Linear and Bilinear Functionals

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1 Linear functionals

Definition 1. A *real linear functional* is a mapping $l(v) : V \to \mathbb{R}$ that is linear with respect to its argument $v \in V$. That is, it must satisfy the properties

$$l(u+v) = l(u) + l(v)$$
$$l(\alpha v) = \alpha l(v)$$

for all $u, v \in V$ and $\alpha \in \mathbb{R}$.

These are the same linearity properties used in the definition of a linear mapping; indeed, a real linear functional is simply a special case of a linear mapping where the range space is $\mathbb{R}$. Complex-valued linear functionals can also be defined. We'll almost always consider only real linear functionals, and will often simply call them "linear functionals." Here are some examples (you should be able to verify each statement).

1. $l(u) = \int_0^1 u(x)\, dx$ is a LF on the space of integrable functions on $[0, 1]$.
2. $l(u) = u(\frac{1}{4})$ is a LF on the space C^0 of continuous functions on $[0, 1]$.
3. $l(u) = \max_x u(x)$ is **not** a LF on C^0
4. Multiplication by a 1-by-N matrix is a LF on $\mathbb{R}^N$.

1.1 Matrix representation of linear functionals

If V is finite dimensional, then upon choosing a basis for V any $l : V \to \mathbb{R}$ can be represented as a $1 \times \dim(V)$ matrix. You can construct the matrix by computing the action of l on each basis vector.

Example 2. We will find the matrix representation for the functional $l(u) = \int_{-1}^1 u(x)\, dx$ defined on P^2, the space of quadratic polynomials. Any vector $u(x) \in P^2$ can be written as a LC of basis vectors, $u(x) = \sum_{i=0}^2 u_i \phi_i(x)$. Apply l to this vector and find

$$l(u) = \sum_{i=0}^{2} u_i \int_{-1}^{1} \phi_i(x)\, dx.$$

Define the 1×3 matrix A with elements $A_{1j} = \int_{-1}^1 \phi_j(x)\, dx$. Then $l(u) = A\mathbf{u}$ where $\mathbf{u}$ is the vector of coefficients in the representation of u in the basis $\{\phi_i\}$. Notice that the matrix A depends on the basis used. For example, if we use the Vandermonde basis $\{1, x, x^2\}$ we have

$$A = \begin{bmatrix} 2 & 0 & \frac{2}{3} \end{bmatrix}$$

whereas if we use the Legendre basis $\{1, x, \frac{3}{2}\left(x^2 - 1\right)\}$ we have

$$A = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}.$$

Notice also that multiplication of a $1 \times N$ matrix A with a vector u is equivalent to the Euclidean inner product between the first (and only) row of A and u. In finite dimensions, then, we can define $\mathbf{c} = \mathrm{row}_1(A)$ and write the action of l as

$$l(u) = \mathbf{c}^{\mathbf{T}}\mathbf{u} = \mathbf{c} \cdot \mathbf{u} = \mathbf{u}^{\mathbf{T}}\mathbf{c}.$$

Again, the elements of $\mathbf{c}$ and $\mathbf{u}$ will depend on the basis used.

2 Bilinear functionals

Bilinear functionals can then be defined in terms of linear functionals: A *real bilinear functional* maps an ordered pair of vectors to the reals, that is a real linear functional with respect to each argument.

Definition 3. A *real bilinear functional* is a mapping $a(u, v) : (u \in U, v \in V) \to \mathbb{R}$ obeying the properties

$$a(u + w, v) = a(u, v) + a(w, v) \quad \forall u, w \in U,\ v \in V$$

$$a(u, v + w) = a(u, v) + a(u, w), \quad \forall u \in U,\ v, w \in V$$

$$a(\alpha u, v) = \alpha a(u, v) \quad \forall \alpha \in \mathbb{R}, u \in U, v \in V$$

$$a(u, \alpha v) = \alpha a(u, v) \quad \forall \alpha \in \mathbb{R}, u \in U, v \in V.$$

Note that these are two of the properties defining a real inner product. Therefore, every real inner product is a real bilinear functional; however, it is not the case that every bilinear functional is an inner product.

Definition 4. A *symmetric bilinear functional* is a bilinear functional such that $a(u, v) = a(v, u)$.

Examples of symmetric BFs include $a(u, v) = \int_0^1 u'(x)v'(x)\,dx$ and $a(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathbf{T}}A\mathbf{y}$ where A is any symmetric matrix.

Definition 5. A *positive definite bilinear functional* is a bilinear functional such that $a(u, u) \geq 0$ for all u, with the equality obtained only when $u = 0$.

A real inner product is a symmetric positive definite bilinear functional. "Symmetric positive definite" appears often enough to have its own acronym: SPD.

2.1 Matrix representations of bilinear functionals

When working in finite dimensions we can represent the arguments $u \in U$ and $v \in V$ in bases. Let $\{\psi_j\}_{j=1}^{N}$ be a basis for V and $\{\phi_i\}_{i=1}^{M}$ be a basis for U. Applying the bilinear functional to u and v and making use of bilinearity gives us

$$a(u, v) = \sum_{i=1}^{M} \sum_{j=1}^{N} v_j u_i a(\phi_i, \psi_j).$$

Defining the $M \times N$ matrix A with elements $A_{ij} = a(\phi_i, \psi_j)$, we recognize that

$$a(u, v) = \mathbf{u}^{\mathbf{T}}A\mathbf{v}.$$

As with linear functionals, the matrix representation will depend on the bases used. If the same bases are used for u and v, and if the functional a is symmetric, then its matrix representation will be symmetric. Notice that a symmetric functional can be represented by a non-symmetric matrix if different bases are chosen for U and V.

2.2 Positive definiteness and eigenvalues

A positive definite bilinear functional has $a(u,u) \geq 0 \; \forall u \neq 0$. Assuming a matrix representation, this condition becomes

$$\mathbf{u}^{\mathbf{T}} A \mathbf{u} \geq 0 \quad \forall \mathbf{u} \neq 0.$$

When a is also symmetric a symmetric matrix representation is possible. Recall that a symmetric matrix has real eigenvalues λ and orthogonal eigenvectors $\mathbf{K}$. Write $\mathbf{u}$ as a LC of the eigenvectors, $\mathbf{u} = \sum_j u_j \mathbf{K_j}$. In this basis, the inequality above is then

$$\sum_i u_i \mathbf{K_i^T} \sum_j u_j A \mathbf{K_j} \geq 0 \quad \forall \mathbf{u} \neq \mathbf{0}$$

which after using $A\mathbf{K} = \lambda \mathbf{K}$ and the orthogonality of the eigenvectors can be reduced to

$$\sum_i \lambda_i u_i^2 \geq 0 \quad \forall \mathbf{u} \neq \mathbf{0}.$$

This will be true iff we have $\lambda_i > 0$ for all i. Thus, a positive definite bilinear functional has positive definite eigenvalues.

3 Quadratic forms

Definition 6. A *quadratic form* is a sum of a linear functional and a symmetric bilinear functional, both applied to the same argument. A standard form is

$$q(u) = l(u) + \frac{1}{2} a(u,u).$$

The factor of $\frac{1}{2}$ is not required (it could be absorbed into the definition of a), but is conventional for reasons that will soon be clear.

An ordinary quadratic function of $x \in \mathbb{R}$ is the simplest case of a quadratic form. In finite dimensions, a quadratic form can be represented as

$$q(\mathbf{u}) = \frac{1}{2} \mathbf{u}^{\mathbf{T}} A \mathbf{u} + \mathbf{u}^{\mathbf{T}} \mathbf{c}.$$

3.1 Linear functionals and inner products

Working in an inner product space V, pick any vector p. The inner product $\langle p, u \rangle$ is a BF of p and u, and is therefore also a LF of u. We can think of any fixed vector p together with a specified inner product as defining a linear functional.

So we can think of inner products as defining linear functionals. Can we go the other direction: can every linear functional acting on V be represented as an inner product with some vector? In finite dimensions, the answer is yes, because of the isomorphism between $1 \times N$ matrices and row vectors. In infinite dimensions, the question is trickier. The conditions under which a linear functional can be represented as an inner product are given by a famous theorem from functional analysis, the Riesz Representation Theorem. When it exists, the vector corresponding to a functional is sometimes called the Riesz representation of that functional.

Example 7. In the example above, we considered the functional $l(u) = \int_{-1}^{1} u(x)\, dx$ on the space P^2. This is simply the unweighted inner product between u and the function $r(x) = 1$, so that

$$l(u) = \langle 1, u \rangle .$$

Exercise 8. Let C^0 be the space of continuous functions on $[-1,1]$ and P^4 be the space of 4-th degree polynomials. We know that P^4 is a subspace of C^0. Define $l : C^0 \to \mathbb{R}$ by $l(u) = u(\frac{1}{3})$.

1. Show that $l(u)$ is linear

2. Using the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$, find a 4-th degree polynomial $r(x)$ such that $\langle r, u \rangle = l(u) = u(\frac{1}{3})$ for all $u \in P^4$. **Hint:** Pick a basis B for P^4. Represent r in that basis, then use the fact that if the equation $\langle r, u \rangle = u(\frac{1}{3})$ holds for all $u \in B$, it will hold for all $u \in P^4$. You will probably want to use Mathematica.

3. Verify that $\langle r, u \rangle = u(\frac{1}{3})$ for the polynomial $u(x) = x^2 + 1$.