Math 3351 - Fall 2009

Final Exam, Take-Home Part

Due: 5 PM Wednesday 16 Dec 2009.

1. You may work together, though you must turn in your own writeup.
2. You may use any notes, books, or other references.
3. You may use a computer
4. The take-home part is worth 40% of the total final exam grade

Problems 1 and 2 will refer to a function $f(\theta)$. In the table below, you'll find a polynomial $p(z)$ indexed by your name. The function $f(\theta)$ is that polynomial evaluated at $z = 2\theta/\pi$. That is, $f(\theta) = p(\frac{2\theta}{\pi})$.

Out[70]=

O. Akogh	$4z^4 - 14z^3 + 17z^2 - 8z$
J. Bailey	$3z^4 - 8z^3 + 9z^2 - 6z$
B. Bryant	$z^4 - 6z^3 + 14z^2 - 14z$
D. Chamness	$2z^4 - 7z^3 + 10z^2 - 7z$
R. Contreras	$z^4 - 3z^2 + 2z$
K. Davis	$2z^4 - 6z^3 + 8z^2 - 6z$
D. Dominguez	$4z^4 - 12z^3 + 11z^2 - 2z$
S. Doss	$3z^4 - 8z^3 + 7z^2 - 2z$
K. Engstrom	$-2z^4 + 11z^3 - 22z^2 + 19z$
S. Grenadier	$-2z^4 + 6z^3 - 8z^2 + 6z$
N. Hamilton	$2z^4 - 8z^3 + 16z^2 - 16z$
J. Haning	$z^4 - 8z^3 + 14z^2 - 8z$
J. Hazelwood	$2z^4 - 10z^3 + 20z^2 - 18z$
J. Hessou	$-2z^3 + 9z^2 - 12z$
N. Igunbor	$-3z^4 + 10z^3 - 16z^2 + 14z$
P. Kelly	$z^4 - 5z^3 + 6z^2 - z$
M. MacHen	$-4z^4 + 15z^3 - 24z^2 + 19z$
F. Madero	$-3z^4 + 12z^3 - 19z^2 + 14z$
R. McClelland	$z^3 + z^2 - 5z$
S. Ouandji	$-z^3 + 4z^2 - 5z$
M. Pare	$-3z^4 + 15z^3 - 23z^2 + 13z$
J. Peters	$2z^4 - 4z^3 + 4z^2 - 4z$
J. Rodriguez	$-3z^4 + 8z^3 - 7z^2 + 2z$
S. Rodriguez	$4z^4 - 20z^3 + 35z^2 - 26z$
C. Veloz	$3z^4 - 14z^3 + 24z^2 - 18z$
J. White	$-4z^3 + 12z^2 - 12z$
J. Wood	$-4z^4 + 17z^3 - 24z^2 + 13z$
J. Yang	$-z^4 + 2z^3 - 4z^2 + 6z$

Problem 1

Assuming that your function $f(\theta)$ can be written as a series

$$f(\theta) = \sum_{n=1}^{\infty} B_n \sin\left((2n-1)\,\theta\right)$$

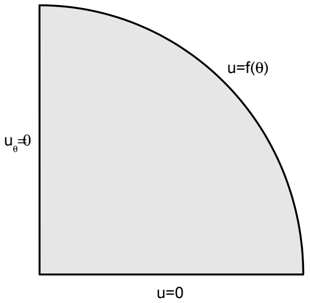
derive a formula for the coefficients B_n. Compute a six-term partial sum of the series, and plot the partial sum and the exact $f(\theta)$ over the interval $[0, \frac{\pi}{2}]$. Verify that the partial sum is a good approximation to the exact function.

Problem 2

Use separation of variables and superposition to solve Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in polar coordinates on the quarter-disk $r \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ shown below.



The boundary conditions are

- $u = 0$ along the line $\theta = 0$
- $u_\theta = \frac{\partial u}{\partial \theta} = 0$ along the line $\theta = \frac{\pi}{2}$
- $u = f(\theta)$ along the outer boundary at $r = 1$
- At the origin, $|u| < \infty$

1. Use separation of variables, superposition, and Fourier analysis to derive an infinite series solution to the problem.
    (a) Explain how certain coefficients in the general solution can be determined to be zero
    (b) Explain how the values of the separation constants are determined
    (c) Explain how the nonzero coefficients can be determined from the function $f(\theta)$.
2. Compute the necessary integrals to find a six-term partial sum approximation to the solution.
3. Create a surface plot of the six-term partial sum. You can do surface plots in polar coordinates using the Mathematica function RevolutionPlot3D.

Problem 3

In polar coordinates with rotational symmetry, the heat conduction equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}.$$

Let the boundary conditions be $u(1,t) = 0$ and $|u(0,t)| < \infty$, and let the initial conditions be $u(r,0) = u_0(r) = 1 - r^2$. This problem will use Bessel functions; you'll need to use either *Mathematica* or a reference book such as *Handbook of Mathematical Functions* to find various integrals and roots of Bessel functions.

1. Separate variables to derive ordinary differential equations for $R(r)$ and $T(t)$. Show that $T(t)$ is an exponential function. Show that $R(r)$ is a Bessel function of order zero (*i.e.*, show that the differential equation for R is Bessel's equation). Explain why the Bessel function of the second kind, Y_0, does not enter the solution.

2. Let $j_{0,n}$ be the n-th root of $J_0(x)$. What is the integral $\int_0^1 J_0(j_{0,n}x)\, J_0(j_{0,m}x)\, x\, dx$ when $m \neq n$? What is the value of this integral when $n = m$?

3. Approximate the solution by one term
$$u_1(r,t) = AJ_0(kr)e^{-k^2 t}$$
and use the boundary conditions and initial conditions to determine the coefficient A and the separation constants k. Use the *smallest* allowed value of k. Be sure to use the correct weight function in any inner products.

4. At $t = 0$, plot the one-term approximation $u_1(r,0)$ and the exact initial condition.

5. Suppose you were to extend the approximation to two terms
$$u_2(r,t) = A_1 J_0(k_1 r)e^{-k_1^2 t} + A_2 J_0(k_2 r)e^{-k_2^2 t}.$$
What would be the value of k_2? At $t = \frac{1}{10}$ compute the ratio
$$\frac{e^{-k_2^2 t}}{e^{-k_1^2 t}}.$$
What does this calculation tell you about the accuracy and usefulness of the one-term approximation?