q"`	rank of $A_1 \cdot A_2$
`"r"`	rank of $\delta\colon A_2 \to \mathrm{Hom}_k(A_1, A_3)$
`"PoincareSeries"`	Poincaré series of R
`"BassSeries"`	Bass series of R

In the example from above one gets:

```
i5 : torAlgData(Q/I)
                                     2    3    4
                            2 + 2T - T  - T  + T
o5 = HashTable{BassSeries => ----------------------  }
                                    2     3    4
                           1 - T - 4T  - 2T  + T
               c => 3
               Class => G
               e => 3
               h => 1
               m => 5
               n => 2
               p => 0
                                                2
                                          (1 + T)
               PoincareSeries => ----------------------
                                            2     3    4
                                   1 - T - 4T  - 2T  + T
               q => 1
               r => 2
```

To facilitate extraction of data from the hash table, the package offers two functions *torAlgDataList* and *torAlgDataPrint* that take as input a quotient ring and a list of keys. In the example from above one gets:

```
i6 : torAlgDataList( Q/I, {"c", "Class", "p", "q", "r", "PoincareSeries"} )

                                     2
                              (1 + T)
o6 = {3, G, 0, 1, 2, ---------------------}
                                 2     3    4
                     1 - T - 4T  - 2T  + T

o6 : List

i7 : torAlgDataPrint( Q/I, {"e", "h", "m", "n", "r"} )

o7 = e=3 h=1 m=5 n=2 r=2
```

As discussed in Remark 6, the computation of Bass numbers may require a reduction modulo a regular sequence. In our implementation such a reduction is attempted if the embedding dimension of the local ring R is more than 3. The procedure involves random choices of ring elements, and hence it may fail. By default, up to 625 attempts are made, and with the function *setAttemptsAtGenericReduction,* one can change the number of attempts. If none of the attempts are successful, then an error message is displayed:

```
i8 : Q = ZZ/2[u,v,w,x,y,z];

i9 : R = Q/ideal(x*y^2,x*y*z,y*z^2,x^4-y^3*z,x*z^3-y^4);

i10 : setAttemptsAtGenericReduction(R,1)

o10 = 1 attempt(s) will be made to compute the Bass numbers via a generic
      reduction

i11 : torAlgClass R
stdio:11:1:(3): error: Failed to compute Bass numbers. You may raise the
    number of attempts to compute Bass numbers via a generic reduction
    with the function setAttemptsAtGenericReduction and try again.

i12 : setAttemptsAtGenericReduction(R,25)

o12 = 625 attempt(s) will be made to compute the Bass numbers via a generic
      reduction

i13 : torAlgClass R

o13 = G(2)
```

Notice that the maximal number of attempts is n^2 where n is the value set with the function *setAttemptsAtGenericReduction*.

Notes. Given Q/I our implementation of Algorithm 5 in *torAlgData* proceeds as follows.

1. Check if a value is set for *attemptsAtBassNumbers*; if not use the default value 25.
2. Initialize the invariants of R (the localization of Q/I at the irrelevant maximal ideal) that are to be returned; see the table in Section 3.
3. Handle the special case where the defining ideal I or Q/I is 0. In all other cases compute the invariants c, e, h, m $(= l + 1)$, and n.
4. If possible, classify R based on c, e, h, m, and n. At this point the implementation deviates slightly from Algorithm 5, as it uses that all rings with $c = 3$ and $h = 2$ are of class $\mathbf{H}(0, 0)$; see [1, 3.5].
5. For rings not classified in step 3 or 4 one has $c = 3$; cf. Remark 4. Compute the Betti numbers β_2, β_3, and β_4, and with the formula from Lemma 1 compute p and q. If possible classify R based on these two invariants.
6. For rings not classified in steps 3–5, compute the Bass numbers μ_{e-2} and μ_{e-1}. If $d = e - 3$ is positive, then the Bass numbers are computed via a reduction modulo a regular sequence of length d as discussed above. Now compute r with the formula from Lemma 1 and classify R.
7. The class of R together with the invariants c, $l = m - 1$, and n determine its Bass and Poincaré series; cf. [1, 2.1].

If I is homogeneous, then various invariants of R can be determined directly from the graded ring Q/I. If I is not homogeneous, and R hence not graded, then functions from the package *LocalRings* are used.

References

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