

Fourier series approximations to a square wave

The square wave is the 2π -periodic extension of the function

$$\begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

In class we showed it can be represented as a Fourier series

$$\sum_{m=1}^{\infty} B_m \sin m x$$

where

$$B_m = \begin{cases} \frac{4}{\pi m} & m \text{ odd} \\ 0 & m \text{ even.} \end{cases}$$

We also showed that this could be written as

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{2m+1}$$

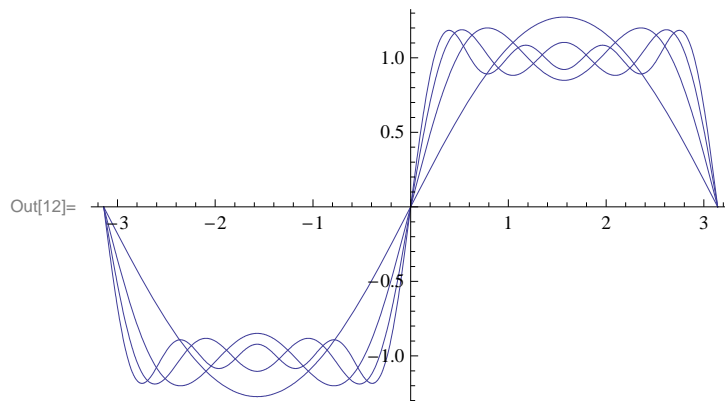
in which form the even terms have been automatically dropped.

We can write this sum as a *Mathematica* function

```
In[3]:= f[M_, x_] := 4 / Pi Sum[Sin[(2 m + 1) x] / (2 m + 1), {m, 0, M}]
```

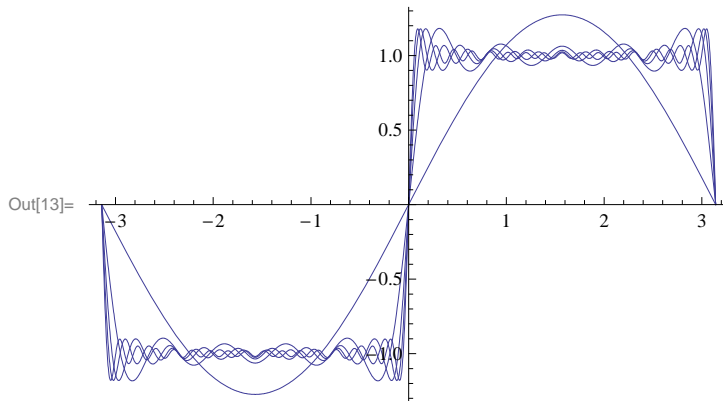
Plot a few partial sums

```
In[12]:= Plot[Table[f[M, x], {M, 0, 3}], {x, -Pi, Pi}]
```



Take more terms

```
In[13]:= Plot[Table[f[M, x], {M, 0, 16, 4}], {x, -Pi, Pi}]
```

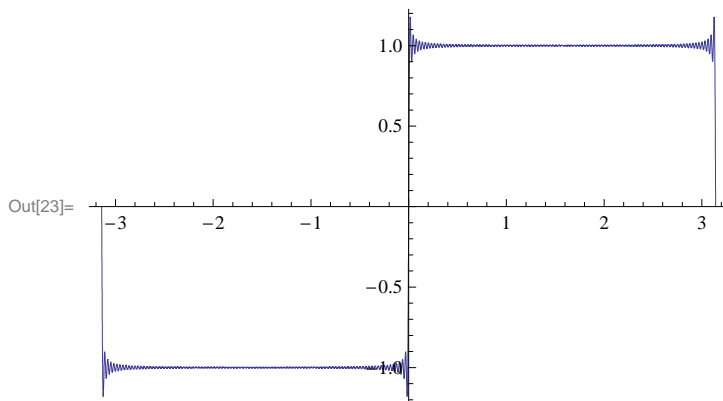


It's looking more and more like a square wave, but notice the persistent overshoot at the end.

Plot a series with a very large number of terms

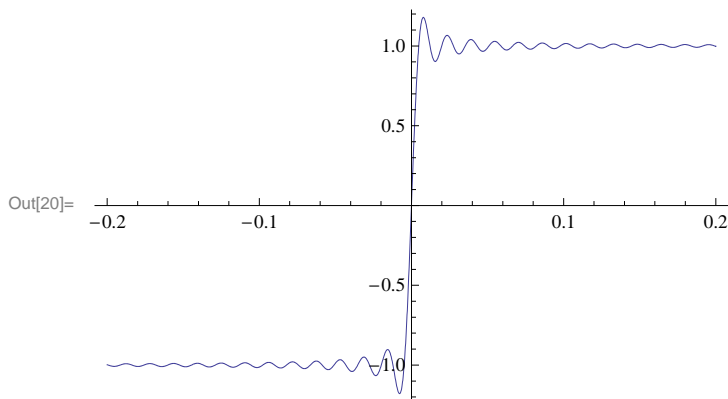
Notice the overshoot at the jump doesn't go away as the number of terms increases. This is the Gibbs phenomenon, and is a generic feature of Fourier series representations of discontinuous functions.

```
In[23]:= Plot[f[100, x], {x, -Pi, Pi}, PlotPoints -> 400]
```



Zoom in on the wiggles near zero

```
In[20]:= Plot[f[200, x], {x, -2/10, 2/10}]
```



Here's a neat trick

To clean the wiggles and reduce the overshoot, apply a low-pass filter to the Fourier coefficients.

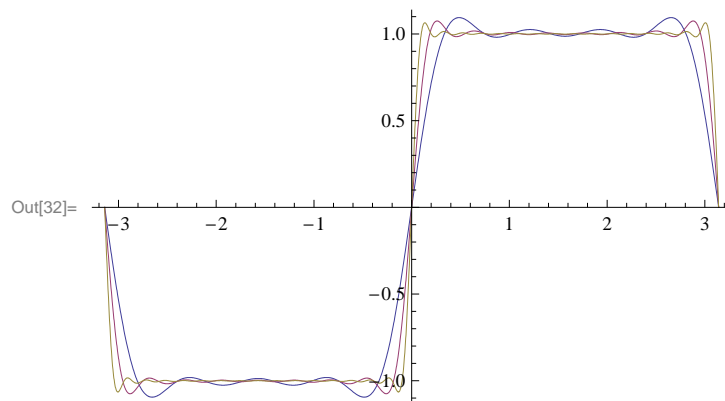
```
In[28]:= filter[ $\omega$ _] = 1 -  $\omega$ ^2
```

```
Out[28]= 1 -  $\omega$ ^2
```

```
In[29]:= g[M_, x_] := 4 / Pi Sum[Sin[(2 m + 1) x] / (2 m + 1) filter[m/M], {m, 0, M}]
```

The series approaches the square wave more slowly, but more smoothly.

```
In[32]:= Plot[{g[4, x], g[8, x], g[16, x]}, {x, -Pi, Pi}]
```



With a large number of terms, the fit is very good.

```
In[31]:= Plot[g[100, x], {x, -Pi, Pi}, PlotPoints -> 400]
```

