Dimensional Analysis and Nondimensional Equations

Math 5310 Fall 2010

The heat conduction equation (usually called simply "the heat equation") in a homogeneous 1D medium is

$$\rho c \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

By the end of this course sequence you'll consider this a pretty easy problem. For now, let's see what we can say about the problem without actually solving it.

Ask the question: suppose a body of size ℓ with thermal conductivity κ, specific heat c, and mass density ρ is initially at uniform temperature u_0, and that it is put in contact with a body of a different temperature u_1. Estimate the time t_c needed for the body to cool to $u \approx u_1$.

Estimating solution properties through dimensional analysis

The one thing we know about t_c is that it is measured in units of *time*. Each parameter entering the problem has its own dimensions: some combination of fundamental dimensions such as length (L), time (T), mass (M), temperature (Θ). We use the following notation to describe the dimensions of a parameter:

$$[p] = \text{dimensions of } p.$$

The parameter ℓ is a length, so we can write

$$[\ell] = L.$$

Similarly,

$$[\rho] = ML^{-3}$$
$$[c] = L^2 T^{-2} \Theta^{-1}$$
$$[\kappa] = MLT^{-3}\Theta^{-1}.$$

Notice that $[XY] = [X]\,[Y]$. The cooling time will be some function of these parameters, and that function must have dimension T. It makes no sense to add quantities of different dimensions, so we must have

$$t_c = \Omega \ell^\alpha \rho^\beta c^\gamma \kappa^\delta$$

where Ω is some dimensionless number. Then

$$[t_c] = T = [\ell]^\alpha\,[\rho]^\beta\,[c]^\gamma\,[\kappa]^\delta$$
$$= L^\alpha M^\beta L^{-3\beta} L^{2\gamma} T^{-2\gamma} \Theta^{-\gamma} M^\delta L^\delta T^{-3\delta} \Theta^{-\delta}$$
$$\alpha - 3\beta + 2\gamma + \delta = 0$$
$$\beta + \delta = 0$$
$$-2\gamma - 3\delta = 1$$
$$\gamma + \delta = 0$$

$$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -3 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\delta = -1$$
$$\gamma = -\frac{1}{2}(1+3\delta) = 1$$
$$\beta = -\delta = 1$$
$$\alpha = -\delta - 2\gamma + 3\beta = 1 - 2 + 3 = 2.$$

Therefore,

$$t_c = \left(\frac{\ell^2 \rho c}{\kappa}\right)\Omega.$$

What does this do for us?

- We don't know Ω, but if we *assume* $\Omega \approx 1$, then we can use this for a quick-and-dirty estimate of t_c. Typically, such estimates are accurate to an order of magnitude or so.
- We now know how the cooling time will change as we change the problem's parameters. For example, doubling the body's size ($\ell \to 2\ell$) while holding everything else constant increases the cooling time by a factor of 4.

Furthermore, we can use this to eliminate all dimensioned parameters from the equation.

Reducing equations to nondimensional form

We now know that $\left[\frac{\ell^2\rho c}{\kappa}\right] = T$. Define the new, dimensionless variables

$$\xi = \frac{x}{\ell}$$
$$\tau = \frac{\kappa t}{\ell^2 \rho c}$$

so that

$$x = \xi\ell, \quad dx = \ell\, d\xi$$
$$t = \tau\ell^2\rho c/\kappa, \quad dt = \ell^2\rho c/\kappa\, d\tau.$$

Then the heat equation is

$$\left(\frac{\kappa}{\rho c \ell^2}\right) \times \rho c \frac{\partial u}{\partial \tau} = \left(\frac{1}{\ell^2}\right) \times \kappa \frac{\partial^2 u}{\partial \xi^2},$$

or after cancellation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2}.$$

We've eliminated all parameters from the problem; if we write $w = u/u_0$ where u_0 is some reference temperature, then the equation is completely dimensionless,

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \xi^2}.$$

If we solve this problem, then we can use that solution for any body (having the same shape) of any length and material properties, simply by rescaling the time, length, and temperature.

Dimensions of common physical properties

Thermal conductivity	[Energy] $L^{-1}T^{-1}\Theta^{-1}$
Energy	ML^2T^{-2}
Force	MLT^{-2}
Velocity	LT^{-1}
Specific heat	[Energy] $M^{-1}\Theta^{-1}$
Density	ML^{-3}
Pressure	$ML^{-1}T^{-2}$