Math 3351 - Fall 2009

Test #1

No calculators.

1. Define the matrix
$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$
   (a) Compute the eigenvectors and eigenvalues of A
   (b) Find a general solution to the differential equation
$$\mathbf{X}' = A\mathbf{X}$$

Prob 1: solution

The eigenvectors of A are the nontrivial solutions $\mathbf{K}$ to the equation $(A - \lambda I)\,\mathbf{K} = 0$. Nontrivial solutions exist if and only if $\det(A - \lambda I) = 0$. The determinant is

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1$$

$$= \lambda^2 + 4\lambda + 3$$

$$= (\lambda + 3)(\lambda + 1).$$

So nontrivial solutions exist when $\lambda = -1$ or $\lambda = -3$. These are the eigenvalues. Having from the eigenvalues, the eigenvectors are obtained by solving $(A - \lambda I)\,\mathbf{K} = 0$.

Eigenvalue $\lambda = 1$:

Solve the linear system

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Gaussian elimination produces a row of zeros (this should not be a surprise, as we know the matrix $A - \lambda I$ is singular when λ is an eigenvalue). The remaining equation is $-x + y = 0$, which has infinitely many solutions: any vector $\begin{bmatrix} x & y \end{bmatrix}^T$ for which $x = y$ will be a solution. Pick $\mathbf{K_1} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ as a representative solution.

Eigenvalue $\lambda = 3$:

Now we solve

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Any vector for which $x = -y$ is a solution. Pick $\mathbf{K_2} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

We can now use the solution to the eigenvalue problem to solve the ODE $\mathbf{X}' = A\mathbf{X}$. The general solution has the form

$$\mathbf{X}(t) = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t}.$$

To verify this: differentiate $\mathbf{X}$ to find $\mathbf{X}' = c_1 \lambda_1 \mathbf{K_1} e^{\lambda_1 t} + c_2 \lambda_2 \mathbf{K}_2 e^{\lambda_2 t}$, which can be rewritten as

$$\mathbf{X}' = c_1 A \mathbf{K}_1 e^{\lambda_1 t} + c_2 A \mathbf{K}_2 e^{\lambda_2 t}.$$

But this is simply $\mathbf{X}' = A\mathbf{X}$. Because the two terms are linearly independent, we can solve for c_1 and c_2 given any initial conditions.

2. The square wave is a periodic extension of the function

$$f(t) = \begin{cases} 0 & -\pi \le t < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} \le t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le t < \pi \end{cases}$$

   (a) Without doing any calculations, you can conclude that the Fourier series for this function has no sine terms. Why?

   (b) Compute the Fourier coefficients for $f(t)$.

Solution:

The function has even symmetry about $t = 0$. The sine has odd symmetry about $t = 0$. The coefficients

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

are therefore all zero.

The coefficients of the cosine terms are

$$A_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos mt \, dt$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos mt \, dt$$

$$= \frac{1}{m\pi} \sin mt \Big|_{-\pi/2}^{\pi/2} = \begin{cases} 1 & m = 0 \\ \frac{-2}{m\pi}(-1)^{(m+1)/2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases} .$$

The Fourier series is then $f(t) = \frac{1}{2} + \frac{2}{\pi}\cos t - \frac{2}{3\pi}\cos 3t + \frac{2}{5\pi}\cos 5t - \frac{2}{7\pi}\cos 7t + \cdots$.

3. Solve the system of linear equations

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

Solution: Form the augmented system

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 4 & 0 \end{bmatrix}$$

and perform Gaussian elimination. Start with $R_3 - 2R_1 \to R_3$,

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & -2 & 0 & -6 \end{bmatrix}$$

which zeros the 3,1 entry. To zero the 2,1 entry, do $R_2 - 2R_1 \to R_2$,

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & 0 & -6 \end{bmatrix}$$

We've zeroed the subdiagonal entries in the first column. Now do the same for the second column, by doing $R_3 - \frac{2}{3}R_2 \to R_3$,

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 2 & -2 \end{bmatrix} .$$

With the system in upper triangular form we can backsubstitute. The last row gives $z = -1$. The second row gives $-3y + 3 = -6$ or $y = 3$. Finally, the first row gives $x + 6 - 2 = 3$ or $x = -1$. All done.

4. Let m and n be integers. Show that $\cos mx$ and $\cos nx$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

when $m \neq n$. You might find useful the identities $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ and $\sin(a \pm b) =$

$\cos a \sin b \pm \sin a \cos b$.**Solution:**

As a preliminary step, add the equations

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

and

$$\cos(a-b) = \cos a \cos b + \sin a \sin b,$$

then divide by two to derive the identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right].$$

This identity lets us rewrite

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

as

$$\frac{1}{2}\int_{-\pi}^{\pi} \left[\cos\left((m+n)\,x\right) + \cos\left((m-n)\,x\right)\right] \, dx.$$

Do the integral to find

$$\langle \cos mx, \cos nx \rangle = \frac{1}{2}\left[\frac{1}{m+n}\sin\left((m+n)\,x\right) + \frac{1}{m-n}\sin\left((m-n)\,x\right)\right]\Bigg|_{-\pi}^{\pi}.$$

Notice there will be trouble when $m = n$, but here we're interested in the case $m \neq n$. The sine of any integer multiple of π is zero, so both terms on the RHS are zero. Therefore, $\cos mx$ and $\cos nx$ are orthogonal when $m \neq n$.

5. Let $\{\phi_0(x), \phi_1(x), \phi_2(x) \cdots\}$ be an infinite set of functions that are orthogonal with respect to some inner product $\langle u, v \rangle$. Assume that a function $f(x)$ can be represented as a series

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x).$$

Use the orthogonality of the ϕ's to derive a formula for computation of the coefficients a_n. Solution:

Take inner products of both sides with $\phi_m(x)$:

$$\langle \phi_m, f \rangle = \left\langle \phi_m, \sum_{n=0}^{\infty} a_n \phi_n \right\rangle$$

$$= \sum_{n=0}^{\infty} \langle \phi_m, a_n \phi_n \rangle$$

$$= \sum_{n=0}^{\infty} a_n \langle \phi_m, \phi_n \rangle.$$

(Why could I do these last two steps?)

Now, by hypothesis the functions ϕ are orthogonal, so $\langle \phi_m, \phi_n \rangle = 0$ when $m \neq n$. Therefore, the only term in the sum that survives is $a_m \langle \phi_m, \phi_m \rangle$, leaving us with

$$\langle \phi_m, f \rangle = a_m \langle \phi_m, \phi_m \rangle$$

from which we can compute

$$a_m = \frac{\langle \phi_m, f \rangle}{\langle \phi_m, \phi_m \rangle}.$$