Math 5310 - Fall 2009

Quiz #5

Suppose you want to approximate the function $f(x) = x^3$ by a quadratic function $u(x) = a_0 + a_1 x + a_2 x^2$ on the interval $[0,1]$. Recall from class that the least-squares quadratic approximation to f is the orthogonal projection of f onto the space $\mathcal{P}^2$. Using the method of orthogonal projection along with the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

derive a linear system of equations for the coefficients a_i, $i = 0, 1, 2$. You need not solve that system of equations.

Solution

The projection of f onto the subspace $\mathcal{P}^2$ will be the function $u \in \mathcal{P}^2$ such that

$$\langle v, u - f \rangle = 0 \quad \forall v \in \mathcal{P}^2.$$

This equation will be true $\forall v \in \mathcal{P}^2$ if it is true on any basis of $\mathcal{P}^2$. We can use the Vandermonde basis $\{1, x, x^2\}$, the members of which we can write as $\phi_i(x) = x^i$. Because $u \in \mathcal{P}^2$, we also know that u is a linear combination of these basis functions,

$$u(x) = \sum_{j=0}^{2} u_j \phi_j(x).$$

Using the distributivity of inner products over addition, the condition for orthogonal projection becomes

$$\langle \phi_i, u \rangle = \langle \phi_i, f \rangle \quad \text{for } i = 0, 1, 2.$$

Using the representation of u in the basis functions ϕ_i gives us

$$\sum_{j=0}^{2} \langle \phi_i, \phi_j \rangle u_j = \langle \phi_i, f \rangle \quad \text{for } i = 0, 1, 2.$$

This is a system of linear equations $A\mathbf{u} = \mathbf{f}$ where $A_{ij} = \langle \phi_i, \phi_j \rangle$ and $f_i = \langle \phi_i, f \rangle$. We can compute the matrix and vector elements by integration

$$A_{ij} = \int_0^1 x^{i+j}\, dx = \frac{1}{i+j+1}$$

and

$$f_i = \int_0^1 x^{i+3}\, dx = \frac{1}{i+4}.$$

Therefore the system is

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{bmatrix}.$$

Though not necessary for the quiz, the solution is $\mathbf{u} = \begin{bmatrix} \frac{1}{20} & -\frac{3}{5} & \frac{3}{2} \end{bmatrix}^T$. The resulting function,

$$u(x) = \frac{1}{20}\left[1 - 12x + 30x^2\right]$$

is plotted against $f(x) = x^3$ in the figure below