Systems of ordinary differential equations: introduction and elementary methods

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Contents

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1 A simple example system

Here's a simple example of a system of differential equations: solve the coupled equations

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 \\ \frac{dy_2}{dt} &= y_1 - 2y_2 \end{aligned} \tag{1}$$

for $y_1(t)$ and $y_2(t)$ given some initial values $y_1(0)$ and $y_2(0)$. We can also write this system of equations with matrix-vector notation as follows: introduce the matrix

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \tag{2}$$

and the vector

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \tag{3}$$

and then the system becomes simply

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \tag{4}$$

with initial value

$$\mathbf{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

Over the next few weeks we'll see how to use this matrix-vector structure to develop some solution methods that are not only practically useful but also give engineering insight into the system behavior. For today, though, we start simple: we'll look at a couple of real-world examples of such a system, and then use some elementary methods to solve it. The elementary methods will later turn out to be special cases of some more advanced, more abstract, methods that use linear algebra to simplify and clarify the calculations.

2 Some real-world problems described by our example equations

2.1 Draining of coupled water tanks

Consider two water tanks (open to the air at the top) joined near their bottoms by a pipe. Additionally, each tank also has a drain pipe. In this configuration water can flow between the tanks as well as out the drains. A schematic of this setup is shown in figure 1.

Recall from hydrostatics that the pressure at the bottom of a water column of height h is $p = \rho g h$ where ρ is the density of water and g is the gravitational acceleration. You may have also seen the Hagen-Poiseuille law for laminar flow through a cylindrical pipe: the volumetric flow rate Q (volume of fluid per unit time) is determined by the pressure difference Δp between the ends of the pipe, the pipe geometry, and the fluid viscosity:

$$Q = \frac{\pi a^4 \Delta p}{8\mu L}$$

where $a$is the pipe radius, $L$the pipe length, and μ the dynamic viscosity of the fluid. Then the flow rate from tank 1 to tank 2 is

$$Q_{12} = \frac{\pi a^4 \rho g}{8\mu L}\left(h_1 - h_2\right).$$

The flow rates in the outlet pipes are

$$Q_{10} = \frac{\pi a^4 \rho g}{8\mu L} h_1$$

and

$$Q_{20} = \frac{\pi a^4 \rho g}{8\mu L} h_2.$$

The volume of water in tank 1 is changing at rate

$$\frac{dV_1}{dt} = -Q_{12} - Q_{10}$$

and that in tank 2 is changing at rate

$$\frac{dV_2}{dt} = Q_{12} - Q_{20}.$$

Noting that the water volume V_i is the column height h_i times the area A of the tank's horizontal cross-section, we find

$$\frac{dh_1}{dt} = \frac{\pi a^4 \rho g}{8\mu LA}\left(-2h_1 + h_2\right)$$

$$\frac{dh_2}{dt} = \frac{\pi a^4 \rho g}{8\mu LA}\left(h_1 - 2h_2\right).$$

The quantity $\frac{1}{\tau} \equiv \frac{\pi a^4 \rho g}{8\mu LA}$ has dimension time^{-1}. Making the transformation to a dimensionless time variable $\hat{t} = t/\tau$, $d\hat{t} = \tau^{-1} dt$ and then thereafter ignoring the "hats" on $\hat{t}$ we find the equations

$$h_1' = -2h_1 + h_2$$
$$h_2' = h_1 - 2h_2$$

or just the system 1.

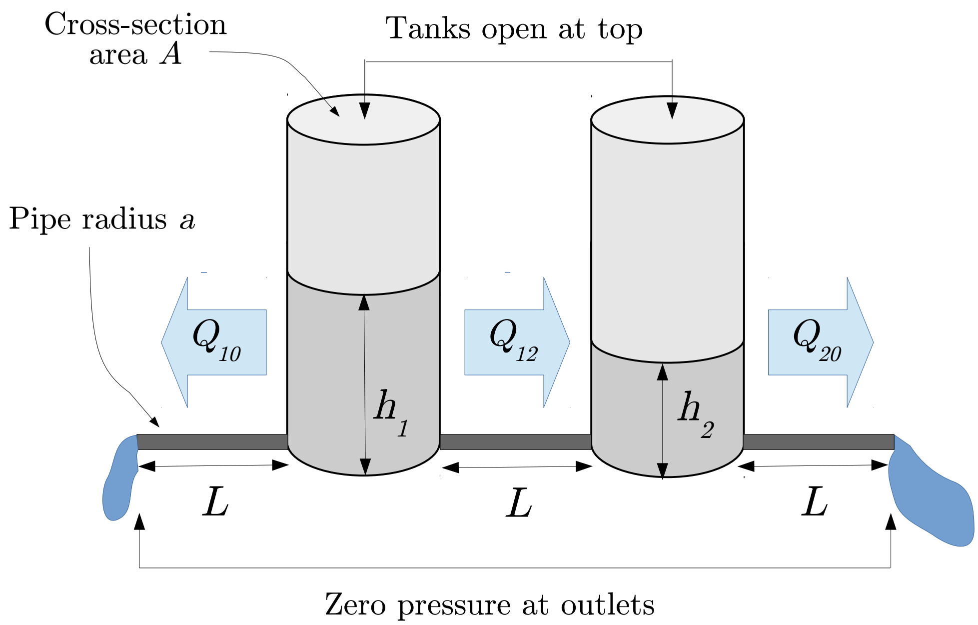

Figure 1: A system of two connected water tanks with outlet pipes. The pipes have length L and radius a. The water levels in the left and right tanks are $h_1(t)$ and $h_2(t)$, respectively. The pressure at the ends of the outlet pipes is zero.

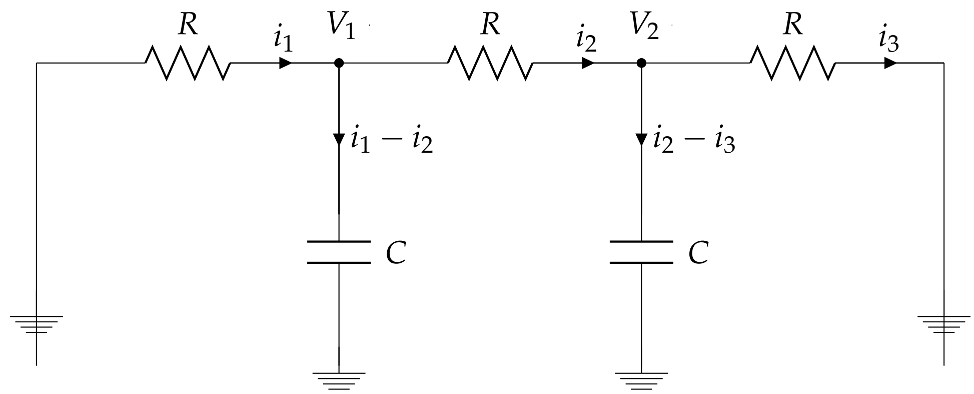

Figure 2: Circuit diagram

2.2 Discharge of a coupled RC circuit

With application of the Kirchhoff laws to the circuit in figure 2, you will find that the potentials V_1 and V_2 obey the system of differential equations

$$RC\frac{dV_1}{dt} = -2V_1 + V_2$$

$$RC\frac{dV_2}{dt} = V_1 - 2V_2.$$

The quantity $\tau^{-1} = \frac{1}{RC}$ has dimension time^{-1}; recall that $(RC)^{-1}$ is the discharge time for a simple RC circuit. Using τ to define a dimensionless time variable as in the previous example, we find the equations

$$V_1' = -2V_1 + V_2$$
$$V_2' = V_1 - 2V_2,$$

exactly the simple system 1 above. Just as the water tanks lose water, the capacitors lose charge (and consequently the voltages at V_1 and V_2 decay to zero).

3 Solving the example system

3.1 Method 1: Conversion to a single second-order equation

It's always possible to rewrite a system of first order ODEs as a single ODE of higher order. Here's how it's done for system 1. Start with the first order system

$$y_1' = -2y_1 + y_2 \tag{5}$$

$$y_2' = y_1 - 2y_2 \tag{6}$$

and differentiate through equation 5:

$$y_1'' = -2y_1' + y_2'.$$

The last term on the right involves y_2; we want to write the equation entirely in terms of y_1. This can be done in two steps. First, use equation 6 to express y_2' in terms of y_1 and y_2

$$y_1'' = -2y_1' + \underbrace{y_1 - 2y_2}_{y_2'}.$$

$$= -2y_1' + y_1 - 2y_2. \tag{7}$$

We've eliminated y_2' but there's still a y_2. But notice that we can use equation 5 to express y_2 in terms of y_1 and y_1':

$$y_2 = y_1' + 2y_1$$

and then use this to replace y_2 in 7,

$$y_1'' = -2y_1' + y_1 - 2\left(y_1' + 2y_1\right)$$

or

$$y_1'' + 4y_1' + 3y_1 = 0.$$

This is a linear second-order system with constant coefficients, which you learned how to solve in DE 1. I won't go through the whole solution procedure here; I'll go only as far as solving the auxiliary equation for the rate constants. The auxiliary equation for this problem is

$$r^2 + 4r + 3 = 0$$

so the rate constants are $r = -1$ and $r = -3$. The solution will be a linear combination of terms involving e^{-t} and e^{-3t}.

The procedure of converting to a higher order equation is fairly messy, and gets worse for systems with more equations or with inhomogeneous terms (which need to be differentiated). Therefore, in practice we'll not use this procedure, but it's important to understand that a system of first order equations is always equivalent to a higher order system.

3.1.1 Writing a higher order equation as a system of first order equations

It's almost always easier to work with a system of first order equations than with a high-order differential equation, so we'll almost never do the procedure above. However, we'll often do the reverse: rewriting a high order equation as a system. It's very easy. Suppose we have

$$y''' + 2y'' + 3y' + 4y = f(t)$$

with initial conditions

$$y(0) = 1, \;\; y'(0) = 2, \;\; y''(0) = -1.$$

Just introduce new variables for each derivative up to (but not including) the order of the equation:

$$y_1 = y$$
$$y_2 = y'$$
$$y_3 = y''.$$

These variables are related as follows:

$$y_1' = y_2$$

$$y_2' = y_3$$

and then the original equation is

$$y_3' = -2y_3 - 3y_2 - 4y_1 + f(t) .$$

So our system is

$$\begin{aligned} y_1' &= & y_2 \\ y_2' &= & y_3 \\ y_3' &= & -4y_1 - 3y_2 - 2y_3 + f(t) \end{aligned}$$

with initial conditions

$$y_1(0) = 1$$

$$y_2(0) = 2$$

$$y_3(0) = -1.$$

In matrix notation the problem is $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t)$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{pmatrix}$$

and

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix}$$

3.2 Method 2: Laplace transforms

We can apply the method of Laplace transforms to systems. The algebra will be messy (we'll see later how it can be simplifed and automated) but the procedure is far simpler than transforming the system to a single higher-order equation. Recall that Laplace transforms convert differential equations to algebraic equations. We're going to convert a system of differential equations to a system of algebraic equations. We have two unknowns, y_1 and y_2, so we're going to find algebraic equations for their Laplace transforms Y_1 and Y_2. Simply Laplace transform both sides of 5 and 6 to find

$$\begin{aligned} sY_1 - y_1(0) &= -2Y_1 + Y_2 \\ sY_2 - y_2(0) &= Y_1 - 2Y_2 \end{aligned} . \tag{8}$$

Rearrange to put the unknowns on the LHS:

$$\begin{aligned} (s+2)Y_1 - Y_2 &= y_1(0) \\ -Y_1 + (s+2)Y_2 &= y_2(0) \end{aligned} . \tag{9}$$

These are two linear equations in two unknowns, Y_1 and Y_2. You know how to solve such a problem: Eliminate Y_2 by adding the second row to $(s+2)$ times the first row,

$$\left((s+2)^2 - 1\right) Y_1 = (s+2)\, y_1(0) + y_2(0)$$

and solve for Y_1:

$$Y_1 = \frac{y_2(0) + (s+2)\, y_1(0)}{s^2 + 4s + 3}.$$

Notice that the denominator is $s^2 + 4s + 3$, and recall that the rate constants are the zeros of the denominator of the Laplace transform. As with the previous method, we once again find solutions involving e^{-t} and e^{-3t}.

I'll leave it to you to solve for Y_2 and take inverse Laplace transforms to find y_1 and y_2.

3.3 Method 3: Decoupling the equations via a change of variables

Finally, we can solve the system of equations easily by spotting a clever change of variables. With the system

$$y_1' = -2y_1 + y_2 \tag{10}$$

$$y_2' = y_1 - 2y_2 \tag{11}$$

notice that if we add the two equations we get

$$\begin{aligned}(y_1 + y_2)' &= -2y_1 + y_1 + y_2 - 2y_2 \\ &= -(y_1 + y_2).\end{aligned}$$

Introduce a new unknown, $z_1 = y_1 + y_2$, and this equation is simply

$$z_1' = -z_1.$$

Next, subtract the two original equations,

$$(y_1 - y_2)' = -3(y_1 - y_2)$$

and introduce $z_2 = y_1 - y_2$ to find

$$z_2' = -3z_2.$$

We've derived an *uncoupled* system of equations:

$$\begin{aligned}z_1' &= -z_1 \\ z_2' &= -3z_2.\end{aligned}$$

These equations are very easy to solve: $z_1 = 2Ae^{-t}$ and $z_2 = 2Be^{-3t}$ (since I know how this will work out I've included a factor of 2 in the definitions of the undetermined constants). We want y_1 and y_2, not z_1 and z_2, so transform back to the original unknowns by solving the equations

$$\begin{aligned}y_1 + y_2 &= z_1 \\ y_1 - y_2 &= z_2\end{aligned}$$

for y_1 and y_2:

$$y_1 = \frac{z_1 + z_2}{2} = Ae^{-t} + Be^{-3t}$$
$$y_2 = \frac{z_1 - z_2}{2} = Ae^{-t} - Be^{-3t}.$$

As before, the constants A and B are found using the initial values for y_1 and y_2.

This is the same result found after transformation to a second order system (as it has to be!), but there was much less work involved. The catch is that we needed to spot the right way to combine the original equations to get an uncoupled system; in a more complicated problem it will be pretty unlikely to spot the right change of variables just by inspection.

A nice feature of this method is that the uncoupled variables have clear physical meaning. Consider the water tank problem. The variable

$$z_1 = y_1 + y_2$$

is simply twice the average height of the two water columns. The variable

$$z_2 = y_1 - y_2$$

is simply the difference in height between the two water columns. The average height decays with rate constant 1, and the difference between the column heights decays with rate constant 3.

4 Worked examples

4.1 Transformation to higher order equation

Example 1. *Transform the system of equations*

$$y_1' = 3y_1 - 2y_2$$
$$y_2' = -y_1 + 4y_2$$

to an equivalent second-order system. Solve the auxiliary equation to determine the rate constants.

Start by differentiating the first equation to get

$$y_1'' = 3y_1' - 2y_2'$$

and then use the second equation to eliminate y_2',

$$y_1'' = 3y_1' - 2\left(-y_1 + 4y_2\right).$$
$$= 3y_1' + 2y_1 - 8y_2$$

Eliminate y_2 using the first equation: $y_2 = \frac{1}{2}\left(3y_1 - y_1'\right)$

$$y_1'' = 3y_1' + 2y_2 - 4\left(3y_1 - y_1'\right)$$

$$y_1'' = 7y_1' - 10y_1$$
$$y_1'' - 7y_1' + 10y_1 = 0.$$

The auxiliary equation is $r^2 - 7r + 10 = 0$, so the rate constants are $r = 2$ and $r = 5$.

4.2 Writing a higher order equation as a first order system

Example 2. *Write the initial value problem*

$$y'' + 2y' + 10y = e^t$$

$$y(0) = 1$$
$$y'(0) = -1$$

as a first order system.

Introduce new variables $y_1 = y$ and $y_2 = y'$. Then the first order system is

$$y_1' = y_2$$

$$y_2' = -10y_1 - 2y_2 + e^t.$$

with initial conditions $y_1(0) = 1$, $y_2(0) = -1$.

Suppose that the equation models a mechanical system, where y is a displacement and y' a velocity. If we introduce $v = y'$ as a new variable, we have the system

$$y' = v$$

$$v' = -10y - 2v + e^t.$$

It's the same set of equations, but with symbols that are more meaningful in a mechanical context.

4.3 Simultaneous Laplace transforms

Example 3. *Use Laplace transforms to solve the system*

$$y_1' = 3y_1 - 2y_2$$

$$y_2' = -y_1 + 4y_2$$

with initial conditions $y_1(0) = 1$, $y_2(0) = -6$.

Take Laplace transforms of both sides to find

$$sY_1 - 1 = 3Y_1 - 2Y_2$$

$$sY_2 + 6 = -Y_1 + 4Y_2.$$

Rearrange:

$$(s-3)\,Y_1 + 2Y_2 = 1$$
$$Y_1 + (s-4)\,Y_2 = -6.$$

Eliminate Y_2 by adding $(s-4)$ times the first equation to -2 times the second:

$$((s-4)\,(s-3) - 2)\,Y_1 = (s-4) + 12$$

and solve for Y_1,

$$Y_1 = \frac{8+s}{(s^2 - 7s + 10)}.$$

Backsubstitute into the first equation to find Y_2

$$Y_2 = \frac{1}{2}\,(1 - (s-3)\,Y_1) = \frac{17-6s}{s^2 - 7s + 10}.$$

Do partial fraction expansions to reveal the rate constants:

$$Y_1 = \frac{13}{3\,(s-5)} - \frac{10}{3\,(s-2)}$$

$$Y_2 = -\frac{5}{3\,(s-2)} - \frac{13}{3\,(s-5)}$$

and then invert the Laplace transforms to find the solution

$$y_1\,(t) = \frac{13}{3}e^{5t} - \frac{10}{3}e^{2t}$$

$$y_2\,(t) = -\frac{10}{3}e^{5t} - \frac{13}{3}e^{2t}.$$

4.4 Decoupling through a change of variables

Example 4. *Show that the change of variables*

$$z_1 = y_1 + y_2$$

$$z_2 = y_1 - 2y_2$$

decouples the system

$$y_1' = 3y_1 - 2y_2$$
$$y_2' = -y_1 + 4y_2.$$

Write z_1' out using the equations for y_1 and y_2:

$$z_1' = (y_1 + y_2)'$$

$$
\begin{aligned}
&= (3y_1 - 2y_2) + (-y_1 + 4y_2) \\
&= 2\,(y_1 + y_2) \\
&= 2z_1.
\end{aligned}
$$

Repeat for z_2:

$$
\begin{aligned}
z_2' &= (y_1 - 2y_2)' \\
&= (3y_1 - 2y_2) - 2\,(-y_1 + 4y_2) \\
&= 5y_1 - 10y_2 \\
&= 5z_2.
\end{aligned}
$$

The equations $z_1' = 2z_1$ and $z_2' = 5z_2$ are uncoupled, and their solutions have rate constants 2 and 5, respectively.

4.5 Simultaneous Laplace transforms using Mathematica

Example 5. *Use Mathematica to solve the linear system in the previous example.* A screenshot from a *Mathematica* notebook is shown in figure 3

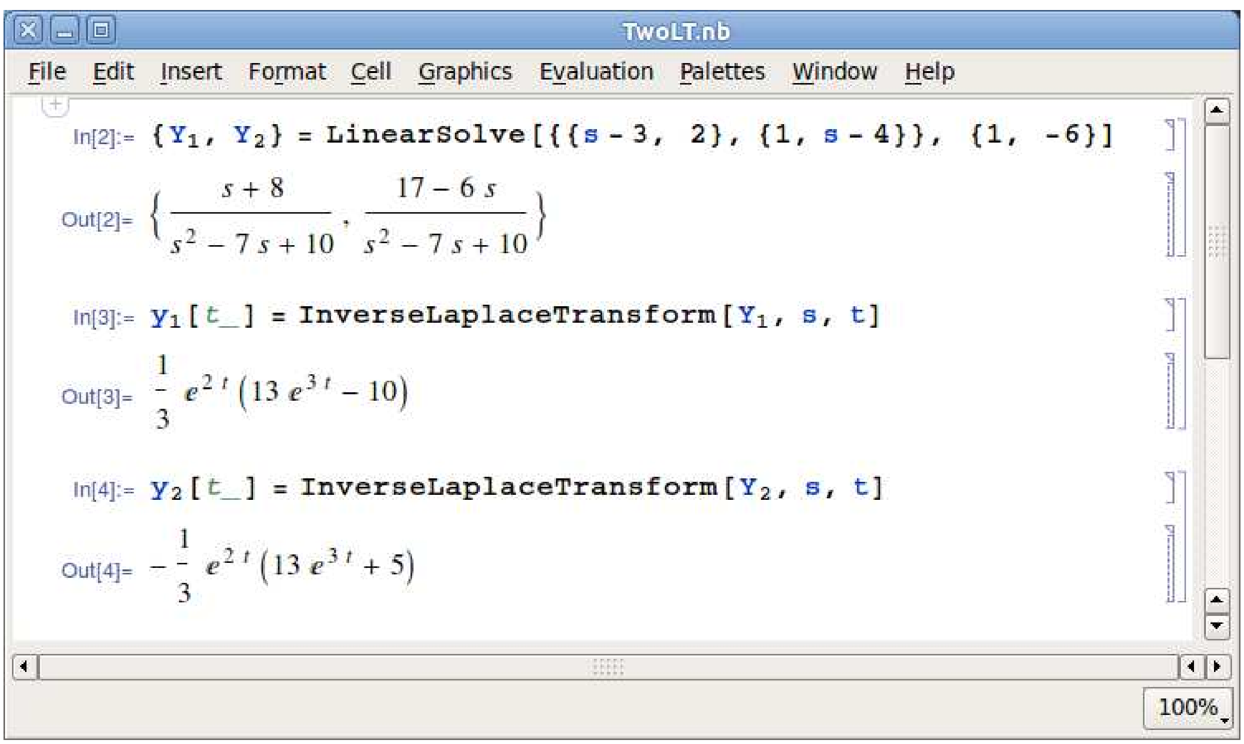

Figure 3: Screenshot of *Mathematica* solution of the problem in example 5.