

Separation of variables with Mathematica

Solve the equation

$$\frac{dy}{dx} = -x(1 + y^2)$$

with initial conditions $y(2)=1$. This is separable with $g(x) = -x$ and $h(y) = \frac{1}{1+y^2}$.

Start by defining *Mathematica* functions for g and h . Note the underscores; they matter.

$$g[x_] = -x$$

$$-x$$

$$h[y_] = 1 / (1 + y^2)$$

$$\frac{1}{y^2 + 1}$$

Define variables for use when we set the initial conditions.

$$x0 = 2$$

$$2$$

$$y0 = 1$$

$$1$$

Doing the integrals

$$G[x_] = Integrate[g[x], x]$$

$$-\frac{x^2}{2}$$

$$H[y_] = Integrate[h[y], y]$$

$$\tan^{-1}(y)$$

Solving the algebraic equation

Solve for y as a function of x . Note the addition of the constant of integration.

$$\text{algSln}[x_, C_] = Solve[H[y] == G[x] + C, y]$$

$$\left\{ \left\{ y \rightarrow \tan\left(\frac{1}{2}(2C - x^2)\right) \right\} \right\}$$

Use the result of the algebraic solution to define a function (a one-parameter family depending on C , actually) representing the solution of the ODE.

```
odeSoln[x_, C_] = y /. algSoln[x][[1]]
```

$$\tan\left(\frac{1}{2}(2C - x^2)\right)$$

Solving for C

Plug the initial conditions into the ODE solution and solve for C.

```
cSoln = Solve[odeSoln[x0, C] == y0, C]
```

```
Solve::ifun :
```

```
Inverse functions are being used by Solve, so some solutions may not be found;
use Reduce for complete solution information. >>
```

$$\left\{\left\{C \rightarrow 2 + \frac{\pi}{4}\right\}\right\}$$

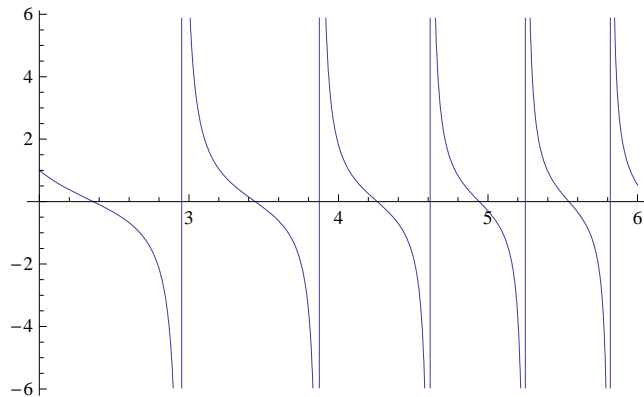
Let's return to the issue of the scary warning message later. For now, forge boldly ahead, substituting $C=2+\frac{\pi}{4}$ into the ODE solution.

```
ivpSoln[x_] = odeSoln[x] /. cSoln[[1]]
```

$$\tan\left(\frac{1}{2}\left(2\left(2 + \frac{\pi}{4}\right) - x^2\right)\right)$$

Here's what the solution looks like

```
Plot[ivpSoln[x], {x, 2, 6}]
```



The solution blows up just below $x=3$. (You should calculate exactly where!) I've continued the graph after that point to highlight the singularity, but **the behavior of the solution to the IVP is undefined once it "blows up."**

Checking the solution

You can use *Mathematica* to simplify checking the solution. Plug in, and let *Mathematica* do the calculus and algebra:

```
LHS = D[ivpSoln[x], x]
```

$$-x \sec^2\left(\frac{1}{2}\left(2\left(2 + \frac{\pi}{4}\right) - x^2\right)\right)$$

$$\text{RHS} = -x (1 + \text{ivpSoln}[x]^2)$$

$$-x \left(\tan^2 \left(\frac{1}{2} \left(2 \left(2 + \frac{\pi}{4} \right) - x^2 \right) \right) + 1 \right)$$

$$\text{RHS} == \text{LHS}$$

$$-x \left(\tan^2 \left(\frac{1}{2} \left(2 \left(2 + \frac{\pi}{4} \right) - x^2 \right) \right) + 1 \right) = -x \sec^2 \left(\frac{1}{2} \left(2 \left(2 + \frac{\pi}{4} \right) - x^2 \right) \right)$$

If you remember your trig identities, you'll see that these are equal. When looking at complicated expressions, the `FullSimplify[]` function is amazingly useful...

$$\text{FullSimplify}[\text{RHS} == \text{LHS}]$$

True

...so the IVP solution is indeed a solution of the ODE. Now check that the IVP solution satisfies the IC:

$$\text{ivpSoln}[x_0] == y_0$$

True

...so our solution passes both the ODE check and the IC check. All done! Well, not quite.

About that warning message

OK, why the scary warning message??? It's because the algebraic equation being solved is

$$\text{odeSoln}[x_0, C] == y_0$$

$$\tan \left(\frac{1}{2} (2C - 4) \right) = 1$$

which we can clean up a little with our friend `FullSimplify[]`

$$\text{FullSimplify}[\text{odeSoln}[x_0, C] == y_0]$$

$$\tan(2 - C) + 1 = 0$$

which has multiple solutions because the tangent function is periodic (with period π). The solution $C = \frac{\pi}{4} + 2$ is the *principal value*, but there are infinitely many other solutions obtained by adding to $\frac{\pi}{4} + 2$ any integer multiple of π .

Taking into account the multiple solutions to the algebraic equation, the IVP solution is actually

$$-\text{Tan}[x^2 / 2 - 2 - \text{Pi} / 4 + n \text{Pi}]$$

$$\tan \left(-\frac{x^2}{2} - n\pi + \frac{\pi}{4} + 2 \right)$$

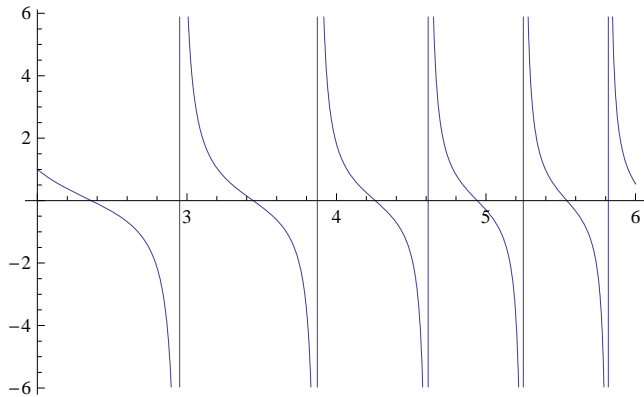
where n is an integer. However, the tangent function is periodic with period π , so the addition of an $n\pi$ phase shift **does not matter**.

Thus the scary warning about inverse function was a false alarm. This will usually be the case for problems of this sort, because we'll be inverting a function to get C , and then putting the value of C right back into the

function. You do things like this all the time in hand calculations, but don't always think about what's happening.

Here's the plot of the solution where we've used $C = 143\pi + 2 + \frac{\pi}{4}$ instead of $C = 2 + \frac{\pi}{4}$. As expected, it is identical to the solution with the principal value of C .

```
Plot[-Tan[x^2/2 - Pi/4 - 2 + 143 Pi], {x, 2, 6}]
```



OK, **now** we're all done.