ASCENT PROPERTIES OF AUSLANDER CATEGORIES

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Abstract. Let R be a homomorphic image of a Gorenstein local ring. Recent work has shown that there is a bridge between Auslander categories and modules of finite Gorenstein homological dimensions over R.

We use Gorenstein dimensions to prove new results about Auslander categories and vice versa. For example, we establish base change relations between the Auslander categories of the source and target rings of a homomorphism $\varphi\colon R \to S$ of finite flat dimension.

Introduction

Transfer of homological properties along ring homomorphisms is already a classical field of study, initiated in [31] and continued in the more recent series [7, 8, 9, 10, 11]. In this paper we investigate ascent properties of modules in the so-called Auslander categories of a commutative noetherian ring.

For a local ring R with a dualizing complex, Avramov and Foxby [8] introduced the Auslander categories $\mathsf{A}(R)$ and $\mathsf{B}(R)$, two subcategories of the derived category of R. This was part of their study of local ring homomorphisms of finite Gorenstein dimension. One theme played in [8] is

(I) *Results for Auslander categories have implications for Gorenstein dimensions*

This is based on the realization that Auslander categories and Gorenstein homological dimensions are close kin [19, 24]. The latter were introduced much earlier by Auslander and Bridger [3, 4] and Enochs, Jenda et. al. [21, 23].

In this paper we continue the theme (I). Let $\varphi\colon R \to S$ be a local homomorphism of rings. Working directly with the definition of A we prove e.g. (2.1)(c):

> **Theorem I.** *Assume that* $\operatorname{fd}\varphi$ *is finite and* S *has a dualizing complex. If* P *is an* R*–module of finite projective dimension and* $\tilde{A} \in \mathsf{A}(S)$ *then* $\mathbf{R}\mathrm{Hom}_R(P, \tilde{A})$ *belongs to* $\mathsf{A}(S)$.

[Here $\mathbf{R}$Hom is the right-derived Hom functor.] For the next result we need the notion of Gorenstein flat modules, which is a generalization of flat modules introduced in [23]. Theorem I has as a consequence (2.4)(b):

> **Corollary I.** *Assume that* $\operatorname{fd}\varphi$ *is finite and* S *has a dualizing complex. If* P *is a projective* R*–module and* $\tilde{A}$ *is a Gorenstein flat* S*–module, then* $\mathrm{Hom}_R(P, \tilde{A})$ *is Gorenstein flat over* S.

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Gorenstein dimensions and Auslander categories are truly two sides of one coin, and the complementary theme

(II) *Results for Gorenstein dimensions have implications for Auslander categories*

turns out to be equally useful. For example, from the definition of Gorenstein flat modules we prove (2.6)(a):

> **Theorem II.** *Assume that* $\operatorname{fd}\varphi$ *is finite. If* $\tilde{F}$ *is a flat* S*–module and* A *is a Gorenstein flat* R*–module, then* $\tilde{F} \otimes_R A$ *is Gorenstein flat over* S*.*

From this one gets (2.8)(a):

> **Corollary II.** *Assume that* $\operatorname{fd}\varphi$ *is finite and that* R *and* S *have dualizing complexes. If* $\tilde{F}$ *is an* S*–module of finite flat dimension and* $A \in \mathsf{A}(R)$ *then* $\tilde{F} \otimes^{\mathbf{L}}_R A$ *belongs to* $\mathsf{A}(S)$*.*

[Here $\otimes^{\mathbf{L}}$ is the left-derived tensor product functor.] We are not aware of any direct proof of Corollary I, i.e. a proof that avoids Theorem I. The same remark applies to Corollary/Theorem II.

Evaluation morphisms are important tools in the study of Auslander categories. Indeed, Theorem I relies on the fact that the tensor evaluation morphism,

$$(*) \qquad \omega^{RS}_{LMN} \colon \operatorname{Hom}_R(L, M) \otimes_S N \to \operatorname{Hom}_R(L, M \otimes_S N),$$

is invertible when L is a projective R–module, M and N are S–modules, and N is finitely generated. In section 3 we give new conditions that ensure invertibility of evaluation morphisms; for example (3.3):

> **Theorem III.** *Assume that* $\operatorname{fd}\varphi$ *is finite. If* L *is finitely generated and Gorenstein flat over* R*,* M *is flat over* S *and* N *is injective over* S*, then* ω^{RS}_{LMN} *in* $(*)$ *is an isomorphism.*

These new isomorphisms have applications beyond the study of Auslander categories, e.g. to formulas of the Auslander–Buchsbaum type: For a finitely generated R–module M of finite flat dimension, the classical Auslander–Buchsbaum formula

$$\sup\{\, m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(k, M) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R M$$

is a special case of (4.3)(a):

$$(\dagger) \qquad \operatorname{depth}_S(N \otimes^{\mathbf{L}}_R M) = \operatorname{depth}_S N + \operatorname{depth}_R M - \operatorname{depth} R,$$

which holds for R–modules M of finite flat dimension and all S–modules N.

Results like Theorem III allow us to prove that (†) also holds for R–modules M of finite Gorenstein flat dimension and S–modules N of finite injective dimension.

As indicated by (†), results in this paper are stated in the language of derived categories; we recall the basic notions in section 0. The prerequisites on Auslander categories and Gorenstein dimensions are given in section 1. Section 2 is devoted to the themes (I) and (II). In section 3 we break to establish certain evaluation isomorphisms and then continue the themes of the previous section. In section 4 we study formulas of the Auslander–Buchsbaum type, and in the final, appendix-like, section 5 we catalogue the ascent results obtained in sections 2 and 3.

0. Notation and prerequisites

All rings in this paper are assumed to be commutative, unital, and non-zero; throughout, R and S denote such rings. All modules are unitary.

(0.1) **Complexes.** We denote by $\mathsf{C}(R)$ the category of R–complexes; that is, chain complexes of R–modules. We use this notation with subscripts $\sqsupset$, $\square$, and $\sqsubset$ to denote the full subcategories of left- and/or right-bounded complexes. E.g.

$$X \;=\; \cdots \longrightarrow X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \longrightarrow \cdots\,,$$

is in $\mathsf{C}_{\sqsupset}(R)$ if and only if $X_\ell = 0$ for $\ell \ll 0$. We use superscripts I, F, P, and fP to indicate that the complexes in question consist of modules which are, respectively, injective, flat, projective, or finite (that is, finitely generated) projective.

The notation $\mathsf{D}(R)$ is used for the derived category of the abelian category of R–modules; see [32, chap. I] or [46, chap. 10]. We use subscripts $\sqsupset$, $\square$, and $\sqsubset$ and superscript f to indicate vanishing and finiteness of homology modules. For the homological supremum and infimum of $X \in \mathsf{D}(R)$ we write $\sup X$ and $\inf X$. Thus, X is in $\mathsf{D}_{\sqsupset}(R)$ if and only if $\inf X > -\infty$.

Since R is commutative, the right derived Hom, $\mathbf{R}\mathrm{Hom}_R(-,-)$, and the left derived tensor product, $- \otimes^{\mathbf{L}}_R -$, are functors (in two variables) in $\mathsf{D}(R)$.

The symbol $\simeq$ denotes quasi-isomorphisms in $\mathsf{C}(R)$ and isomorphisms in $\mathsf{D}(R)$.

The category of complexes of (R,S)–bimodules is denoted $\mathsf{C}(R,S)$. We write $\mathsf{D}(R,S)$ for the derived category of the abelian category of (R,S)–bimodules, and we use sub- and superscripts on $\mathsf{D}(R,S)$ as we do on $\mathsf{D}(R)$.

(0.2) **Homological dimensions.** We use abbreviations pd, id, and fd for projective, injective, and flat dimension of complexes. By $\mathsf{P}(R)$, $\mathsf{I}(R)$, and $\mathsf{F}(R)$ we denote the full subcategories of $\mathsf{D}_{\square}(R)$ whose objects are complexes of finite projective/injective/flat dimension.

The (left derived) tensor product is left-adjoint to the (right derived) Hom functor; this gives the adjunction isomorphism(s). This and other standard isomorphisms, associativity and commutativity of tensor products, are used freely.

The, in general not invertible, evaluation morphisms shall play a key role in several proofs. For later reference, we recall a selection of conditions under which they are invertible.

(0.3) **Evaluation morphisms in C.** *Let $X \in \mathsf{C}(R)$, $Y \in \mathsf{C}(R,S)$ and $Z \in \mathsf{C}(S)$. Then $\mathrm{Hom}_S(Y,Z)$, $\mathrm{Hom}_R(X,Y)$, and $Y \otimes_S Z$ belong to $\mathsf{C}(R,S)$; the canonical maps*

$$\theta^{RS}_{XYZ}\colon X \otimes_R \mathrm{Hom}_S(Y,Z) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(X,Y),Z) \quad \textit{and}$$

$$\omega^{RS}_{XYZ}\colon \mathrm{Hom}_R(X,Y) \otimes_S Z \longrightarrow \mathrm{Hom}_R(X, Y \otimes_S Z)$$

are morphisms in $\mathsf{C}(R,S)$ and functorial in X, Y, and Z.

If two of the complexes X, Y, and Z are bounded, then the Hom evaluation morphism θ^{RS}_{XYZ} is invertible under each of the following extra conditions:

(a) *$X \in \mathsf{C}^{\mathrm{fP}}(R)$; or*

(b) *R is noetherian, $X \in \mathsf{C}^{\mathrm{f}}(R)$, and $Z \in \mathsf{C}^{\mathrm{I}}(S)$.*

If two of the complexes X, Y, and Z are bounded, then the tensor evaluation morphism ω^{RS}_{XYZ} is invertible under each of the following extra conditions:

(c) $X \in \mathsf{C}^{\mathrm{fP}}(R)$;
(d) *R is noetherian, $X \in \mathsf{C}^{\mathrm{f}}(R)$, and $Z \in \mathsf{C}^{\mathrm{F}}(S)$;*
(e) *$Z \in \mathsf{C}^{\mathrm{fP}}(S)$; or*
(f) *S is noetherian, $Z \in \mathsf{C}^{\mathrm{f}}(S)$, and $X \in \mathsf{C}^{\mathrm{P}}(R)$.*

Proof. Conditions (a)–(d) can be traced back to [25, sec. 0, 5, and 9], [6, lem. 4.4], and [2, thm. 1 and 2]. We have not found references for (e) and (f), so we include the argument:

(e): Under the boundedness conditions the morphism ω^{RS}_{XYZ} will, in each degree, be a finite sum of evaluation morphisms of modules X_h, Y_i, and Z_j. Thus, it is sufficient to deal with the module case. When Z is a finite projective module it is a direct summand in a finite free module S^β. By additivity of the involved functors it suffices to establish the isomorphism for $Z = S^\beta$, and that follows immediately from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(X,Y) \otimes_S S^\beta & \xrightarrow{\ \omega^{RS}_{XYS^\beta}\ } & \mathrm{Hom}_R(X, Y \otimes_S S^\beta)\ . \\
\Big\downarrow \cong & & \cong \Big\downarrow \\
\mathrm{Hom}_R(X,Y)^\beta & \xrightarrow{\ \cong\ } & \mathrm{Hom}_R(X, Y^\beta)
\end{array}$$

(f): As above it suffices to deal with the module case and we may assume that X is free, $X = R^{(\Lambda)}$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(R^{(\Lambda)},Y) \otimes_S Z & \xrightarrow{\ \omega^{RS}_{R^{(\Lambda)}YZ}\ } & \mathrm{Hom}_R(R^{(\Lambda)}, Y \otimes_S Z) \\
\Big\downarrow \cong & & \cong \Big\downarrow \\
Y^\Lambda \otimes_S Z & \longrightarrow & (Y \otimes_S Z)^\Lambda
\end{array}$$

When S is noetherian and Z is finite, the lower horizontal homomorphism is invertible by [15, ch. II, exerc. 2] or [22, thm. 3.2.22]; thus $\omega^{RS}_{R^{(\Lambda)}YZ}$ is an isomorphism. □

(0.4) **Evaluation morphisms in D.** *Let $X \in \mathsf{D}(R)$, $Y \in \mathsf{D}(R,S)$ and $Z \in \mathsf{D}(S)$. Then $\mathbf{R}\mathrm{Hom}_S(Y,Z)$, $\mathbf{R}\mathrm{Hom}_R(X,Y)$, and $Y \otimes^{\mathbf{L}}_S Z$ are representable by complexes of (R,S)–bimodules. The canonical R– and S–linear maps,*

$$\theta^{RS}_{XYZ}\colon X \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_S(Y,Z) \longrightarrow \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X,Y),Z) \quad \text{and}$$

$$\omega^{RS}_{XYZ}\colon \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes^{\mathbf{L}}_S Z \longrightarrow \mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathbf{L}}_S Z)$$

are functorial in X, Y, and Z.

If R is noetherian, then the Hom evaluation morphism θ^{RS}_{XYZ} is invertible, provided that:

(a) *$X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}_{\Box}(R,S)$; or*
(b) *$X \in \mathsf{D}^{\mathrm{f}}_{\Box}(R)$, $Y \in \mathsf{D}_{\sqsubset}(R,S)$, and $Z \in \mathsf{I}(S)$.*

If R is noetherian, then the tensor evaluation morphism ω^{RS}_{XYZ} is invertible, provided that:

(c) *$X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}_{\Box}(R,S)$; or*
(d) *$X \in \mathsf{D}^{\mathrm{f}}_{\Box}(R)$, $Y \in \mathsf{D}_{\sqsubset}(R,S)$, and $Z \in \mathsf{F}(S)$.*

If S is noetherian, then the tensor evaluation morphism ω^{RS}_{XYZ} is invertible, provided that:

(e) $Z \in \mathsf{P}^{\mathrm{f}}(S)$ *and* $Y \in \mathsf{D}_{\sqsubset}(R,S)$*; or*

(f) $X \in \mathsf{P}(R)$, $Y \in \mathsf{D}_{\sqsubset}(R,S)$, *and* $Z \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(S)$.

Proof. Conditions (a)–(d) can be traced back to [25, sec. 0, 5, and 9], [6, lem. 4.4], and [2, thm. 1 and 2]. Parts (e) and (f) follow from (0.3); they have similar proofs and we only write out the details for (f):

Since S is noetherian $Z \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(S)$ has a resolution by finite free S–modules, $\mathsf{C}^{\mathrm{fP}}(S) \ni L \xrightarrow{\simeq} Z$. As $X \in \mathsf{P}(R)$ there also exists a bounded projective resolution, $\mathsf{C}^{\mathrm{P}}_{\square}(R) \ni P \xrightarrow{\simeq} X$. Let Y' be an appropriate truncation of Y, then $Y' \in \mathsf{C}_{\square}(R,S)$ is isomorphic to Y in $\mathsf{D}(R,S)$. Now the tensor-evaluation morphism ω^{RS}_{XYZ} in $\mathsf{D}(R,S)$ is represented by

$$\mathrm{Hom}_R(P,Y') \otimes_S L \xrightarrow{\omega^{RS}_{PY'L}} \mathrm{Hom}_R(P, Y' \otimes_S L),$$

and by (0.3)(e) this map is an isomorphism in $\mathsf{C}(R,S)$. $\square$

All results in this paper are phrased in a *relative* setting, that is, they refer to a homomorphism $\varphi\colon R \to S$ of rings. We refer to the situation $\varphi = 1_R$ as *the absolute case.* Complexes over S are considered as R–complexes with the action given by φ.

Again, we recall for later reference the ascent properties of the classical homological dimensions. To distinguish S–modules from R–modules we mark the former with a tilde, e.g. $\tilde{N}$. This praxis is applied whenever convenient.

(0.5) **Ascent for modules.** *Let $\varphi\colon R \to S$ be a homomorphism of rings. The following hold:*

- *If $\tilde{F}$ is a flat S–module and F a flat R–module, then $\tilde{F} \otimes_R F$ is flat over S*
- *If $\tilde{P}$ is a projective S–module and P is a projective R–module, then $\tilde{P} \otimes_R P$ is projective over S*
- *If $\tilde{F}$ is a flat S–module and I is an injective R–module, then $\mathrm{Hom}_R(\tilde{F}, I)$ is injective over S*
- *If F is a flat R–module and $\tilde{I}$ is an injective S–module, then $\mathrm{Hom}_R(F, \tilde{I})$ is injective over S*

If S is noetherian, also the following hold:

- *If $\tilde{I}$ is an injective S–module and F a flat R–module, then $\tilde{I} \otimes_R F$ is injective over S*
- *If P is a projective R–module and $\tilde{F}$ a flat S–module, then $\mathrm{Hom}_R(P, \tilde{F})$ is flat over S*
- *If $\tilde{I}$ is an injective S–module and I an injective R–module, then $\mathrm{Hom}_R(\tilde{I}, I)$ is flat over S*

Proof. All seven results are folklore and are straightforward to verify; see also [35]. As an example, consider the penultimate one: The projective module P is a direct summand in a free R–module; that makes $\mathrm{Hom}_R(P, \tilde{F})$ a direct summand in a product of flat S–modules and hence flat, as S is noetherian. (This and the two neighboring results can also be proved using the evaluation morphisms from (0.3).)

Also note that the fourth is a consequence of the isomorphism,

$$\operatorname{Hom}_S(-,\operatorname{Hom}_R(F,\tilde{I}))\cong\operatorname{Hom}_S(-\otimes_R F,\tilde{I}),$$

which is an easily verified variant of standard adjointness. □

(0.6) **Ascent for complexes.** Let $\varphi\colon R\to S$ be a homomorphism of rings. The results in (0.5) imply similar ascent results for complexes of finite homological dimension. In short and suggestive notation we write them as:

- $\mathsf{F}(S)\otimes^{\mathbf{L}}_R\mathsf{F}(R)\subseteq\mathsf{F}(S)$
- $\mathsf{P}(S)\otimes^{\mathbf{L}}_R\mathsf{P}(R)\subseteq\mathsf{P}(S)$
- $\mathbf{R}\operatorname{Hom}_R(\mathsf{F}(S),\mathsf{I}(R))\subseteq\mathsf{I}(S)$
- $\mathbf{R}\operatorname{Hom}_R(\mathsf{P}(R),\mathsf{I}(S))\subseteq\mathsf{I}(S)$

If S is noetherian, we also have:

- $\mathsf{I}(S)\otimes^{\mathbf{L}}_R\mathsf{F}(R)\subseteq\mathsf{I}(S)$
- $\mathbf{R}\operatorname{Hom}_R(\mathsf{P}(R),\mathsf{F}(S))\subseteq\mathsf{F}(S)$
- $\mathbf{R}\operatorname{Hom}_R(\mathsf{I}(S),\mathsf{I}(R))\subseteq\mathsf{F}(S)$

(0.7) **Local rings and homomorphisms.** We say that $(R,\mathfrak{m},k)$ is local, if R is noetherian and local with maximal ideal $\mathfrak{m}$ and residue field k. A homomorphism of rings $\varphi\colon(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is said to be local if $\varphi(\mathfrak{m})\subseteq\mathfrak{n}$.

(0.8) **Homomological dimensions of homomorphisms.** Let $\varphi\colon R\to S$ be a homomorphism of rings. The flat dimension of φ is by definition the flat dimension of S considered as a module over R with the action given by φ. That is, $\operatorname{fd}\varphi=\operatorname{fd}_R S$. The projective and injective dimensions of φ are defined similarly.

Note that in the case where φ is a local homomorphism, our definition of $\operatorname{pd}\varphi$ differs from the one in [38, def. 4.2]. However $\operatorname{pd}\varphi$ in our sense, and $\operatorname{pd}\varphi$ in the sense of [38, def. 4.2] are simultaneously finite.

Transfer of homological properties along homomorphisms is already a classical field of study. A basic observation is [8, prop. (4.6)(b)]: If $\operatorname{fd}\varphi$ is finite, then

$$\big(\mathsf{P}(S)\subseteq\big)\,\mathsf{F}(S)\subseteq\mathsf{F}(R)\quad\text{and}\quad\mathsf{I}(S)\subseteq\mathsf{I}(R).$$

In (0.9) below we use this to establish a useful variant of (0.6).

The literature emphasizes the study of homomorphisms of finite flat dimension; largely, we follow this tradition, as it is well founded: Let $\varphi\colon R\to S$ be a homomorphism of noetherian rings.

- If the projective dimension of φ is finite then so is $\operatorname{fd}\varphi$, and the converse holds if R has finite Krull dimension; see [39, prop. 6] and [45, thm. II.3.2.6].
- If φ is local, then the injective dimension of φ is finite if and only if $\operatorname{fd}\varphi$ is finite and R is Gorenstein. This was only established recently, in [12, thm. 13.2], though the surjective case goes back to [44, (II.5.5)].
- If R has finite Krull dimension, then φ has finite injective dimension if and only if $\operatorname{fd}\varphi$ is finite and R is Gorenstein at any contraction $\mathfrak{p}=\mathfrak{q}\cap R$ of a prime ideal in S. This follows from the local case above.

The next lemma resembles the last part of (0.6); the difference is that the noetherian assumption has been moved from S to R, while the complexes all have S–structures.

(0.9) **Lemma.** *Let* $\varphi\colon R \to S$ *be a homomorphism of rings with* $\operatorname{fd}\varphi$ *finite. If* R *is noetherian, then the following hold:*

- $\mathsf{I}(S) \otimes^{\mathbf{L}}_S \mathsf{F}(S) \subseteq \mathsf{I}(R)$
- $\mathbf{R}\mathrm{Hom}_S(\mathsf{P}(S), \mathsf{F}(S)) \subseteq \mathsf{F}(R)$
- $\mathbf{R}\mathrm{Hom}_S(\mathsf{I}(S), \mathsf{I}(S)) \subseteq \mathsf{F}(R)$

Proof. For the last assertion let $\tilde{I}, \tilde{J} \in \mathsf{I}(S)$. We must show that for any finite R–module M, the complex $M \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(\tilde{J}, \tilde{I})$ is in $\mathsf{D}_\square(R)$. By (0.4)(b) we have

$$M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_S(\tilde{J}, \tilde{I}) \simeq \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M, \tilde{J}), \tilde{I}),$$

and the latter complex is bounded as $\tilde{I} \in \mathsf{I}(S)$ and $\tilde{J} \in \mathsf{I}(S) \subseteq \mathsf{I}(R)$ by (0.8).

The other assertions have similar proofs. □

1. Gorenstein dimensions and Auslander categories

This paper pivots on the interplay between (semi-)dualizing complexes, their Auslander categories, and Gorenstein homological dimensions.

Semi-dualizing complexes and Auslander categories came up in studies of ring homomorphisms [8] and are used to detect the Gorenstein [17] and Cohen–Macaulay [34] properties of rings. This section recaps the relevant definitions and results.

(1.1) **Gorenstein dimensions.** Gorenstein projective, injective and flat modules are defined in terms of so-called complete resolutions:

- An R–module A is *Gorenstein projective* if there exists an exact complex $\boldsymbol{P}$ of projective modules, such that $A \cong \mathrm{Coker}(P_1 \to P_0)$ and $\mathrm{H}(\mathrm{Hom}_R(\boldsymbol{P}, Q)) = 0$ for all projective R–modules Q. Such a complex $\boldsymbol{P}$ is called a *complete projective resolution* (of A).
- An R–module B is *Gorenstein injective* if there exists an exact complex $\boldsymbol{I}$ of injective modules, such that $B \cong \mathrm{Ker}(I_0 \to I_{-1})$ and $\mathrm{H}(\mathrm{Hom}_R(J, \boldsymbol{I})) = 0$ for all injective R–modules J. Such a complex $\boldsymbol{I}$ is called a *complete injective resolution* (of B).
- An R–module A is *Gorenstein flat* if there exists an exact complex $\boldsymbol{F}$ of flat modules, such that $A \cong \mathrm{Coker}(F_1 \to F_0)$ and $\mathrm{H}(J \otimes_R \boldsymbol{F}) = 0$ for all injective R–modules J. Such a complex $\boldsymbol{F}$ is called a *complete flat resolution* (of A).

These definitions from [21, 23] generalize and dualize the notion of G–dimension 0 modules from [3, 4]; see [16, thm. (4.2.5) and (5.1.11)].

By taking resolutions, one defines the *Gorenstein projective* and *Gorenstein flat dimension* of right-bounded complexes in $\mathsf{D}(R)$ and the *Gorenstein injective dimension* of left-bounded complexes. For details see [16, def. (4.4.2), (5.2.2), and (6.2.2)]. All projective modules are Gorenstein projective, so the Gorenstein projective dimension of an R–complex $X \in \mathsf{D}_\sqsupset(R)$ is a finer invariant than the usual projective dimension; that is, $\mathrm{Gpd}_R X \leq \mathrm{pd}_R X$. Similarly, injective and flat modules are Gorenstein injective and Gorenstein flat, so we have inclusions

$$\mathsf{P}(R) \subseteq \mathsf{GP}(R), \quad \mathsf{I}(R) \subseteq \mathsf{GI}(R) \quad \text{and} \quad \mathsf{F}(R) \subseteq \mathsf{GF}(R).$$

Here $\mathsf{GP}(R)$ denotes the full subcategory of bounded complexes of finite Gorenstein projective dimension; $\mathsf{GI}(R)$ and $\mathsf{GF}(R)$ are defined similarly. See [33, 16, 19] for details on Gorenstein dimensions.

(1.2) **Auslander Categories.** Assume that R is noetherian. A *semi-dualizing complex* for R is a complex $C \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ such that the homothety morphism,

$$R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C,C),$$

is invertible in $\mathsf{D}(R)$; cf. [17]. Note that R is a semi-dualizing complex for itself.

If, in addition, $C \in \mathsf{I}(R)$, then C is a *dualizing complex* for R, cf. [32, V.§2].

Let C be a semi-dualizing complex for R, and consider the adjoint pair of functors

$$(*) \qquad \mathsf{D}(R) \underset{\mathbf{R}\mathrm{Hom}_R(C,-)}{\overset{C\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathsf{D}(R).$$

The Auslander categories with respect to C, denoted ${}_C\mathsf{A}(R)$ and ${}_C\mathsf{B}(R)$, are the full subcategories of $\mathsf{D}_{\square}(R)$ whose objects are specified as follows:

$${}_C\mathsf{A}(R) = \left\{ X \in \mathsf{D}_{\square}(R) \;\middle|\; \begin{array}{l} \eta_X \colon X \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \text{ is an iso-} \\ \text{morphism in } \mathsf{D}(R), \text{ and } C \otimes^{\mathbf{L}}_R X \in \mathsf{D}_{\square}(R) \end{array} \right\},$$

$${}_C\mathsf{B}(R) = \left\{ Y \in \mathsf{D}_{\square}(R) \;\middle|\; \begin{array}{l} \varepsilon_Y \colon C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,Y) \xrightarrow{\simeq} Y \text{ is an isomor-} \\ \text{phism in } \mathsf{D}(R), \text{ and } \mathbf{R}\mathrm{Hom}_R(C,Y) \in \mathsf{D}_{\square}(R) \end{array} \right\},$$

where η and ε denote the unit and counit of the pair $(C \otimes^{\mathbf{L}}_R -, \mathbf{R}\mathrm{Hom}_R(C,-))$. These categories were introduced in [8, 17].

The Auslander categories are triangulated subcategories of $\mathsf{D}(R)$, and the adjoint pair in $(*)$ restricts to an equivalence between them,

$${}_C\mathsf{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(C,-)}{\overset{C\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} {}_C\mathsf{B}(R).$$

By [17, prop. (4.4)] there are inclusions $\mathsf{F}(R) \subseteq {}_C\mathsf{A}(R)$ and $\mathsf{I}(R) \subseteq {}_C\mathsf{B}(R)$.

The relation between Auslander categories and Gorenstein dimensions is established in [24, 47, 19]: If D is a dualizing complex for R, then

$$(1.2.1) \qquad {}_D\mathsf{A}(R) = \mathsf{GP}(R) = \mathsf{GF}(R) \quad \text{and} \quad {}_D\mathsf{B}(R) = \mathsf{GI}(R).$$

2. Ascent properties

The first result below should be compared to (0.6).

(2.1) **Proposition.** *Let $\varphi\colon R \to S$ be a homomorphism of rings. If S is noetherian and $\tilde{C}$ is a semi-dualizing complex for S, then the following hold:*

(a) *If $\tilde{A} \in {}_{\tilde{C}}\mathsf{A}(S)$ and $F \in \mathsf{F}(R)$, then $\tilde{A} \otimes^{\mathbf{L}}_R F \in {}_{\tilde{C}}\mathsf{A}(S)$*

(b) *If $\tilde{B} \in {}_{\tilde{C}}\mathsf{B}(S)$ and $F \in \mathsf{F}(R)$, then $\tilde{B} \otimes^{\mathbf{L}}_R F \in {}_{\tilde{C}}\mathsf{B}(S)$*

(c) *If $P \in \mathsf{P}(R)$ and $\tilde{A} \in {}_{\tilde{C}}\mathsf{A}(S)$, then $\mathbf{R}\mathrm{Hom}_R(P, \tilde{A}) \in {}_{\tilde{C}}\mathsf{A}(S)$*

(d) *If $P \in \mathsf{P}(R)$ and $\tilde{B} \in {}_{\tilde{C}}\mathsf{B}(S)$, then $\mathbf{R}\mathrm{Hom}_R(P, \tilde{B}) \in {}_{\tilde{C}}\mathsf{B}(S)$*

(e) *If $\tilde{A} \in {}_{\tilde{C}}\mathsf{A}(S)$ and $I \in \mathsf{I}(R)$, then $\mathbf{R}\mathrm{Hom}_R(\tilde{A}, I) \in {}_{\tilde{C}}\mathsf{B}(S)$*

(f) *If $\tilde{B} \in {}_{\tilde{C}}\mathsf{B}(S)$ and $I \in \mathsf{I}(R)$, then $\mathbf{R}\mathrm{Hom}_R(\tilde{B}, I) \in {}_{\tilde{C}}\mathsf{A}(S)$*

For the important special case of a dualizing complex, absolute versions of parts (a), (b), (e), and (f) appear in [16, (6.4.13)] and certain relative versions in [41].

Proof. (c): First note that $\mathbf{R}\mathrm{Hom}_R(P,\tilde{A})$ belongs to $\mathsf{D}_\square(S)$, as P has finite projective dimension over R and $\tilde{A} \in \mathsf{D}_\square(S)$. To see that also $\tilde{C} \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(P,\tilde{A})$ is homologically bounded, we employ (0.4)(f) to get an isomorphism,

$$\tilde{C} \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(P,\tilde{A}) \xrightarrow[\simeq]{\omega} \mathbf{R}\mathrm{Hom}_R(P,\tilde{C} \otimes^{\mathbf{L}}_S \tilde{A}).$$

The latter complex is homologically bounded as $\tilde{C} \otimes^{\mathbf{L}}_S \tilde{A}$ is so. Finally, the commutative diagram

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(P,\tilde{A}) & \xrightarrow{\eta_{\mathbf{R}\mathrm{Hom}_R(P,\tilde{A})}} & \mathbf{R}\mathrm{Hom}_S(\tilde{C},\tilde{C} \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(P,\tilde{A})) \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(P,\eta_{\tilde{A}})}\downarrow \simeq & & \simeq \downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}_S(\tilde{C},\omega)} \\
\mathbf{R}\mathrm{Hom}_R(P,\mathbf{R}\mathrm{Hom}_S(\tilde{C},\tilde{C} \otimes^{\mathbf{L}}_S \tilde{A})) & \xrightarrow[\text{swap}]{\simeq} & \mathbf{R}\mathrm{Hom}_S(\tilde{C},\mathbf{R}\mathrm{Hom}_R(P,\tilde{C} \otimes^{\mathbf{L}}_S \tilde{A}))
\end{array}$$

shows that the unit $\eta_{\mathbf{R}\mathrm{Hom}_R(P,\tilde{A})}$ is invertible in $\mathsf{D}(R)$.

The proof of (d) is similar and also uses (0.4)(f); the proofs of (a), (b), (e), and (f) are also similar and rely on standard conditions for invertibility of tensor and Hom evaluation morphisms, cf. (0.4). □

The next example shows why (0.6) and Proposition (2.1) give no results about $\mathsf{I}(R) \otimes^{\mathbf{L}}_R \mathsf{I}(R)$ and $\mathbf{R}\mathrm{Hom}_R(\mathsf{I}(R),\mathsf{P}(R))$.

(2.2) **Example.** Let k be a field and consider the zero-dimensional local ring

$$R = k[\![X,Y]\!]/(X^2,XY,Y^2).$$

We write the residue classes $x = [X]$ and $y = [Y]$; since $(x) \cap (y) = (0)$, the ring R is not Gorenstein, but it has an injective dualizing module $D = \mathrm{E}_R(k)$. Furthermore, the maximal ideal (x,y) is isomorphic to k^2, so we have an exact sequence $0 \to k^2 \to R \to k \to 0$. Applying $\mathrm{Hom}_R(-,D)$ we get:

$$(*) \qquad 0 \to k \to D \to k^2 \to 0.$$

(a) Since R is not Gorenstein, $D \notin {}_D\mathsf{A}(R)$ by [16, thm. (3.3.5)], and hence $D \otimes^{\mathbf{L}}_R D$ is not in ${}_D\mathsf{B}(R)$; cf. [8, thm. (3.2)]. Of course, this argument is valid over any non-Gorenstein local ring with a dualizing complex; over the ring in question it is even true that $D \otimes^{\mathbf{L}}_R D$ is not homological bounded, in particular $D \otimes^{\mathbf{L}}_R D$ is not in ${}_D\mathsf{A}(R)$ either. To see this, we assume that $\mathrm{Tor}^R_m(D,D) = 0$ for $m \geq$ some m_0 and use $(*)$ to derive a contradiction. Applying $D \otimes^{\mathbf{L}}_R -$ to $(*)$ we get a long exact sequence of Tor-modules, which shows that $\mathrm{Tor}^R_m(D,k) \cong \mathrm{Tor}^R_{m+1}(D,k^2)$ for $m \geq m_0$. Thus, the Betti numbers of D satisfy the relation $\beta^R_{m_0}(D) = 2^t\beta^R_{m_0+t}(D)$ for $t \geq 0$. But this implies that they must vanish from m_0, in particular D must have finite projective dimension, which is tantamount to R being Gorenstein.

(b) Since R is not Gorenstein, $\mathbf{R}\mathrm{Hom}_R(D,R)$ does not belong to ${}_D\mathsf{A}(R)$ cf. [8, thm. (3.2)] and [16, thm. (3.3.5)]. Again, this argument is valid for any non-Gorenstein local ring with a dualizing complex. For this specific ring the complex $\mathbf{R}\mathrm{Hom}_R(D,R)$ is not even bounded, and in particular not in ${}_D\mathsf{B}(R)$. One can show this by applying $\mathbf{R}\mathrm{Hom}_R(-,R)$ to $(*)$ and then argue as in (a).

These arguments actually show that any finite module in ${}_D\mathsf{A}(R)$ has finite flat dimension, and any finite module in ${}_D\mathsf{B}(R)$ has finite injective dimension. The first part, at least, is well-known as R is Golod and not a hypersurface, cf. [13, exa. 3.5(2)]. However, there is a direct argument that applies to arbitrary modules; it is a non-finite version of [48, prop. 2.4] and equivalent to [37, prop. 6.1(2)]:

(2.3) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be local with $\mathfrak{m}^2 = 0$. If R is not Gorenstein, then any Gorenstein flat R–module is free, and any Gorenstein injective R–module is injective.*

Proof. Let A be Gorenstein flat. Since the maximal ideal of R is nilpotent [20, prop. 3 and 15] gives a projective cover of A, that is, an exact sequence

$$(*) \qquad 0 \to K \to P \to A \to 0,$$

where P is projective and $K \subseteq \mathfrak{m}P$. The module K is a k–vector space, because $\mathfrak{m}^2 = 0$, and Gorenstein flat by exactness of $(*)$. If $K \neq 0$ this implies that k is Gorenstein flat, which contradicts the assumption that R is not Gorenstein. Hence, $K = 0$ and A is isomorphic to P, which is free as R is local, cf. [40, thm. 2].

If B is Gorenstein injective, $\mathrm{Hom}_R(B, \mathrm{E}_R(k))$ is Gorenstein flat by [19, prop. 5.1] and hence flat. Since $\mathrm{E}_R(k)$ is faithful, this implies that B is injective. □

Parts (a) and (c) in the next proposition are relative versions of [19, cor. 5.2 and prop. 5.1]. The proofs presented here require existence of a dualizing complex $\tilde{D}$ for S so that results about Auslander categories from Proposition (2.1) may be applied to $\tilde{C} = \tilde{D}$. We do not know of a proof that does not use dualizing complexes.

(2.4) **Proposition.** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\mathrm{fd}\,\varphi$ finite. If S is noetherian and admits a dualizing complex, then the following hold:*

(a) *If $\tilde{B}$ is Gorenstein injective over S and F is flat over R, then $\tilde{B} \otimes_R F$ is Gorenstein injective over S*

(b) *If P is projective over R and $\tilde{A}$ is Gorenstein flat over S, then $\mathrm{Hom}_R(P, \tilde{A})$ is Gorenstein flat over S*

(c) *If $\tilde{B}$ is Gorenstein injective over S and I is injective over R, then $\mathrm{Hom}_R(B, I)$ is Gorenstein flat over S*

Proof. The three assertions have similar proofs; we only write out part (b): Let $\tilde{D}$ be a dualizing complex for S. The module $\mathrm{Hom}_R(P, \tilde{A})$ represents $\mathbf{R}\mathrm{Hom}_R(P, \tilde{A})$, so $\mathrm{Gfd}_S \mathrm{Hom}_R(P, \tilde{A})$ is finite by Proposition (2.1)(c) and (1.2.1). Actually,

$$\mathrm{Gfd}_S \mathrm{Hom}_R(P, \tilde{A}') \leq d := \mathrm{FFD}(S) < \infty$$

for any Gorenstein flat S–module $\tilde{A}'$ by [19, thm. 3.5] and [32, cor. V.7.2]. Here $\mathrm{FFD}(S)$ is the finitistic flat dimension of S, which is defined as

$$\mathrm{FFD}(S) = \sup\{\mathrm{fd}_S \tilde{M} \mid \tilde{M} \text{ is an } S\text{–module of finite flat dimension}\}.$$

Consider a piece of a complete flat resolution of $\tilde{A}$:

$$0 \to \tilde{A} \to \tilde{F}_0 \to \cdots \to \tilde{F}_{d-1} \to \tilde{A}' \to 0;$$

also $\tilde{A}'$ is Gorenstein flat. Applying the exact functor $\mathrm{Hom}_R(P, -)$ gives an exact sequence of S–modules,

$$0 \to \mathrm{Hom}_R(P, \tilde{A}) \to \mathrm{Hom}_R(P, \tilde{F}_0) \to \cdots \to \mathrm{Hom}_R(P, \tilde{F}_{d-1}) \to \mathrm{Hom}_R(P, \tilde{A}') \to 0.$$

By (0.5) the modules $\mathrm{Hom}_R(P, \tilde{F}_\ell)$ are flat over S, whence $\mathrm{Hom}_R(P, \tilde{A})$ is Gorenstein flat over S by [33, thm. 3.14]. □

(2.5) **Remark.** Proposition (2.4) demonstrates how ascent properties of Auslander categories yield ascent results for Gorenstein dimensions, and we do not know any other way to prove these results. However, information also flows in the opposite direction. The following Lemma (2.6) about Gorenstein dimensions is the corner stone in the proof of the subsequent Theorem (2.8) concerning Auslander categories. Again, we are not aware of any proof of (2.8) that does not use the connection to Gorenstein dimensions.

(2.6) **Lemma.** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\operatorname{fd}\varphi$ finite, then the following hold:*

(a) *If $\tilde{F}$ is flat over S and A is Gorenstein flat over R, then $\tilde{F} \otimes_R A$ is Gorenstein flat over S.*

(b) *If $\tilde{P}$ is projective over S and B is Gorenstein injective over R, then $\operatorname{Hom}_R(\tilde{P}, B)$ is Gorenstein injective over S, provided that $\mathsf{P}(S) \subseteq \mathsf{P}(R)$.*

(c) *If A is Gorenstein flat over R and $\tilde{I}$ is injective over S, then $\operatorname{Hom}_R(A, \tilde{I})$ is Gorenstein injective over S.*

Proof. (a): Let $\boldsymbol{Q}$ be a complete flat resolution of A. The module $\tilde{F}$ has finite flat dimension over R, so the complex $\tilde{F} \otimes_R \boldsymbol{Q}$ of flat S–modules, cf. (0.5), is exact by [19, lem. 2.3]. For any injective S–module $\tilde{J}$ we have exactness of $\tilde{J} \otimes_S (\tilde{F} \otimes_R \boldsymbol{Q}) \cong \tilde{F} \otimes_S (\tilde{J} \otimes_R \boldsymbol{Q})$, as $\tilde{J} \in \mathsf{I}(S) \subseteq \mathsf{I}(R)$, so $\tilde{F} \otimes_R \boldsymbol{Q}$ is a complete flat resolution over S.

(b): Let $\boldsymbol{I}$ be a complete injective resolution of B, then $\operatorname{Hom}_R(\tilde{P}, \boldsymbol{I})$ is a complex of injective S–modules, cf. (0.5), and exact as $\tilde{P} \in \mathsf{P}(S) \subseteq \mathsf{P}(R)$. For any injective S–module $\tilde{J}$, the complex $\operatorname{Hom}_S(\tilde{J}, \operatorname{Hom}_R(\tilde{P}, \boldsymbol{I})) \cong \operatorname{Hom}_S(\tilde{P}, \operatorname{Hom}_R(\tilde{J}, \boldsymbol{I}))$ is exact, as $\tilde{J} \in \mathsf{I}(S) \subseteq \mathsf{I}(R)$.

(c): Let $\boldsymbol{F}$ be a complete flat resolution of A. By (0.5) the complex $\operatorname{Hom}_R(\boldsymbol{F}, \tilde{I})$ consists of injective S–modules. To see that it is exact, write $\operatorname{Hom}_R(\boldsymbol{F}, \tilde{I}) \cong \operatorname{Hom}_S(\boldsymbol{F} \otimes_R S, \tilde{I})$. Exactness then follows as $S \in \mathsf{F}(R)$; cf. [19, lem. 2.3]. For any injective S–module $\tilde{J}$, we have $\operatorname{Hom}_S(\tilde{J}, \operatorname{Hom}_R(\boldsymbol{F}, \tilde{I})) \cong \operatorname{Hom}_S(\tilde{J} \otimes_R \boldsymbol{F}, \tilde{I})$, which is exact as $\tilde{J} \in \mathsf{I}(S) \subseteq \mathsf{I}(R)$. □

(2.7) **Ascent results for complexes.** Just like the classical ascent results for modules, (0.5), the lemma above gives rise to ascent results for complexes, similar to (0.6). For example, if $\varphi\colon R \to S$ is of finite flat dimension, then

(a) $$\mathsf{F}(S) \otimes^{\mathbf{L}}_R \mathsf{GF}(R) \subseteq \mathsf{GF}(S).$$

Indeed, let $\tilde{F}$ and A be bounded complexes of, respectively, flat S–modules and Gorenstein flat R–modules. Since the modules in $\tilde{F}$ have finite flat dimension over R, the bounded complex $\tilde{F} \otimes_R A$ represents $\tilde{F} \otimes^{\mathbf{L}}_R A$ by [19, cor. 2.16]. Lemma (2.6)(a) — and the fact that direct sums of Gorenstein flat modules are Gorenstein flat [33, prop. 3.2] — now shows that $\tilde{F} \otimes_R A$ is a complex of Gorenstein flat S–modules; thus it belongs to $\mathsf{GF}(S)$. We even get a bound on the dimension:

(b) $$\operatorname{Gfd}_S(\tilde{F} \otimes^{\mathbf{L}}_R A) \leq \operatorname{fd}_S \tilde{F} + \operatorname{Gfd}_R A \quad \text{for } \tilde{F} \in \mathsf{F}(S) \text{ and } A \in \mathsf{GF}(R).$$

A series of ascent results for modules — including (2.4) and (2.6) — are summed up in tables (5.5) and (5.6). We do not write out the corresponding results for complexes, let alone the bounds on homological dimensions. Rather, we leave it to the reader to derive further results like (a) and (b) above from the module versions in (5.5) and (5.6). Certain results of this kind have been established under more restrictive conditions in [16, sec. 6.4] and [41, sec. 3].

(2.8) **Theorem.** *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings with $\operatorname{fd}\varphi$ finite. If R and S have dualizing complexes D and $\tilde{D}$, respectively, then the following hold:*

(a) *If $\tilde{F} \in \mathsf{F}(S)$ and $A \in {}_D\mathsf{A}(R)$, then $\tilde{F} \otimes^{\mathbf{L}}_R A \in {}_{\tilde{D}}\mathsf{A}(S)$*

(b) *If $\tilde{P} \in \mathsf{P}(S)$ and $B \in {}_D\mathsf{B}(R)$, then $\mathbf{R}\mathrm{Hom}_R(\tilde{P}, B) \in {}_{\tilde{D}}\mathsf{B}(S)$*

(c) *If $A \in {}_D\mathsf{A}(R)$ and $\tilde{I} \in \mathsf{I}(S)$, then $\mathbf{R}\mathrm{Hom}_R(A, \tilde{I}) \in {}_{\tilde{D}}\mathsf{B}(S)$*

Proof. Recall that in the presence of dualizing complexes, we have ${}_D\mathsf{A}(R) = \mathsf{GF}(R)$ and ${}_{\tilde{D}}\mathsf{A}(S) = \mathsf{GF}(S)$. Part (a) is now a reformulation of (2.7)(a) above. Also (b) and (c) are straightforward consequences of Lemma (2.6). For part (b) note that $\mathsf{P}(S) \subseteq \mathsf{F}(R)$ since $\operatorname{fd}\varphi$ is finite, and furthermore $\mathsf{F}(R) = \mathsf{P}(R)$ as R has a dualizing complex; see e.g. [26, proof of cor. 3.4]. □

We should like to stress that no assumptions are made in Theorem (2.8) about an explicit connection between the dualizing complexes D and $\tilde{D}$. If, on the other hand, $\tilde{D}$ is the base change of D, then it is elementary to verify (2.8) from the definitions of Auslander categories; see e.g. [17, prop. (5.8)(a)].

3. Evaluation morphisms

The main results of this section — Theorems (3.1) and (3.2) — give new sufficient conditions for invertibility of evaluation morphisms. To get a feeling for the nature of these results, compare Theorem (3.1)(a) below to (0.4)(a): The condition on the left-hand complex, X, has been relaxed from finite projective dimension to finite Gorenstein projective dimension, and in return conditions of finite homological dimension have been imposed on the other two complexes.

However special the conditions in (3.1) and (3.2) may seem, the theorems have interesting applications; these are explored in (3.5)–(3.7) below and further in section 4. The proofs of (3.1) and (3.2) are deferred to the end of the section.

(3.1) **Theorem (Hom evaluation).** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\operatorname{fd}\varphi$ finite. For complexes $X \in \mathsf{D}(R)$ and $Y, Z \in \mathsf{D}(S)$ the Hom evaluation morphism*

$$\theta^{RS}_{XYZ} :\ X \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_S(Y, Z) \longrightarrow \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X, Y), Z)$$

is an isomorphism in $\mathsf{D}(S)$, provided that

(a) *$X \in \mathsf{GP}^{\mathrm{f}}(R)$, $Y \in \mathsf{P}(S)$, $Z \in \mathsf{F}(S)$, and R is noetherian.*

For complexes $U, V \in \mathsf{D}(S)$ and $W \in \mathsf{D}(R)$ the morphism

$$\theta^{SR}_{UVW} :\ U \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(V, W) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(U, V), W)$$

is an isomorphism in $\mathsf{D}(S)$ provided that

(b) *$U \in \mathsf{I}^{\mathrm{f}}(S)$, $V \in \mathsf{I}(S)$, $W \in \mathsf{GI}(R)$, S is noetherian, and $\mathsf{F}(S) \subseteq \mathsf{P}(R)$*[1].

---

[1]In Theorems (3.1) and (3.2) we encounter two requirements:

$$\mathsf{P}(S) \subseteq \mathsf{P}(R) \quad \text{and} \quad \mathsf{F}(S) \subseteq \mathsf{P}(R);$$

(3.2) **Theorem (Tensor evaluation).** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\operatorname{fd}\varphi$ finite. For complexes $U, V \in \mathsf{D}(S)$ and $W \in \mathsf{D}(R)$ the tensor evaluation morphism*

$$\omega_{UVW}^{SR} : \mathbf{R}\mathrm{Hom}_S(U,V) \otimes_R^{\mathbf{L}} W \longrightarrow \mathbf{R}\mathrm{Hom}_S(U, V \otimes_R^{\mathbf{L}} W)$$

is an isomorphism in $\mathsf{D}(S)$ if either:

(a) *$U \in \mathsf{I}^{\mathrm{f}}(S)$, $V \in \mathsf{I}(S)$, $W \in \mathsf{GF}(R)$, and S is noetherian; or*

(b) *$U \in \mathsf{I}(S)$, $V \in \mathsf{I}(S)$, $W \in \mathsf{GF}^{\mathrm{f}}(R)$, and R is noetherian.*

For complexes $X \in \mathsf{D}(R)$ and $Y, Z \in \mathsf{D}(S)$ the morphism

$$\omega_{XYZ}^{RS} : \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes_S^{\mathbf{L}} Z \longrightarrow \mathbf{R}\mathrm{Hom}_R(X, Y \otimes_S^{\mathbf{L}} Z)$$

is an isomorphism in $\mathsf{D}(S)$ if:

(c) *$X \in \mathsf{GP}^{\mathrm{f}}(R)$, $Y \in \mathsf{F}(S)$, $Z \in \mathsf{I}(S)$, and R is noetherian;*

(d) *$X \in \mathsf{GP}(R)$, $Y \in \mathsf{F}(S)$, $Z \in \mathsf{I}^{\mathrm{f}}(S)$, S is noetherian, and $\mathsf{F}(S) \subseteq \mathsf{P}(R)^1$; or*

(d') *$X \in \mathsf{GP}(R)$, $Y \in \mathsf{P}(S)$, $Z \in \mathsf{I}^{\mathrm{f}}(S)$, S is noetherian, and $\mathsf{P}(S) \subseteq \mathsf{P}(R)^1$.*

(3.3) **Observation.** Let R be noetherian and G be a finite Gorenstein projective R–module; let $\tilde{F}$ be a flat S–module and $\tilde{I}$ an injective S–module. Then $\mathbf{R}\mathrm{Hom}_R(G,\tilde{F}) \otimes_S^{\mathbf{L}} \tilde{I}$ is represented by $\mathrm{Hom}_R(G,\tilde{F}) \otimes_S \tilde{I}$ and $\mathbf{R}\mathrm{Hom}_R(G,\tilde{F} \otimes_S^{\mathbf{L}} \tilde{I})$ by $\mathrm{Hom}_R(G,\tilde{F} \otimes_S \tilde{I})$; thus (3.2)(d) implies that

$$\omega_{G\tilde{F}\tilde{I}}^{RS} : \mathrm{Hom}_R(G,\tilde{F}) \otimes_S \tilde{I} \longrightarrow \mathrm{Hom}_R(G,\tilde{F} \otimes_S \tilde{I})$$

is an isomorphism of S–modules. Similar remarks apply to the other parts of Theorems (3.1) and (3.2).

Now we turn to applications of Theorems (3.1) and (3.2).

(3.4) **Remarks.** The connection between Auslander categories and Gorenstein dimensions, as captured by (1.2.1), allows transfer of information between these two realms. This is the theme of section 2 and, consequently, each result in that section is phrased as either a statement about Auslander categories or a statement about Gorenstein dimensions. Thus, the hybrid statements in (3.5) below call for a comment:

---

in fact we already met the former in Lemma (2.6). In the absolute case, the first one is void while the second says that flat modules have finite projective dimension.

It is clear that the first is weaker than the second and tantamount to $\operatorname{pd}\varphi$ being finite. By [39, prop. 6], the second requirement is satisfied when:

(1) $\operatorname{fd}\varphi$ is finite and $\mathrm{FPD}(R)$ is finite; or

(2) $\operatorname{pd}\varphi$ is finite and $\mathrm{FPD}(S)$ is finite.

Here $\mathrm{FPD}(R)$ is the finitistic projective dimension of R, which is defined as

$$\mathrm{FPD}(R) = \sup\{\operatorname{pd}_R M \mid M \text{ is an } R\text{–module of finite projective dimension}\}.$$

Recall that over a noetherian ring, the finitistic projective dimension, FPD, is equal to the Krull dimension; see [14, cor. 5.5] and [45, thm. II.3.2.6].

The conditions (1) and (2) are actually independent: Let Q denote Nagata's noetherian regular ring of infinite Krull dimension [43, example 1, p. 203] and consider the natural inclusion $k \hookrightarrow Q$, where k is the field over which Q is built. Clearly, $k \hookrightarrow Q$ satisfies (1) but not (2). Now let $\mathfrak{m}$ be any maximal ideal of Q and consider the projection $Q \twoheadrightarrow Q/\mathfrak{m}$. Since Q is regular, every finite Q–module has finite projective dimension, cf. [29, cor. 3], ergo $Q \twoheadrightarrow Q/\mathfrak{m}$ satisfies (2) but not (1).

Recall that behind the results in Theorem (2.8), e.g. part (a):

$$\mathsf{F}(S) \otimes^{\mathbf{L}}_R {}_D\mathsf{A}(R) \subseteq {}_{\tilde{D}}\mathsf{A}(S),$$

lie stronger inclusions derived from Lemma (2.6); in this case (2.7)(a):

$$\mathsf{F}(S) \otimes^{\mathbf{L}}_R \mathsf{GF}(R) \subseteq \mathsf{GF}(S).$$

Thus, Theorem (2.8)(a) could have been phrased as a hybrid,

$$(\star) \qquad \mathsf{F}(S) \otimes^{\mathbf{L}}_R \mathsf{GF}(R) \subseteq {}_{\tilde{D}}\mathsf{A}(S),$$

and in that form it does not require a dualizing complex for R.

In view of (0.6) it is natural to seek a result like:

$$(?) \qquad \mathsf{I}(S) \otimes^{\mathbf{L}}_R \mathsf{GF}(R) \subseteq \mathsf{GI}(S) \quad (S \text{ noetherian}).$$

We do not know if (?) holds in general, but through an application of Theorem (3.2) we obtain the weaker hybrid (3.5)(a):

$$\mathsf{I}(S) \otimes^{\mathbf{L}}_R \mathsf{GF}(R) \subseteq {}_{\tilde{D}}\mathsf{B}(S) \quad (S \text{ noetherian with a dualizing complex}).$$

Embedded herein are results that can be stated purely in terms of Auslander categories or Gorenstein dimensions; we write them out in Corollaries (3.6) and (3.7).

Note that also $(\star)$, which in the discussion above appears a consequence of (1.2.1) and (2.7)(a), can be derived easily from Theorem (3.2): Let $\tilde{F} \in \mathsf{F}(S)$ and $A \in \mathsf{GF}(R)$. By [17, prop. (4.4)] the unit $\eta_{\tilde{F}}$ is an isomorphism, and by (0.6) the complex $\tilde{D} \otimes^{\mathbf{L}}_S \tilde{F}$ belongs to $\mathsf{I}(S)$. Now (3.2)(a) and the commutative diagram below shows that $\eta_{\tilde{F}\otimes^{\mathbf{L}}_R A}$ is invertible.

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_S(\tilde{D}, \tilde{D} \otimes^{\mathbf{L}}_S (\tilde{F} \otimes^{\mathbf{L}}_R A)) & \xleftarrow{\eta_{\tilde{F}\otimes^{\mathbf{L}}_R A}} & \tilde{F} \otimes^{\mathbf{L}}_R A \\
\text{associativity} \downarrow \simeq & & \simeq \downarrow \eta_{\tilde{F}} \otimes^{\mathbf{L}}_R A \\
\mathbf{R}\mathrm{Hom}_S(\tilde{D}, (\tilde{D} \otimes^{\mathbf{L}}_S \tilde{F}) \otimes^{\mathbf{L}}_R A) & \xleftarrow[\omega^{SR}_{\tilde{D}(\tilde{D}\otimes^{\mathbf{L}}_S \tilde{F})A}]{\simeq} & \mathbf{R}\mathrm{Hom}_S(\tilde{D}, \tilde{D} \otimes^{\mathbf{L}}_S \tilde{F}) \otimes^{\mathbf{L}}_R A
\end{array}$$

We emphasize that (1.2.1) plays no part in the proof of Theorem (3.2).

(3.5) **Proposition.** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\operatorname{fd}\varphi$ finite, and assume that S is noetherian with a dualizing complex $\tilde{D}$. The following hold:*

(a) *If $\tilde{I} \in \mathsf{I}(S)$ and $A \in \mathsf{GF}(R)$ then $\tilde{I} \otimes^{\mathbf{L}}_R A \in {}_{\tilde{D}}\mathsf{B}(S)$*

If we also assume that $\mathsf{F}(S) \subseteq \mathsf{P}(R)$, then the following hold:

(b) *If $A \in \mathsf{GP}(R)$ and $\tilde{F} \in \mathsf{F}(S)$ then $\mathbf{R}\mathrm{Hom}_R(A, \tilde{F}) \in {}_{\tilde{D}}\mathsf{A}(S)$*

(c) *If $\tilde{I} \in \mathsf{I}(S)$ and $B \in \mathsf{GI}(R)$ then $\mathbf{R}\mathrm{Hom}_R(\tilde{I}, B) \in {}_{\tilde{D}}\mathsf{A}(S)$*

Proof. (c): Under the assumptions we have $\tilde{I} \in \mathsf{I}(R)$, so $\mathbf{R}\mathrm{Hom}_R(\tilde{I}, B)$ is bounded by [19, cor. 2.12]. From Theorem (3.1)(b) we get an isomorphism,

$$\tilde{D} \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(\tilde{I}, B) \xrightarrow[\simeq]{\theta^{SR}_{\tilde{D}\tilde{I}B}} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(\tilde{D}, \tilde{I}), B),$$

where the right-hand side is bounded by [19, cor. 2.12], as

$$\mathbf{R}\mathrm{Hom}_S(\tilde{D}, \tilde{I}) \in \mathsf{F}(S) \subseteq \mathsf{P}(R).$$

by (0.6) and the assumptions. Finally, the commutative diagram,

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(\tilde{I},B) & \xrightarrow{\eta_{\mathbf{R}\mathrm{Hom}_R(\tilde{I},B)}} & \mathbf{R}\mathrm{Hom}_S(\tilde{D},\tilde{D}\otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(\tilde{I},B)) \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\varepsilon_{\tilde{I}},B)}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathbf{R}\mathrm{Hom}_S(\tilde{D},\theta^{SR}_{\tilde{D}\tilde{I}B})} \\
\mathbf{R}\mathrm{Hom}_R(\tilde{D}\otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_S(\tilde{D},\tilde{I}),B) & \underset{\text{adj.}}{\xrightarrow{\simeq}} & \mathbf{R}\mathrm{Hom}_S(\tilde{D},\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(\tilde{D},\tilde{I}),B))
\end{array}$$

shows that the unit η on $\mathbf{R}\mathrm{Hom}_R(\tilde{I},B)$ is an isomorphism in $\mathsf{D}(S)$, as desired.

The proofs of the first two parts are similar; part (a) relies on Theorem (3.2)(a) and part (b) on (3.2)(d). ☐

(3.6) **Corollary.** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\mathrm{fd}\,\varphi$ finite, and assume that R and S are noetherian with dualizing complexes D and $\tilde{D}$, respectively. The following hold:*

(a) *If $\tilde{I} \in \mathsf{I}(S)$ and $A \in {}_D\mathsf{A}(R)$ then $\tilde{I}\otimes^{\mathbf{L}}_R A \in {}_{\tilde{D}}\mathsf{B}(S)$*

(b) *If $A \in {}_D\mathsf{A}(R)$ and $\tilde{F} \in \mathsf{F}(S)$ then $\mathbf{R}\mathrm{Hom}_R(A,\tilde{F}) \in {}_{\tilde{D}}\mathsf{A}(S)$*

(c) *If $\tilde{I} \in \mathsf{I}(S)$ and $B \in {}_D\mathsf{B}(R)$ then $\mathbf{R}\mathrm{Hom}_R(\tilde{I},B) \in {}_{\tilde{D}}\mathsf{A}(S)$*

Proof. Since R is noetherian and has a dualizing complex, we have $\mathsf{F}(R) = \mathsf{P}(R)$ by [26, proof of cor. 3.4] and hence the condition $\mathsf{F}(S) \subseteq \mathsf{P}(R)$ is satisfied. The assertions now follow from Proposition (3.5) in view of (1.2.1). ☐

As a second corollary of (3.5) we get ascent results for Gorenstein dimensions:

(3.7) **Corollary.** *Let $\varphi\colon R \to S$ be a homomorphism of rings with $\mathrm{fd}\,\varphi$ finite. If S is noetherian and has a dualizing complex, then the following hold:*

(a) *If $\tilde{I}$ is injective over S and A is Gorenstein flat over R, then $\tilde{I}\otimes_R A$ is Gorenstein injective over S*

If we also assume that $\mathsf{F}(S) \subseteq \mathsf{P}(R)$, then the following hold:

(b) *If A is Gorenstein projective over R and $\tilde{F}$ is flat over S, then $\mathrm{Hom}_R(A,\tilde{F})$ is Gorenstein flat over S*

(c) *If $\tilde{I}$ is injective over S and B is Gorenstein injective over R, then $\mathrm{Hom}_R(\tilde{I},B)$ is Gorenstein flat over S*

Proof. The three parts have similar proofs; we write out part (b): By the assumptions $\tilde{F} \in \mathsf{P}(R)$. For any Gorenstein projective R–module A', $\mathbf{R}\mathrm{Hom}_R(A',\tilde{F})$ is isomorphic to $\mathrm{Hom}_R(A',\tilde{F})$ by [19, cor. 2.10], so this module has finite Gorenstein flat dimension over S by Proposition (3.5)(b) and (1.2.1). Consequently,

$$\mathrm{Gfd}_S\,\mathrm{Hom}_R(A',\tilde{F}) \leq d := \mathrm{FFD}(S) < \infty,$$

where the first inequality is by [19, thm. 3.5] and the second follows from [32, cor. V.7.2]. Now consider a piece of a complete projective resolution of A:

$$0 \to A' \to P_{d-1} \to \cdots \to P_0 \to A \to 0.$$

Also A' is Gorenstein projective, and the functor $\mathrm{Hom}_R(-,\tilde{F})$ leaves this sequence exact, as $\mathrm{pd}_R\,\tilde{F}$ is finite. In the ensuing sequence,

$$0 \to \mathrm{Hom}_R(A,\tilde{F}) \to \mathrm{Hom}_R(P_0,\tilde{F}) \to \cdots \to \mathrm{Hom}_R(P_{d-1},\tilde{F}) \to \mathrm{Hom}_R(A',\tilde{F}) \to 0,$$

the S–modules $\operatorname{Hom}_R(P_\ell,\tilde{F})$ are flat, cf. (0.5), and it follows that $\operatorname{Hom}_R(A,\tilde{F})$ is Gorenstein flat over S by [33, cor. 3.14]. □

Proof of Theorem (3.1). (a): Choose resolutions

$$\mathsf{C}^{\mathrm{P}}_{\sqsupset}(S) \ni \tilde{P} \xrightarrow{\simeq} Y \quad \text{and} \quad Z \xrightarrow{\simeq} \tilde{J} \in \mathsf{C}^{\mathrm{I}}_{\sqsubset}(S)$$

and a complex $\mathsf{C}^{\mathrm{F}}_{\square}(S) \ni \tilde{F} \simeq Z$. Since R is noetherian, we can choose a bounded complex $G \simeq X$ of finite Gorenstein projective R–modules. Further, because $\operatorname{fd}\varphi$ is finite, the modules in the bounded complex $\operatorname{Hom}_S(\tilde{P},\tilde{F})$ are of finite flat dimension over R by Lemma (0.9). The modules in G are Gorenstein flat by [16, thm. (4.2.6) and (5.1.11)], so the complex $G \otimes_R \operatorname{Hom}_S(\tilde{P},\tilde{F})$ represents $X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_S(Y,Z)$ by [19, cor. 2.16]. As $\tilde{F} \simeq \tilde{J}$ there is a quasi-isomorphism $\tilde{F} \xrightarrow{\simeq} \tilde{J}$, cf. [6, 1.4.I], which by [19, thm. 2.15] induces another quasi-isomorphism,

$$(*) \qquad G \otimes_R \operatorname{Hom}_S(\tilde{P},\tilde{F}) \xrightarrow{\simeq} G \otimes_R \operatorname{Hom}_S(\tilde{P},\tilde{J}).$$

Here we use that the modules in $\operatorname{Hom}_S(\tilde{P},\tilde{J})$ have finite injective dimension over R. Thus, also the right-hand side in $(*)$ represents $X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_S(Y,Z)$; furthermore the complex $\operatorname{Hom}_S(\operatorname{Hom}_R(G,\tilde{P}),\tilde{J})$ represents $\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_R(X,Y),Z)$ by [19, cor. 2.10 and rmk. 2.11], whence θ^{RS}_{XYZ} in $\mathsf{D}(S)$ is represented by

$$G \otimes_R \operatorname{Hom}_S(\tilde{P},\tilde{J}) \xrightarrow{\theta^{RS}_{G\tilde{P}\tilde{J}}} \operatorname{Hom}_S(\operatorname{Hom}_R(G,\tilde{P}),\tilde{J}),$$

which is an isomorphism by (0.3)(b).

(b): Using that S is noetherian, we choose resolutions

$$\mathsf{C}^{\mathrm{fP}}_{\sqsupset}(S) \ni \tilde{L} \xrightarrow{\simeq} U \xrightarrow{\simeq} \tilde{J} \in \mathsf{C}^{\mathrm{I}}_{\square}(S) \quad \text{and} \quad V \xrightarrow{\simeq} \tilde{I} \in \mathsf{C}^{\mathrm{I}}_{\square}(S).$$

We also choose a bounded complex $B \simeq W$ of Gorenstein injective R–modules. The complex $\operatorname{Hom}_R(\operatorname{Hom}_S(\tilde{J},\tilde{I}),B)$ represents $\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(U,V),W)$ by [19, cor. 2.12], as all the modules in $\operatorname{Hom}_S(\tilde{J},\tilde{I})$ have finite projective dimension over R. The last claim relies on (0.5) and the assumption $\mathsf{F}(S) \subseteq \mathsf{P}(R)$. By [19, thm. 2.9(b)] the composite $\tilde{L} \xrightarrow{\simeq} \tilde{J}$ induces a quasi-isomorphism,

$$\operatorname{Hom}_R(\operatorname{Hom}_S(\tilde{L},\tilde{I}),B) \xrightarrow{\simeq} \operatorname{Hom}_R(\operatorname{Hom}_S(\tilde{J},\tilde{I}),B),$$

so also the left-hand side represents $\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(U,V),W)$. Furthermore, $\tilde{L} \otimes_S \operatorname{Hom}_R(\tilde{I},B)$ represents $U \otimes^{\mathbf{L}}_S \mathbf{R}\operatorname{Hom}_R(V,W)$ by [19, cor. 2.12], and the morphism θ^{SR}_{UVW} in $\mathsf{D}(S)$ is represented by

$$\tilde{L} \otimes_S \operatorname{Hom}_R(\tilde{I},B) \xrightarrow{\theta^{SR}_{\tilde{L}\tilde{I}B}} \operatorname{Hom}_R(\operatorname{Hom}_S(\tilde{L},\tilde{I}),B),$$

which is an isomorphism by (0.3)(a). □

Proof of Theorem (3.2). (a) and (b): Choose resolutions

$$\mathsf{C}^{\mathrm{P}}_{\sqsupset}(S) \ni \tilde{P} \xrightarrow{\simeq} U \xrightarrow{\simeq} \tilde{J} \in \mathsf{C}^{\mathrm{I}}_{\square}(S) \quad \text{and} \quad V \xrightarrow{\simeq} \tilde{I} \in \mathsf{C}^{\mathrm{I}}_{\square}(S)$$

and a bounded complex $A \simeq W$ of Gorenstein flat R–modules. As $\operatorname{fd}\varphi$ is finite and either S or R is noetherian, the bounded complex $\operatorname{Hom}_S(\tilde{J},\tilde{I})$ consists of modules of finite flat dimension over R; cf. (0.5) and (0.9). Therefore, the complex $\operatorname{Hom}_S(\tilde{J},\tilde{I}) \otimes_R A$ represents $\mathbf{R}\operatorname{Hom}_S(U,V) \otimes^{\mathbf{L}}_R W$ by [19, cor. 2.16]. The composite $\tilde{P} \xrightarrow{\simeq} \tilde{J}$ induces a quasi-isomorphism $\operatorname{Hom}_S(\tilde{J},\tilde{I}) \xrightarrow{\simeq} \operatorname{Hom}_S(\tilde{P},\tilde{I})$ between left-bounded complexes of flat and injective S–modules. By (0.8) these modules have finite flat or finite injective dimension over R, so by [19, thm. 2.15(b)] there is a quasi-isomorphism

$$\operatorname{Hom}_S(\tilde{J},\tilde{I})\otimes_R A \xrightarrow{\simeq} \operatorname{Hom}_S(\tilde{P},\tilde{I})\otimes_R A;$$

in particular, also the latter complex represents $\mathbf{R}\operatorname{Hom}_S(U,V)\otimes^{\mathbf{L}}_R W$. Furthermore, $\tilde{I}\otimes_R A$ represents $V\otimes^{\mathbf{L}}_R W$, again by [19, cor. 2.16], so $\operatorname{Hom}_S(\tilde{P},\tilde{I}\otimes_R A)$ represents $\mathbf{R}\operatorname{Hom}_S(U,V\otimes^{\mathbf{L}}_R W)$, and the morphism ω^{SR}_{UVW} in $\mathsf{D}(S)$ is represented by

$$\operatorname{Hom}_S(\tilde{P},\tilde{I})\otimes_R A \xrightarrow{\omega^{SR}_{\tilde{P}\tilde{I}A}} \operatorname{Hom}_S(\tilde{P},\tilde{I}\otimes_R A).$$

If $U\in\mathsf{I}^{\mathrm{f}}(S)$ and S is noetherian, we may assume that $\tilde{P}\in\mathsf{C}^{\mathrm{fP}}_{\sqsupset}(S)$, and $\omega^{SR}_{\tilde{P}\tilde{I}A}$ is then invertible by (0.3)(c). Similarly, if R is noetherian and $W\in\mathsf{GF}^{\mathrm{f}}(R)$, we may assume that all modules in A are finite, and then $\omega^{SR}_{\tilde{P}\tilde{I}A}$ is invertible by (0.3)(f).

(c) and (d): Choose complexes

$$\mathsf{C}^{\mathrm{F}}_{\square}(S)\ni\tilde{F}\simeq Y \quad\text{and}\quad \mathsf{C}^{\mathrm{P}}_{\sqsupset}(S)\ni\tilde{Q}\xrightarrow{\simeq} Z\xrightarrow{\simeq}\tilde{J}\in\mathsf{C}^{\mathrm{I}}_{\square}(S)$$

and a bounded complex $A\simeq X$ of Gorenstein projective R–modules. In the case of (d) we may assume $\tilde{Q}\in\mathsf{C}^{\mathrm{fP}}_{\sqsupset}(S)$ as S is noetherian; and in the case of (c) we may assume that the modules in A are finite as R is noetherian. We claim that

(1) the complex $\operatorname{Hom}_R(A,\tilde{F})\otimes_S\tilde{Q}$ represents $\mathbf{R}\operatorname{Hom}_R(X,Y)\otimes^{\mathbf{L}}_S Z$; and
(2) the complex $\operatorname{Hom}_R(A,\tilde{F}\otimes_S\tilde{Q})$ represents $\mathbf{R}\operatorname{Hom}_R(X,Y\otimes^{\mathbf{L}}_S Z)$.

The modules in $\tilde{F}$ have finite flat dimension over R by (0.8), and if $\mathsf{F}(S)\subseteq\mathsf{P}(R)$ they even have finite projective dimension over R. Hence (1) follows from [19, cor. 2.10 and rmk. 2.11]. To prove (2) we note that as either S or R is noetherian, (0.5) or (0.9) implies that all modules in the bounded complex $\tilde{F}\otimes_S\tilde{J}$ have finite injective dimension over R. Therefore [19, cor. 2.10] gives that the complex $\operatorname{Hom}_R(A,\tilde{F}\otimes_S\tilde{J})$ represents $\mathbf{R}\operatorname{Hom}_R(X,Y\otimes^{\mathbf{L}}_S Z)$. The composite quasi-isomorphism $\tilde{Q}\xrightarrow{\simeq}\tilde{J}$ induces a quasi-isomorphism $\tilde{F}\otimes_S\tilde{Q}\xrightarrow{\simeq}\tilde{F}\otimes_S\tilde{J}$ between right-bounded complexes. The modules in $\tilde{F}\otimes_S\tilde{Q}$ have finite flat dimension over R according to (0.8), and if $\mathsf{F}(S)\subseteq\mathsf{P}(R)$ they even have finite projective dimension over R. By [19, thm. 2.8(b) and rmk. 2.11] this quasi-isomorphism is preserved by $\operatorname{Hom}_R(A,-)$.

In total, this shows that the morphism ω^{RS}_{XYZ} in $\mathsf{D}(S)$ is represented by

$$\operatorname{Hom}_R(A,\tilde{F})\otimes_S\tilde{Q}\xrightarrow{\omega^{RS}_{A\tilde{F}\tilde{Q}}}\operatorname{Hom}_R(A,\tilde{F}\otimes_S\tilde{Q}).$$

If $\tilde{Q}\in\mathsf{C}^{\mathrm{fP}}_{\sqsupset}(S)$ then $\omega^{RS}_{A\tilde{F}\tilde{Q}}$ is then invertible by (0.3)(e); this proves (d). If all the modules in A are finite then $\omega^{RS}_{A\tilde{F}\tilde{Q}}$ is an isomorphism by (0.3)(d); this proves (c).

The proof of (d') is similar to the proof of (d) and thus omitted. □

4. Auslander–Buchsbaum formulas

We now turn attention to formulas of the Auslander–Buchsbaum type. As in the previous sections, we consider a relative situation. That is, when X is an R–complex and Y an S–complex, we relate the S–depth of $X\otimes^{\mathbf{L}}_R Y$ to the depths of X and Y over R and S, respectively.

(4.1) **Depth and width.** The invariants depth and width for complexes over a local ring $(S,\mathfrak{n},l)$ can be computed in a number of different ways as demonstrated in [27]. Here we shall only need two of them. Let $K=\mathrm{K}_S[x_1,\ldots,x_e]$ be the Koszul complex on a set of generators $x_1, \dots, x_e$ for $\mathfrak{n}$. For an S–complex Y the *depth* and *width* are given by:

$$\begin{aligned} \operatorname{depth}_S Y &= -\sup \mathbf{R}\mathrm{Hom}_S(l, Y) \\ &= -\sup \mathbf{R}\mathrm{Hom}_S(K, Y) \\ &= e - \sup (K \otimes_S Y), \text{ and} \end{aligned} \tag{4.1.1}$$

$$\operatorname{width}_S Y = \inf (l \otimes^{\mathbf{L}}_S Y) = \inf (K \otimes_S Y). \tag{4.1.2}$$

It is easy to prove the inequalities:

$$\operatorname{depth}_S Y \geq -\sup Y \quad \text{and} \quad \operatorname{width}_S Y \geq \inf Y. \tag{4.1.3}$$

If $M \neq 0$ is an $\mathfrak{n}$–torsion module, that is, $M = \bigcup_{n=1}^{\infty} (0 : \mathfrak{n}^n)_M$, then $\mathrm{Hom}_S(l, M)$ is non-zero. This has the following consequence:

(4.1.4) If $\mathrm{H}(Y)$ is degree-wise $\mathfrak{n}$–torsion, then $\operatorname{depth}_S Y = -\sup Y$.

For a much stronger statement see [27, lem. 2.8].

(4.2) **Remark.** If $(R, \mathfrak{m}, k)$ is local and $X \in \mathsf{D}^{\mathrm{f}}_{\Box}(R)$, then

$$\operatorname{pd}_R X = \operatorname{fd}_R X = \sup (k \otimes^{\mathbf{L}}_R X) = -\operatorname{depth}_R(k \otimes^{\mathbf{L}}_R X), \tag{$*$}$$

where the last equality follows from (4.1.4), as all the modules in $\mathrm{H}(k \otimes^{\mathbf{L}}_R X)$ are annihilated by $\mathfrak{m}$. The classical Auslander–Buchsbaum formula states that if this number $(*)$ is finite, then it equals

$$\operatorname{depth} R - \operatorname{depth}_R X = -(\operatorname{depth}_R k + \operatorname{depth}_R X - \operatorname{depth} R).$$

Thus one recovers the classical Auslander–Buchsbaum formula by setting $Y = k$ and $\varphi = 1_R$ in Theorem (4.3)(a). This illustrates the point of view that (4.3)(a) is the Auslander–Buchsbaum formula for X with coefficients in Y (or vice versa).

Given any local homomorphism φ, Theorem (4.3) gives Auslander–Buchsbaum type formulas for a module X of finite classical homological dimension (fd, pd, or id) with coefficients in an arbitrary module Y. In Theorem (4.4) we relax the conditions on X and, in return, impose conditions on Y and φ to obtain similar formulas for modules of finite Gorenstein homological dimension.

(4.3) **Theorem.** *Let $\varphi\colon R \to S$ be a local homomorphism of rings. Then the following hold:*

(a) *Let $Y \in \mathsf{D}_{\sqsubset}(S)$ and $X \in \mathsf{D}_{\sqsubset}(R)$. If $Y \in \mathsf{F}(R)$ or $X \in \mathsf{F}(R)$ then*
$$\operatorname{depth}_S(Y \otimes^{\mathbf{L}}_R X) = \operatorname{depth}_S Y + \operatorname{depth}_R X - \operatorname{depth} R.$$

(b) *Let $X \in \mathsf{D}_{\sqsubset}(R)$ and $Y \in \mathsf{D}_{\sqsupset}(S)$. If $X \in \mathsf{P}(R)$ or $Y \in \mathsf{I}(R)$ then*
$$\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(X, Y) = \operatorname{depth}_R X + \operatorname{width}_S Y - \operatorname{depth} R.$$

(c) *Let $Y \in \mathsf{D}_{\sqsubset}(S)$ and $X \in \mathsf{D}_{\sqsupset}(R)$. If $Y \in \mathsf{P}(R)$ or $X \in \mathsf{I}(R)$ then*
$$\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(Y, X) = \operatorname{depth}_S Y + \operatorname{width}_R X - \operatorname{depth} R.$$

The absolute version already exists in [36, thm. 2.1], [25, (12.8) and (12.20)], and [18, thm. (4.14)]. This relative version is joint work between Srikanth Iyengar and Lars Winther Christensen. The proof is deferred to the end of this section.

The isomorphisms from (3.1) and (3.2) propel the main result of this section:

(4.4) **Theorem.** *Let $\varphi\colon R \to S$ be a local homomorphism with $\operatorname{fd}\varphi$ finite. The following hold:*

(a) *For $Y \in \mathsf{I}(S)$ and $X \in \mathsf{GF}(R)$ there is an equality:*

$$\operatorname{depth}_S(Y \otimes^{\mathbf{L}}_R X) = \operatorname{depth}_S Y + \operatorname{depth}_R X - \operatorname{depth} R.$$

(b) *For $X \in \mathsf{GP}(R)$ and $Y \in \mathsf{P}(S)$ there is an equality:*

$$\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(X,Y) = \operatorname{depth}_R X + \operatorname{width}_S Y - \operatorname{depth} R.$$

(c) *For $Y \in \mathsf{I}(S)$ and $X \in \mathsf{GI}(R)$ there is an equality:*

$$\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(Y,X) = \operatorname{depth}_S Y + \operatorname{width}_R X - \operatorname{depth} R.$$

(4.5) **Remark.** Comparison of (4.3)(a) to (4.4)(a) raises two questions:

- Does (4.3)(a) hold if one assumes that $Y \in \mathsf{F}(S)$ instead of $Y \in \mathsf{F}(R)$?
- Does (4.4)(a) hold without the assumption $Y \in \mathsf{I}(S)$?

The answer to the first question is negative: Let $(R,\mathfrak{m},k)$ be non-regular and φ be the canonical projection $R \twoheadrightarrow k$. Then $k \in \mathsf{F}(k)$ but $k \notin \mathsf{F}(R)$, and

$$\operatorname{depth}_k(k \otimes^{\mathbf{L}}_R k) = -\sup \mathbf{R}\mathrm{Hom}_k(k, k \otimes^{\mathbf{L}}_R k) = -\sup(k \otimes^{\mathbf{L}}_R k) = -\infty,$$

while $\operatorname{depth}_k k + \operatorname{depth}_R k - \operatorname{depth} R = -\operatorname{depth} R$.

Also the second question has a negative answer: Let $S = R$ be Gorenstein but not regular (and φ be the identity map). Then $k \in \mathsf{GF}(R)$ but $k \notin \mathsf{I}(R)$, and

$$\operatorname{depth}_R(k \otimes^{\mathbf{L}}_R k) = -\sup\,(k \otimes^{\mathbf{L}}_R k) = -\infty,$$

by (4.1.4), while $\operatorname{depth}_R k + \operatorname{depth}_R k - \operatorname{depth} R = -\operatorname{depth} R$.

Proof of Theorem (4.4). (a): We let $\mathfrak{n}$ denote the unique maximal ideal of S and l be the residue field. Furthermore, let K be the Koszul complex on a set of generators for $\mathfrak{n}$. In the following sequence of isomorphisms, the first and last are by Hom evaluation (0.4)(a), and the middle one is by Theorem (3.2)(a). This theorem applies because the complex $\mathrm{Hom}_S(K,\mathrm{E}_S(l))$ has finite injective dimension and finite (length) homology modules:

$$\begin{aligned} K \otimes^{\mathbf{L}}_S (\mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l),Y) \otimes^{\mathbf{L}}_R X) &\simeq \mathbf{R}\mathrm{Hom}_S(\mathrm{Hom}_S(K,\mathrm{E}_S(l)),Y) \otimes^{\mathbf{L}}_R X \\ &\simeq \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(K,\mathrm{E}_S(l)), Y \otimes^{\mathbf{L}}_R X) \\ &\simeq K \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l), Y \otimes^{\mathbf{L}}_R X) \end{aligned}$$

Because K is depth sensitive, cf. (4.1.1), this isomorphism implies an equality:

$$(*) \qquad \operatorname{depth}_S(\mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l),Y) \otimes^{\mathbf{L}}_R X) \;=\; \operatorname{depth}_S \mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l), Y \otimes^{\mathbf{L}}_R X).$$

The complex $\mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l),Y)$ is in $\mathsf{F}(S)$, and hence also in $\mathsf{F}(R)$ as $\operatorname{fd}\varphi$ is finite. Therefore the left-hand side of $(*)$ is equal to:

$$\begin{aligned} &\operatorname{depth}_S \mathbf{R}\mathrm{Hom}_S(\mathrm{E}_S(l),Y) + \operatorname{depth}_R X - \operatorname{depth} R \;= \\ &\operatorname{width}_S \mathrm{E}_S(l) + \operatorname{depth}_S Y + \operatorname{depth}_R X - \operatorname{depth} R \end{aligned}$$

by Theorem (4.3)(a) and [16, lem. (A.6.4)]. By [27, prop. 4.6] the right-hand side of $(*)$ equals

$$\operatorname{width}_S \mathrm{E}_S(l) + \operatorname{depth}_S(Y \otimes^{\mathbf{L}}_R X),$$

and the desired formula follows.

Similar arguments establish parts (b) and (c): Part (b) uses Theorem (3.2)(d') and (4.3)(b), while part (c) relies on Theorem (3.1)(b) and (4.3)(c). □

(4.6) **Corollary.** *Let* $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ *be a local homomorphism with* $\operatorname{fd}\varphi$ *finite. The following hold:*

(a) *If* $X\in\mathsf{GF}(R)$ *then* $\sup(\mathrm{E}_S(l)\otimes^{\mathbf{L}}_R X)=\operatorname{depth}R-\operatorname{depth}_R X$.

(b) *If* $X\in\mathsf{GP}(R)$ *then* $-\inf\mathbf{R}\mathrm{Hom}_R(X,\widehat{S})=\operatorname{depth}R-\operatorname{depth}_R X$.

(c) *If* $X\in\mathsf{GI}(R)$ *then* $-\inf\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X)=\operatorname{depth}R-\operatorname{width}_R X$.

Proof. For part (a) we set $Y=\mathrm{E}_S(l)$ in Theorem (4.4)(a) to obtain

$$-\operatorname{depth}_R(\mathrm{E}_S(l)\otimes^{\mathbf{L}}_R X)=\operatorname{depth}R-\operatorname{depth}_R X.$$

Observe that the homology modules of $\mathrm{E}_S(l)\otimes^{\mathbf{L}}_R X$ are $\mathfrak{m}$–torsion, as φ is local and $\mathrm{E}_R(l)$ is $\mathfrak{n}$–torsion, and apply (4.1.4). Part (b) follows from (a) as $\mathsf{GP}(R)\subseteq\mathsf{GF}(R)$:

$$\begin{aligned}-\inf\mathbf{R}\mathrm{Hom}_R(X,\widehat{S})&=-\inf\mathbf{R}\mathrm{Hom}_R(X,\mathrm{Hom}_S(\mathrm{E}_S(l),\mathrm{E}_S(l)))\\&=-\inf\mathbf{R}\mathrm{Hom}_S(X\otimes^{\mathbf{L}}_R\mathrm{E}_S(l),\mathrm{E}_S(l))\\&=\sup(\mathrm{E}_S(l)\otimes^{\mathbf{L}}_R X).\end{aligned}$$

To prove (c) we need the representations of local (co)homology from [1, 30] as summed up in [28, (2.6)]:

$$\mathbf{L}\Lambda^{\mathfrak{n}}\,\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X)\simeq\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{n}}\,\mathrm{E}_S(l),X)\simeq\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X).$$

Consequently,

$$\begin{aligned}-\inf\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X)&=-\inf(\mathbf{L}\Lambda^{\mathfrak{n}}\,\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X))\\&=-\operatorname{width}_S\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_S(l),X)\\&=\operatorname{depth}R-\operatorname{width}_R X\end{aligned}$$

by [28, thm. (2.11)] and Theorem (4.4)(c). □

A finite Gorenstein projective module G over a local ring R has $\operatorname{depth}_R G=\operatorname{depth}R$. The next corollary extends this equality to non-finite modules.

(4.7) **Corollary.** *Let* $(R,\mathfrak{m},k)$ *be local. If* A *is a Gorenstein flat module of finite depth, then* $\operatorname{depth}_R A=\operatorname{depth}R$. *Similarly, if* B *is a Gorenstein injective module of finite width, then* $\operatorname{width}_R B=\operatorname{depth}R$.

Proof. By (4.6)(c) we have $-\inf\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),B)=\operatorname{depth}R-\operatorname{width}_R B$. If $\operatorname{width}_R B$ is finite, so is $-\inf\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),B)$, and since $\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),B)$ is represented by the module $\mathrm{Hom}_R(\mathrm{E}_R(k),B)$, cf. [19, cor. 2.12], the infimum must be zero. This proves the second statement; the proof of the first one is similar. □

We close this section with the proof of Theorem (4.3). It is broken down into six steps, the first of which is a straightforward generalization of the argument in [27, thm. 2.4] to the relative situation. The third step uses (0.4)(f) and is considerably shorter than the proof of the absolute version in [18, thm. (4.14)].

Proof of Theorem (4.3). 1° First we assume that $X\in\mathsf{F}(R)$ and $Y\in\mathsf{D}_{\sqsubset}(S)$. The second equality in the next computation follows by tensor evaluation (0.4)(d); the third holds as all l–vector spaces become vector spaces over k through the local homomorphism φ,

$$\begin{aligned}\operatorname{depth}_S(Y \otimes^{\mathbf{L}}_R X) &= -\sup \mathbf{R}\mathrm{Hom}_S(l, Y \otimes^{\mathbf{L}}_R X)\\ &= -\sup (\mathbf{R}\mathrm{Hom}_S(l, Y) \otimes^{\mathbf{L}}_R X)\\ &= -\sup (\mathbf{R}\mathrm{Hom}_S(l, Y) \otimes_k (k \otimes^{\mathbf{L}}_R X))\\ &= -\sup \mathbf{R}\mathrm{Hom}_S(l, Y) - \sup (k \otimes^{\mathbf{L}}_R X)\\ &= \operatorname{depth}_S Y - \sup (k \otimes^{\mathbf{L}}_R X).\end{aligned}$$

In particular, for $S = R = Y$ and $\varphi = 1_R$ we get $\operatorname{depth}_R X = \operatorname{depth} R - \sup (k \otimes^{\mathbf{L}}_R X)$; combining this with the equality above, the desired equality follows.

2° Next we assume that $Y \in \mathsf{F}(R)$ and $X \in \mathsf{D}_{\sqsubset}(R)$. Let $K = \mathrm{K}_S[x_1, \ldots, x_e]$ be the Koszul complex on a set of generators for $\mathfrak{n}$. In the next computation, the first and last equalities are by (4.1.1), while the second and penultimate ones follow by (4.1.4). (Since φ is local, the homology modules of $K \otimes_S Y$ and $K \otimes^{\mathbf{L}}_S (Y \otimes^{\mathbf{L}}_R X) \simeq (K \otimes_S Y) \otimes^{\mathbf{L}}_R X$ are $\mathfrak{m}$–torsion, even annihilated by $\mathfrak{m}$ cf. [42, thm. 16.4].) The Koszul complex K consists of finite free S–modules, so also $K \otimes_S Y$ belongs to $\mathsf{F}(R)$; the third equality below is therefore the absolute version of the formula already established in 1°.

$$\begin{aligned}\operatorname{depth}_S(Y \otimes^{\mathbf{L}}_R X) &= e - \sup (K \otimes^{\mathbf{L}}_S (Y \otimes^{\mathbf{L}}_R X))\\ &= e + \operatorname{depth}_R((K \otimes_S Y) \otimes^{\mathbf{L}}_R X)\\ &= e + \operatorname{depth}_R(K \otimes_S Y) + \operatorname{depth}_R X - \operatorname{depth} R\\ &= e - \sup (K \otimes_S Y) + \operatorname{depth}_R X - \operatorname{depth} R\\ &= \operatorname{depth}_S Y + \operatorname{depth}_R X - \operatorname{depth} R.\end{aligned}$$

This concludes the proof of part (a). The arguments establishing (b) and (c) are intertwined, and the whole argument is divided into four steps (3°–6°).

3° We establish the equality in (b) under the assumption that $X \in \mathsf{P}(R)$ and $Y \in \mathsf{D}_{\sqsupset}(S)$. The second equality in the next computation follows by tensor evaluation, (0.4)(f), and the third by adjunction:

$$\begin{aligned}\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(X, Y) &= \inf (\mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_S l)\\ &= \inf \mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathbf{L}}_S l)\\ &= \inf \mathrm{Hom}_k(X \otimes^{\mathbf{L}}_R k, Y \otimes^{\mathbf{L}}_S l)\\ &= \inf (Y \otimes^{\mathbf{L}}_S l) - \sup (X \otimes^{\mathbf{L}}_R k).\end{aligned}$$

Since $X \in \mathsf{P}(R) \subseteq \mathsf{F}(R)$ we have $-\sup (X \otimes^{\mathbf{L}}_R k) = \operatorname{depth}_R X - \operatorname{depth} R$, as established in 1° above, and by (4.1.2) we have $\inf (Y \otimes^{\mathbf{L}}_S l) = \operatorname{width}_S Y$.

4° Based on 3° we can prove the equality in (c) under the assumptions that $Y \in \mathsf{P}(R)$ and $X \in \mathsf{D}_{\sqsupset}(R)$:

$$\begin{aligned}\operatorname{width}_S \mathbf{R}\mathrm{Hom}_R(Y, X) &= \inf (K \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(Y, X))\\ &= \operatorname{width}_R(K \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(Y, X))\\ &= \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(\mathrm{Hom}_S(K, Y), X)\\ &= \operatorname{depth}_R \mathrm{Hom}_S(K, Y) + \operatorname{width}_R X - \operatorname{depth} R\\ &= -\sup \mathrm{Hom}_S(K, Y) + \operatorname{width}_R X - \operatorname{depth} R\\ &= \operatorname{depth}_S Y + \operatorname{width}_R X - \operatorname{depth} R.\end{aligned}$$

The first equality is by (4.1.2), the second by (4.1.4), and the third by Hom evaluation (0.4)(a). The complex $\mathrm{Hom}_S(K,Y)$ is in $\mathsf{P}(R)$, so the fourth equality follows from what we have already proved in 3° with $\varphi = 1_R$. The penultimate equality is by (4.1.4), as $\mathrm{Hom}_S(K,Y)$ is shift-isomorphic to $K \otimes_S Y$, and hence its homology is annihilated by $\mathfrak{m}$. The last equality is by (4.1.1).

5° We prove (c) under the assumption that $X \in \mathsf{I}(R)$ and $Y \in \mathsf{D}_{\sqsubset}(S)$. The second equality below follows by Hom evaluation (0.4)(b) and the third by adjunction,

$$\begin{aligned}\mathrm{width}_S\, \mathbf{R}\mathrm{Hom}_R(Y,X) &= \inf\,(l \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(Y,X)) \\ &= \inf \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(l,Y),X) \\ &= \inf \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_S(l,Y),\mathbf{R}\mathrm{Hom}_R(k,X)) \\ &= \inf \mathbf{R}\mathrm{Hom}_R(k,X) - \sup \mathbf{R}\mathrm{Hom}_S(l,Y) \\ &= \inf \mathbf{R}\mathrm{Hom}_R(k,X) + \mathrm{depth}_S\, Y.\end{aligned}$$

In particular, with $S = R = Y$ and $\varphi = 1_R$ we get $\mathrm{width}_R\, X = \inf \mathbf{R}\mathrm{Hom}_R(k,X) + \mathrm{depth}\, R$, and the desired formula follows. This concludes to proof of part (c).

6° A computation similar to the one performed in 4° — but this time based on (0.4)(e) and the absolute case $\varphi = 1_R$ of 5° — proves (b) under the assumption that $Y \in \mathsf{I}(R)$ and $X \in \mathsf{D}_{\sqsubset}(R)$. This concludes the proof. □

5. Catalogues

In this final section we catalogue ascent properties of Auslander categories and Gorenstein dimensions. It summarizes the results proved in sections 2 and 3 and fills the gaps that become apparent when the results are presented systematically.

(5.1) **Ascent cross tables.** Let $\varphi\colon R \to S$ be a homomorphism of rings. The cross tables below sum up ascent properties of Auslander categories.

The results from sections 2 and 3 give general results for half of the combinations considered in these tables. For the other half, we provide references to counterexamples (cntrex) and in some cases to interesting special cases (sp case).

In (a) we assume that S is noetherian and $\tilde{C}$ a semi-dualizing complex for S.

(a)

$- \otimes^{\mathbf{L}}_R -$	$\mathsf{F}(R)$	$\mathsf{I}(R)$
${}_{\tilde{C}}\mathsf{A}(S)$ S noetherian	${}_{\tilde{C}}\mathsf{A}(S)$ by (2.1)(a)	cntrex/sp case see (5.2)/(5.2)(a)
${}_{\tilde{C}}\mathsf{B}(S)$ S noetherian	${}_{\tilde{C}}\mathsf{B}(S)$ by (2.1)(b)	cntrex see (2.2)(a)

In (b) we assume that R and S are noetherian, $\mathrm{fd}\,\varphi$ is finite, and D and $\tilde{D}$ are dualizing complexes for R and S, respectively.

(b)

$- \otimes^{\mathbf{L}}_R -$ $\mathrm{fd}\,\varphi$ finite	${}_D\mathsf{A}(R)$ R noetherian	${}_D\mathsf{B}(R)$ R noetherian
$\mathsf{F}(S)$ S noetherian	${}_{\tilde{D}}\mathsf{A}(S)$ by (2.8)(a)	cntrex/sp case see (5.3)(a)/(5.2)(b)
$\mathsf{I}(S)$ S noetherian	${}_{\tilde{D}}\mathsf{B}(S)$ by (3.6)(a)	cntrex see (2.2)(a)

In (c) and (d) we assume that S is noetherian and $\tilde{C}$ semi-dualizing for S.

(c)

$\mathbf{R}\mathrm{Hom}_R(-,-)$	${}_{\tilde{C}}\mathsf{A}(S)$ S noetherian	${}_{\tilde{C}}\mathsf{B}(S)$ S noetherian
$\mathsf{P}(R)$	${}_{\tilde{C}}\mathsf{A}(S)$ by (2.1)(c)	${}_{\tilde{C}}\mathsf{B}(S)$ by (2.1)(d)
$\mathsf{I}(R)$	cntrex see (2.2)(b)	cntrex/sp case see (5.2)/(5.2)(a)

(d)

$\mathbf{R}\mathrm{Hom}_R(-,-)$	$\mathsf{P}(R)$	$\mathsf{I}(R)$
${}_{\tilde{C}}\mathsf{A}(S)$ S noetherian	cntrex/sp case see (5.2)/(5.2)(a)	${}_{\tilde{C}}\mathsf{B}(S)$ by (2.1)(e)
${}_{\tilde{C}}\mathsf{B}(S)$ S noetherian	cntrex see (2.2)(b)	${}_{\tilde{C}}\mathsf{A}(S)$ by (2.1)(f)

In (e) and (f) we assume that R and S are noetherian, $\mathrm{fd}\,\varphi$ is finite, and D and $\tilde{D}$ are dualizing complexes for R and S, respectively.

(e)

$\mathbf{R}\mathrm{Hom}_R(-,-)$ $\mathrm{fd}\,\varphi$ finite	${}_D\mathsf{A}(R)$ R noetherian	${}_D\mathsf{B}(R)$ R noetherian
$\mathsf{P}(S)$ S noetherian	cntrex/sp case see (5.3)(b)/(5.2)(b)	${}_{\tilde{D}}\mathsf{B}(S)$ by (2.8)(b)
$\mathsf{I}(S)$ S noetherian	cntrex see (2.2)(b)	${}_{\tilde{D}}\mathsf{A}(S)$ by (3.6)(c)

(f)

$\mathbf{R}\mathrm{Hom}_R(-,-)$ $\mathrm{fd}\,\varphi$ finite	$\mathsf{P}(S)$ S noetherian	$\mathsf{I}(S)$ S noetherian
${}_D\mathsf{A}(R)$ R noetherian	${}_{\tilde{D}}\mathsf{A}(S)$ by (3.6)(b)	${}_{\tilde{D}}\mathsf{B}(S)$ by (2.8)(c)
${}_D\mathsf{B}(R)$ R noetherian	cntrex see (2.2)(b)	cntrex/sp case see (5.3)(c)/(5.2)(b)

(5.2) **Observation.** Let R be noetherian and C be a semi-dualizing complex for R. In general, the combinations

$$ {}_C\mathsf{A}(R) \otimes^{\mathbf{L}}_R \mathsf{I}(R), \quad \mathbf{R}\mathrm{Hom}_R({}_C\mathsf{A}(R), \mathsf{P}(R)) \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_R(\mathsf{I}(R), {}_C\mathsf{B}(R)) $$

do not even end up in $\mathsf{D}_{\square}(R)$, and in particular not in ${}_C\mathsf{A}(R)$ or ${}_C\mathsf{B}(R)$. Indeed, let R be the ring from Example (2.2) and set $C = R$. The dualizing module D belongs to both $\mathsf{I}(R)$ and ${}_R\mathsf{A}(R) = \mathsf{D}_{\square}(R)$, and R belongs to both $\mathsf{P}(R)$ and ${}_R\mathsf{B}(R) = \mathsf{D}_{\square}(R)$, but $D \otimes^{\mathbf{L}}_R D$ as well as $\mathbf{R}\mathrm{Hom}_R(D, R)$ is unbounded.

(a): Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings. Let D be a dualizing complex for R and assume that $\tilde{D} = S \otimes^{\mathbf{L}}_R D$ is dualizing[2] for S. An S–complex is

[2]Local homomorphisms with this property are called *quasi-Gorenstein* and were first studied in [8].

then in ${}_{\tilde{D}}\mathsf{A}(S)$ or ${}_{\tilde{D}}\mathsf{B}(S)$ if and only if it belongs to ${}_{D}\mathsf{A}(R)$ or ${}_{D}\mathsf{B}(R)$, respectively; see [8, proof of cor. (7.9)]. Applying Corollary (3.6) to $\varphi = 1_R$ now yields

- ${}_{\tilde{D}}\mathsf{A}(S) \otimes^{\mathbf{L}}_R \mathsf{I}(R) \subseteq {}_{\tilde{D}}\mathsf{B}(S)$,
- $\mathbf{R}\mathrm{Hom}_R({}_{\tilde{D}}\mathsf{A}(S), \mathsf{P}(R)) \subseteq {}_{\tilde{D}}\mathsf{A}(S)$, and
- $\mathbf{R}\mathrm{Hom}_R(\mathsf{I}(R), {}_{\tilde{D}}\mathsf{B}(S)) \subseteq {}_{\tilde{D}}\mathsf{A}(S)$.

(b): Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings with $\mathrm{fd}\,\varphi$ finite. If R has a dualizing complex, D, then $\mathsf{P}(R) = \mathsf{F}(R)$, e.g. by [26, proof of cor. 3.4], and hence it is immediate by the absolute version of Prop. (2.1)(b,c,f), cf. (0.8), that

- $\mathsf{F}(S) \otimes^{\mathbf{L}}_R {}_{D}\mathsf{B}(R) \subseteq {}_{D}\mathsf{B}(R)$,
- $\mathbf{R}\mathrm{Hom}_R(\mathsf{P}(S), {}_{D}\mathsf{A}(R)) \subseteq {}_{D}\mathsf{A}(R)$, and
- $\mathbf{R}\mathrm{Hom}_R({}_{D}\mathsf{B}(R), \mathsf{I}(S)) \subseteq {}_{D}\mathsf{A}(R)$.

Under the additional assumption that $\tilde{D} = S \otimes^{\mathbf{L}}_R D$ is dualizing[3] for S, it follows, as above, that e.g. the combination $\mathsf{F}(S) \otimes^{\mathbf{L}}_R {}_{D}\mathsf{B}(R)$ even ends up in ${}_{\tilde{D}}\mathsf{B}(S)$.

It is easy to see that this is not the general behavior:

(5.3) **Example.** Let R be a field and consider the flat map

$$\varphi:\ R \longrightarrow S = R[\![X,Y]\!]/(X^2, XY, Y^2).$$

R is Gorenstein, so $D = R$ is dualizing for R and ${}_{D}\mathsf{A}(R) = {}_{D}\mathsf{B}(R) = \mathsf{D}_{\Box}(R)$. The ring S has a dualizing complex $\tilde{D} \simeq \mathbf{R}\mathrm{Hom}_R(S,R)$ but is not Gorenstein, see Example (2.2). In particular, S is not in ${}_{\tilde{D}}\mathsf{B}(S)$, and $\tilde{D}$ is not in ${}_{\tilde{D}}\mathsf{A}(S)$. Nevertheless,

(a) $$\mathsf{F}(S) \otimes^{\mathbf{L}}_R {}_{D}\mathsf{B}(R) \ni S \otimes^{\mathbf{L}}_R R \simeq S \notin {}_{\tilde{D}}\mathsf{B}(S),$$

(b) $$\mathbf{R}\mathrm{Hom}_R(\mathsf{P}(S), {}_{D}\mathsf{A}(R)) \ni \mathbf{R}\mathrm{Hom}_R(S,R) \notin {}_{\tilde{D}}\mathsf{A}(S), \text{ and}$$

(c) $$\mathbf{R}\mathrm{Hom}_R({}_{D}\mathsf{B}(R), \mathsf{I}(S)) \ni \mathbf{R}\mathrm{Hom}_R(R,\tilde{D}) \simeq \tilde{D} \notin {}_{\tilde{D}}\mathsf{A}(S).$$

(5.4) **Remark.** In general one cannot expect ascent results involving two modules of finite Gorenstein dimension or two Auslander categories: Let R be a local non-regular Gorenstein ring with residue field k, e.g. $R = k[X]/(X^2)$. In this case, k has finite Gorenstein projective dimension and finite Gorenstein injective dimension; in particular, $k \in {}_{D}\mathsf{A}(R) = {}_{D}\mathsf{B}(R) = \mathsf{D}_{\Box}(R)$, but both $k \otimes^{\mathbf{L}}_R k$ and $\mathbf{R}\mathrm{Hom}_R(k,k)$ are unbounded as $\mathrm{pd}_R\, k = \mathrm{fd}_R\, k = \mathrm{id}_R\, k = \infty$.

[3]Local homomorphisms with this property are called *Gorenstein* and were first studied in [5].

(5.5) **Ascent table I.** Let $\varphi\colon R \to S$ be a homomorphism of rings. The table below sums up ascent results for Gorenstein dimensions pertaining to that situation. We use abbreviated notation, e.g. "G–projective/S" means a Gorenstein projective S–module. Note that the "G" always lives over S.

	Ascent result			Requirements
(a)	*G–flat/S* $\otimes$ *flat/R*	is	*G–flat/S*	
(b)	*G–projective/S* $\otimes$ *projective/R*	is	*G–projective/S*	
(c)	*G–injective/S* $\otimes$ *flat/R*	is	*G–injective/S*	S is noetherian and has a dualizing complex
(c')	*G–injective/S* $\otimes$ *f.g. projective/R*	is	*G–injective/S*	
(d)	Hom(*flat/R*, *G–injective/S*)	is	*G–injective/S*	$\mathsf{F}(R) = \mathsf{P}(R)$ or $\mathsf{F}(S) = \mathsf{P}(S)$
(e)	Hom(*projective/R*, *G–injective/S*)	is	*G–injective/S*	
(f)	Hom(*projective/R*, *G–flat/S*)	is	*G–flat/S*	S is noetherian and has a dualizing complex
(f')	Hom(*f.g. projective/R*, *G–flat/S*)	is	*G–flat/S*	
(g)	Hom(*f.g. projective/R*, *G–projective/S*)	is	*G–projective/S*	
(h)	Hom(*G–flat/S*, *injective/R*)	is	*G–injective/S*	
(i)	Hom(*G–injective/S*, *injective/R*)	is	*G–flat/S*	S is noetherian and has a dualizing complex

Comments: The requirements in (d) hold if R or S is noetherian with finite Krull dimension; see footnote 1 to Theorem (3.1).

In the absolute case, $\varphi = 1_R$, the two requirements in (5.5)(d) are the same.

(5.6) **Ascent table II.** Let $\varphi\colon R \to S$ be a homomorphism with $\operatorname{fd}\varphi$ finite. The table below sums up ascent results pertaining to that situation. We use the abbreviated notation from (5.5); note that in this table the "G" always lives over R.

	Ascent result			Requirements
(a)	*flat*/$S \otimes$ *G–flat*/R	is	*G–flat*/S	
(a')	*flat*/$S \otimes$ *G–projective*/R	is	*G–flat*/S	S is noetherian and $\mathsf{F}(S) \subseteq \mathsf{P}(R)$
(b)	*projective*/$S \otimes$ *G–projective*/R	is	*G–projective*/S	$\mathsf{P}(S) \subseteq \mathsf{P}(R)$
(c)	*injective*/$S \otimes$ *G–flat*/R	is	*G–injective*/S	S is noetherian with a dualizing complex
(c')	*injective*/$S \otimes$ *f.g. G–projective*/R	is	*G–injective*/S	R is noetherian
(d)	Hom(*G–flat*/R, *injective*/S)	is	*G–injective*/S	
(e)	Hom(*G–projective*/R, *injective*/S)	is	*G–injective*/S	S is noetherian and $\mathsf{F}(S) \subseteq \mathsf{P}(R)$
(e')	Hom(*f.g. G–projective*/R, *injective*/S)	is	*G–injective*/S	R is noetherian
(f)	Hom(*G–projective*/R, *flat*/S)	is	*G–flat*/S	S is noetherian with a dualizing complex and $\mathsf{F}(S) \subseteq \mathsf{P}(R)$
(f')	Hom(*f.g. G–projective*/R, *flat*/S)	is	*G–flat*/S	R is noetherian
(g)	Hom(*f.g. G–projective*/R, *projective*/S)	is	*G–projective*/S	R is noetherian
(h)	Hom(*flat*/S, *G–injective*/R)	is	*G–injective*/S	(1) $\mathsf{F}(S) = \mathsf{P}(S) \subseteq \mathsf{P}(R)$; or (2) $\mathsf{F}(S) \subseteq \mathsf{P}(R)$, and R or S is noetherian
(h')	Hom(*projective*/S, *G–injective*/R)	is	*G–injective*/S	$\mathsf{P}(S) \subseteq \mathsf{P}(R)$
(i)	Hom(*injective*/S, *G–injective*/R)	is	*G–flat*/S	S is noetherian with a dualizing complex and $\mathsf{F}(S) \subseteq \mathsf{P}(R)$

Comments: Part (b) is an unpublished result of Hans-Bjørn Foxby.

Note that with the exception of (c), (f) and (i), which require a dualizing complex for S, all results in this table hold when R and S are noetherian and R has finite Krull dimension; see the footnote 1 to Theorem (3.1).

In the absolute case, $\varphi = 1_R$, the first requirement in (5.6)(h) is weaker than the second.

(5.7) **Remarks.** Four results in Table II, namely (5.6)(c',e',f',g), deal with finite Gorenstein projective modules over a noetherian ring. Recall that a finite module over such a ring is Gorenstein projective if and only if it is Gorenstein flat if and only if it is totally reflexive (in the sense of [13]), cf. [16, thm. (4.2.6) and (5.1.11)].

Over a noetherian ring where every flat module has finite projective dimension, e.g. a ring of finite Krull dimension, every Gorenstein projective module is Gorenstein flat, cf. [33, prop. 3.4]. Thus, for example, (5.5)(a) includes the result:

G–projective/S ⊗ flat/R is G–flat/S when S is noetherian and $\mathsf{F}(S) = \mathsf{P}(S)$.

By the same token, (5.6)(a') also holds when R is noetherian with $\mathsf{F}(R) = \mathsf{P}(R)$; in that setting it is a special case of (5.6)(a).

Proofs for ascent table I. Parts (c), (f), and (i) are proved in Prop. (2.4).

The proofs of parts (a), (b), (c'), (e), (f'), (g), and (h) are entirely functorial and follow the pattern from the proof of Lemma (2.6).

This leaves only (d): Let F be flat over R and $\tilde{B}$ be Gorenstein injective over S. Under either assumption, the module $F \otimes_R S$ has finite projective dimension over S, so there exists an exact sequence $0 \to \tilde{P}_d \to \cdots \to \tilde{P}_0 \to F \otimes_R S \to 0$, where the $\tilde{P}$'s are projective over S. The functor $\mathrm{Hom}_S(-, \tilde{B})$ leaves this sequence exact,

$$0 \to \mathrm{Hom}_S(F \otimes_R S, \tilde{B}) \to \mathrm{Hom}_S(\tilde{P}_0, \tilde{B}) \to \cdots \to \mathrm{Hom}_S(\tilde{P}_d, \tilde{B}) \to 0.$$

It follows that $\mathrm{Gid}_S \mathrm{Hom}_S(F \otimes_R S, \tilde{B}) \leq d$, as each module $\mathrm{Hom}_S(\tilde{P}_\ell, \tilde{B})$ is Gorenstein injective over S by (e); note that d is independent of $\tilde{B}$. Next, consider a piece of a complete injective resolution of $\tilde{B}$, say $0 \to \tilde{B}' \to \tilde{I}_{d-1} \to \cdots \to \tilde{I}_0 \to \tilde{B} \to 0$. Also $\tilde{B}'$ is Gorenstein injective, so $\mathrm{Gid}_S \mathrm{Hom}_S(F \otimes_R S, \tilde{B}') \leq d$, and applying $\mathrm{Hom}_S(F \otimes_R S, -)$ we get the exact sequence

$$0 \to \mathrm{Hom}_S(F \otimes_R S, \tilde{B}') \to \mathrm{Hom}_S(F \otimes_R S, \tilde{I}_{d-1}) \to \cdots$$
$$\cdots \to \mathrm{Hom}_S(F \otimes_R S, \tilde{I}_0) \to \mathrm{Hom}_S(F \otimes_R S, \tilde{B}) \to 0.$$

The modules $\mathrm{Hom}_S(F \otimes_R S, \tilde{I}_\ell)$ are injective over S, by (0.5), and hence $\mathrm{Hom}_S(F \otimes_R S, \tilde{B}) \cong \mathrm{Hom}_R(F, \tilde{B})$ is Gorenstein injective over S. □

Proofs for ascent table II. Parts (c), (f), and (i) are proved in Corollary (3.7); parts (a), (d), and (h') are proved in Lemma (2.6).

(a'): Let $\boldsymbol{P}$ be complete projective resolution over R and $\tilde{F}$ a flat S–module. The complex $\tilde{F} \otimes_R \boldsymbol{P}$ is exact and consists of flat S–modules. Now let $\tilde{J}$ be any S–injective module; by adjunction we have

$$\mathrm{Hom}_{\mathbb{Z}}(\tilde{J} \otimes_S (\tilde{F} \otimes_R \boldsymbol{P}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(\boldsymbol{P}, \mathrm{Hom}_{\mathbb{Z}}(\tilde{J} \otimes_S \tilde{F}, \mathbb{Q}/\mathbb{Z})).$$

Since S is noetherian, the module $\mathrm{Hom}_{\mathbb{Z}}(\tilde{J} \otimes_S \tilde{F}, \mathbb{Q}/\mathbb{Z})$ is S–flat and hence in $\mathsf{P}(R)$. This shows that $\tilde{F} \otimes_R \boldsymbol{P}$ is a complete flat resolution over S.

(b): Let $\tilde{P}$ be a projective S–module and $\boldsymbol{P}$ a complete projective resolution over R. The complex $\tilde{P} \otimes_R \boldsymbol{P}$ consists of projective S–modules, and it is exact, as $\tilde{P} \in \mathsf{P}(S) \subseteq \mathsf{F}(R)$. For any projective S–module $\tilde{Q}$, we have $\mathrm{Hom}_S(\tilde{P} \otimes_R \boldsymbol{P}, \tilde{Q}) \cong \mathrm{Hom}_S(\tilde{P}, \mathrm{Hom}_R(\boldsymbol{P}, \tilde{Q}))$, which is exact as $\tilde{Q} \in \mathsf{P}(S) \subseteq \mathsf{P}(R)$.

(c'): Let $\boldsymbol{L}$ be a complete resolution by finite free R–modules (R is noetherian). By [16, lem. (5.1.10)], $\boldsymbol{L}$ is also a complete flat resolution. Since $\mathrm{fd}\,\varphi$ is finite, the injective dimension of $\tilde{I}$ over R is finite, so $\tilde{I} \otimes_R \boldsymbol{L}$ is an exact complex of injective

S–modules. To see that $\tilde{I} \otimes_R \boldsymbol{L}$ is a complete injective resolution over S, let $\tilde{J}$ be any injective S–module, and apply (0.3)(e) to obtain

$$(*) \qquad \mathrm{Hom}_S(\tilde{J}, \tilde{I} \otimes_R \boldsymbol{L}) \cong \mathrm{Hom}_S(\tilde{J}, \tilde{I}) \otimes_R \boldsymbol{L}.$$

By Lemma (0.9) the module $\mathrm{Hom}_S(\tilde{J}, \tilde{I}) \simeq \mathbf{R}\mathrm{Hom}_S(\tilde{J}, \tilde{I})$ has finite flat dimension over R. The complete flat resolution $\boldsymbol{L}$ remains exact when tensored by a module in $\mathsf{F}(R)$, cf. [19, lem. 2.3], so the complex in $(*)$ is exact.

(e): Let G be a Gorenstein projective R–module and $\tilde{I}$ an injective S–module. Under the assumptions, $G \otimes_R S$ is Gorenstein flat over S by (a') and $\mathrm{Hom}_R(G, \tilde{I}) \cong \mathrm{Hom}_S(G \otimes_R S, \tilde{I})$ is then Gorenstein injective over S by (5.5)(h).

(e'): Let $\boldsymbol{L}$ be a complete resolution by finite free R–modules and $\tilde{I}$ be an injective S–module. The complex $\mathrm{Hom}_R(\boldsymbol{L}, \tilde{I})$ consists of injective S–modules, and it is exact as $\tilde{I} \in \mathsf{I}(S) \subseteq \mathsf{I}(R)$. Let $\tilde{J}$ be any injective S–module; swap gives $\mathrm{Hom}_S(\tilde{J}, \mathrm{Hom}_R(\boldsymbol{L}, \tilde{I})) \cong \mathrm{Hom}_R(\boldsymbol{L}, \mathrm{Hom}_S(\tilde{J}, \tilde{I}))$. This complex is exact by [16, prop. (4.1.3)], as $\mathrm{Hom}_S(\tilde{J}, \tilde{I}) \simeq \mathbf{R}\mathrm{Hom}_S(\tilde{J}, \tilde{I}) \in \mathsf{F}(R)$ by Lemma (0.9).

(f'): Let $\boldsymbol{L}$ be a complete resolution by finite free R–modules and $\tilde{F}$ be a flat S–module. The complex $\mathrm{Hom}_R(\boldsymbol{L}, \tilde{F})$ consists of flat S–modules and is exact by [16, prop. (4.1.3)] as $\tilde{F} \in \mathsf{F}(S) \subseteq \mathsf{F}(R)$. Let $\tilde{J}$ be any injective S–module; tensor evaluation (0.3)(c) gives an isomorphism $\tilde{J} \otimes_S \mathrm{Hom}_R(\boldsymbol{L}, \tilde{F}) \cong \mathrm{Hom}_R(\boldsymbol{L}, \tilde{F} \otimes_S \tilde{J})$. This complex is exact, as the module $\tilde{F} \otimes_S \tilde{J} \simeq \tilde{F} \otimes^{\mathbf{L}}_S \tilde{J} \in \mathsf{I}(R)$ by Lemma (0.9).

(g): Let $\boldsymbol{L}$ be a complete resolution by finite free R–modules. If $\tilde{P}$ is a projective S–module, the complex $\mathrm{Hom}_R(\boldsymbol{L}, \tilde{P})$ consists of projective S–modules and is exact by [16, prop. (4.1.3)]. Let $\tilde{Q}$ be any projective S–module; Hom evaluation (0.3)(a) gives an isomorphism $\mathrm{Hom}_S(\mathrm{Hom}_R(\boldsymbol{L}, \tilde{P}), \tilde{Q}) \cong \boldsymbol{L} \otimes_R \mathrm{Hom}_S(\tilde{P}, \tilde{Q})$. By Lemma (0.9) we get $\mathrm{Hom}_S(\tilde{P}, \tilde{Q}) \simeq \mathbf{R}\mathrm{Hom}_S(\tilde{P}, \tilde{Q}) \in \mathsf{F}(R)$, and thus $\boldsymbol{L} \otimes_R \mathrm{Hom}_S(\tilde{P}, \tilde{Q})$ is exact by [16, prop. (4.1.3)].

(h): First we assume (1). Let $\tilde{F}$ be flat over S, let B be Gorenstein injective over R and consider $\mathrm{Hom}_R(\tilde{F}, B) \cong \mathrm{Hom}_S(\tilde{F}, \mathrm{Hom}_R(S, B))$. Since $\mathsf{P}(S) \subseteq \mathsf{P}(R)$ the module $\mathrm{Hom}_R(S, B)$ is Gorenstein injective over S by (h'), and since $\mathsf{F}(S) = \mathsf{P}(S)$, it follows by (5.5)(d) applied to $\varphi = 1_R$ that $\mathrm{Hom}_S(\tilde{F}, \mathrm{Hom}_R(S, B))$ is Gorenstein injective over S.

Next we assume (2). Let $\boldsymbol{I}$ be a complete injective resolution over R, then the complex $\mathrm{Hom}_R(\tilde{F}, \boldsymbol{I})$ consists of injective S–modules, and it is exact as $\tilde{F}$ belongs to $\mathsf{F}(S) \subseteq \mathsf{P}(R)$. For any injective S–module $\tilde{J}$ the complex $\mathrm{Hom}_S(\tilde{J}, \mathrm{Hom}_R(\tilde{F}, \boldsymbol{I})) \cong \mathrm{Hom}_R(\tilde{J} \otimes_S \tilde{F}, \boldsymbol{I})$ is exact, as $\tilde{J} \otimes_S \tilde{F} \in \mathsf{I}(R)$ by (0.6), if S is noetherian, or by Lemma (0.9), if R is noetherian. □

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