Weakly Perfect Generalized Ordered Spaces

by

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Abstract: A space X is weakly perfect if each closed subset of X contains a dense subset that is a G_δ-subset of X. This property was introduced by Kočinac and later studied by Heath. We provide three mechanisms for constructing ZFC examples of spaces that are weakly perfect but not perfect. Some of our examples are compact linearly ordered spaces, while others are types of Michael lines. Our constructions begin with special subsets of the usual unit interval, e.g., perfectly meager subsets. We conclude by giving a new and strictly internal topological characterization of perfectly meager subsets of $[0,1]$, namely that a topological space X is homeomorphic to a perfectly meager subset of $[0,1]$ if and only if X is a zero-dimensional separable metrizable space with the property that every subset $A \subset X$ contains a countable set B that is dense in A and is a G_δ-subset of X.

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1. Introduction.

Recall that a topological space X is *perfect* if each closed subset of X is a G_δ-subset of X. Over the years, many different generalizations of this notion have appeared. Almost all of them are, for a generalized ordered space X, equivalent to the statement "X is perfect"[BHL]. But there is one very interesting exception, namely the property "weakly perfect" introduced by Kočinac in [K1] and [K2], and studied by Heath [H].

According to Kočinac[K1], a space X is *weakly perfect* if each closed subset C of X contains a set D having:

a) D is a G_δ-subset of X; and
b) the closure of D in X is C.

Kočinac observed [K2] that the original definition is equivalent to the assertion that every open subset U of X is the interior of some F_σ-subset of X.

The usual space of countable ordinals is weakly perfect (see 2.2) so that even among linearly ordered spaces, "weakly perfect" and "perfect" are very different properties. It is harder to see that the two concepts are distinct among really nice spaces, e.g., among Lindelöf spaces or compact Hausdorff spaces. Kočinac [K1] used the set-theoretic principle $\diamondsuit$ to construct a compact Hausdorff space that is weakly perfect but not perfect. Subsequently, R.W. Heath [H] showed that there is a ZFC example of a Lindelöf quasi-developable space that is weakly perfect and not perfect, commented that there is a linearly ordered space that is weakly perfect but not perfect, and constructed a ZFC example of a compact Hausdorff space that is weakly perfect but not perfect. Section 3 of this paper sharpens the examples provided by Kočinac and Heath, providing a family of ZFC examples of compact *linearly ordered* topological spaces that are *hereditarily* weakly perfect but not perfect. We use a point-splitting process to construct a compact LOTS $X(P)$ for each subset $P \subset [0,1]$ and prove that $X(P)$ is weakly perfect but not perfect if and only if P is a subset of $[0,1]$ that is *perfectly meager*, i.e., $P \cap K$ is a first category subset of the subspace K whenever K is a closed, dense-in-self subset of the real line.

The space $X(P)$ contains interesting subspaces, and we study some of them in Sections 4 and 5. Starting with any subspace P of the unit interval, in Section 4 we construct a natural subspace $Y(P) \subset X(P)$ that uses only points of P in its construction (and does not invoke properties of the set $[0,1]-P$ that play an important role in the study of $X(P)$). It is surprising that we can show that $Y(P)$ is weakly perfect if and only if $X(P)$ is weakly perfect. Then, in Section 5, we start with any subset $P \subset [0,1]$ and study certain Michael line constructions that yield natural subspaces $M(P)$ of $X(P)$. We characterize when $M(P)$ is weakly perfect in terms of properties of P. Our characterization resembles, but is strictly weaker than, the definition of perfectly meager. We use the $M(P)$ construction to obtain generalized ordered examples of quasi-developable, hereditarily weakly perfect spaces that are not metrizable. Assuming the existence of an ω_1-scale, we construct such spaces that are Lindelöf and hereditarily weakly perfect, thereby extending certain results of Heath [H] mentioned above. Finally, in Section 6 we combine our results to give a new, internal, and strictly topological characterization of perfectly meager subsets of $[0,1]$, namely:

Proposition: *A topological space X is homeomorphic to a perfectly meager subset of the unit interval $I = [0,1]$ if and only if X is a zero-dimensional, separable metric space and for every $A \subset X$ there is a countable set $B \subset A$ that is dense in A and is a G_δ-subset of X.*

Readers of Sections 4 and 5 will note that the spaces $X(P)$, $Y(P)$, and $M(P)$ studied in this paper are hereditarily weakly perfect whenever they are weakly perfect. Our proofs of that fact depend in central ways on the close relation between our spaces and the unit interval, a totally bounded metric space. We do not know whether the property "weak perfect" is hereditary in arbitrary GO-spaces. It would be enough to show that every dense subspace of a weakly perfect GO-space is weakly perfect.

Recall that a *linearly ordered topological space* (LOTS) is a linearly ordered set $(X, <)$ equipped with the usual open interval topology of $<$. By a *generalized ordered space*, or GO-space, we mean a linearly ordered set equipped with a Hausdorff topology that has a base of open, convex subsets, where we say that a set C is *convex (in X)* provided for any $a < b < c$ in X, if $\{a, c\} \subset C$ then $b \in C$. It is known that the class of GO-spaces coincides with the class of subspaces of linearly ordered topological spaces. It will be important to distinguish between subsets of a space X that are *relatively discrete*, i.e., discrete in their subspace topologies, and those subsets that are both closed and discrete. Other notation and terminology will follow [E] and [L].

We want to thank the referee of this paper for suggestions that significantly shortened several of our proofs.

2. Preliminary results and examples on weakly perfect GO-spaces

It is difficult to find topological properties that follow from the weakly perfect property, even in ordered spaces. However, one can show:

2.1 **Lemma**: *Suppose that X is a weakly perfect GO-space. Then X is first countable.*

Proof: Let X be a weakly perfect GO-space and let $a \in X$. Then the closed set $C = \{a\}$ must contain a dense set D that is a G_δ-subset of X. Then $D = C$, so each point is a G_δ-subset of X. Hence X is first countable. $\square$

It is known [L] that any perfect GO-space is paracompact. The situation for weakly perfect GO-spaces is quite different as can be seen from the fact that the usual space of countable ordinals is weakly perfect. That follows from our next lemma.

2.2 **Lemma**: *Let X be a topological space that is scattered, hereditarily collectionwise Hausdorff, and in which each point is a G_δ-set. Then X is weakly perfect. In particular, any first-countable, scattered GO-space is weakly perfect.*

Proof: Let C be any closed subset of X, and let $D = \{x \in C : \{x\}$ is relatively open in $C\}$. Then D is dense in C because X is scattered. In addition, because X is hereditarily collectionwise Hausdorff, there is a pairwise disjoint collection $\mathcal{U} = \{U(x) : x \in D\}$ of open subsets of X with $x \in U(x)$ and, because points are G_δ-sets, there is a sequence $U(x,n)$ of open subsets of $U(x)$ such that $\bigcap\{U(x,n) : n \geq 1\} = \{x\}$ for each $x \in D$. Let $V(n) = \bigcup\{U(x,n) : x \in D\}$. Then $D = \bigcap\{V(n) : n \geq 1\}$, so D is the required G_δ-subset of X. $\square$

2.3 **Example**: The usual space $[0, \omega_1[$ of countable ordinals is weakly perfect, countably compact, and not perfect or paracompact. The usual ordinal space $[0, \omega_1]$ is not weakly perfect, in the light of (2.1)

2.4 **Remark**: Arhangel'skii and Kočinac asked in [AK] whether the spread of a countably compact, weakly perfect space must be countable. The usual space of countable ordinals provides a negative answer.

2.5 **Remark**: In the light of (2.3), it is unlikely that there is an interesting characterization of paracompactness in weakly perfect GO-spaces that goes beyond repeating the familiar characterization of paracompactness in arbitrary ordered spaces, namely that the space in question does not contain a closed subspace that is a topological copy of a stationary subset of an uncountable regular cardinal κ. It is worth pointing out that, even though weakly perfect GO-spaces are first countable, one must consider stationary sets in cardinals other than $\kappa = \omega_1$ when applying that characterization to weakly perfect GO-spaces, because consistently there are $E(\omega_2)$ sets. Such a set (in its subspace topology from ω_2) would be weakly perfect and would fail to be paracompact, even though it would not contain any stationary subset of ω_1.

2.6 **Example**: The lexicographic square $X = [0,1] \times [0,1]$ is a compact LOTS that is not weakly perfect. To see that X is not weakly perfect, consider the closed subset $C = ([0,1] \times \{0,1\})$ of X. Suppose that $D = \bigcap\{G(n) : n \geq 1\} \subset C$ is a G_δ subset of X whose closure is C. Then D is Čech-complete. Furthermore, because $D \subset C$, the collection $\mathcal{G}(n) = \{H : H \text{ is a convex component of } G(n)\}$ separates points of the subspace D. Thus D is Čech-complete, paracompact, and has a G_δ-diagonal, so that D is completely metrizable [E]. But that is impossible because, by a Baire category argument, C has no such subspace. (See [BLP].) $\square$

The existence of weakly perfect LOTS that are not perfect (e.g., the countable ordinals or the spaces in Section 3, below) suggests two variations on a well-known question from the theory of perfect GO-spaces. First, is it true that every *perfect* GO-space can be topologically embedded in a weakly perfect LOTS? Second, is it true that every weakly perfect GO-space can be topologically embedded in a weakly perfect LOTS? (We say "topologically embedded" to emphasize that there is no requirement that the orders of the original GO-space and the larger LOTS are compatible.)

The following examples show that the X^* and $L(X)$ constructions, the best-known linearly ordered extensions of a GO-space X, might, or might not, embed a perfect GO-space into a weakly perfect LOTS. To describe those two extensions, let $\mathcal{T}$ be the given topology of X and let $\mathcal{I}$ be the usual open interval topology induced on X by the given linear ordering. Define subsets of X as follows:

$R = \{x \in X : [x \rightarrow [\, \in \mathcal{T} - \mathcal{I}\}$,

$L = \{x \in X :]\leftarrow, x] \in \mathcal{T} - \mathcal{I}\}$.

Let $\mathbb{N}$ be the set of all natural numbers and define

$$L(X) = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\})$$

and

$$X^* = (X \times \{0\}) \cup (\{(x, -n) : x \in R,\ n \in \mathbb{N}\}) \cup (\{(x, n) : x \in L,\ n \in \mathbb{N}\}).$$

Order both sets lexicographically and endow each with the open interval topology of the lexicographic order. Then X is naturally homeomorphic to a dense subspace of the LOTS $L(X)$ and to a closed subspace of the LOTS X^*. It is important to note that $L(X)$ is a subset, but not generally a subspace, of X^*. It is easy to verify:

2.7 **Example**: Let X be the Sorgenfrey line. Then $L(X)$ is perfect (because it is separable), but X^* is not weakly perfect. To see that X^* is not weakly perfect, suppose that the closed subset $X \times \{0\}$ contains a dense subset S that is a G_δ-set in X^*, say $S = \bigcap\{G(n) : n \geq 1\}$. Then S would necessarily be a subset of the set of end points of the convex components of the sets $G(n)$, so that S would be countable. But that is impossible because $X \times \{0\}$ is a topological copy of the Sorgenfrey line which is a Baire space, and a dense-in-itself Baire space cannot contain any dense, countable G_δ-set. $\square$

2.8 **Example**: Let Y be the GO-space obtained by isolating every limit ordinal in the usual space $[0, \omega_1[$. Then Y is metrizable and hence perfect, so that Y^* is also metrizable. But $L(Y)$ is a copy of the usual space of countable ordinals. Thus $L(Y)$ is weakly perfect but not perfect.

2.9 **Example**: Let $Z = \mathbb{R} \times [0, 1[$, where $\mathbb{R}$ is the set of all real numbers, and modify the lexicographic order topology by declaring $[(x, 0), \rightarrow [$ to be open in Z for each $x \in \mathbb{R}$. Then Z is a topological sum of copies of the usual space $[0, 1[$, so that Z is metrizable. Hence so is Z^*. The space $L(Z)$ is the lexicographically ordered set $Z \cup \{(x, -1) : x \in \mathbb{R}\}$ with the usual open interval topology. To see that $L(Z)$ is not weakly perfect, consider the closed subset $C = \{(x, -1) : x \in \mathbb{R}\}$ of $L(Z)$. This subspace is homeomorphic to the usual Sorgenfrey line, so it is a Baire space. If $L(Z)$ were weakly perfect, there would be a G_δ-subset $S \subset L(Z)$ that is a dense subset of C. As in (2.7), the set S would be countable, and that is impossible because the Sorgenfrey line has no dense, countable G_δ-subsets. $\square$

2.10 **Example**: Let W be the topological sum of the spaces X and Z from Examples 2.7 and 2.9, ordered in such a way that X precedes Z. Then W is perfect and neither W^* nor $L(W)$ is weakly perfect. Because of well-known minimality properties of the construction of W^* and $L(W)$, it follows that the perfect GO-space W cannot embed as a closed, or dense, subset of any weakly perfect LOTS whose order extends the given ordering of W. However, that does not answer the question above about topological embedding in a weakly perfect LOTS. $\square$

3. A compact, hereditarily weakly perfect LOTS that is not perfect.

In this section we will describe a point-splitting process that always constructs compact linearly ordered spaces, and we will give necessary and sufficient conditions in Theorem 3.10 for that process to yield compact spaces that are weakly perfect but not perfect. Our results will produce a family of ZFC examples of compact linearly ordered spaces that are (hereditarily) weakly perfect but not perfect. The best previously known example of this type was a compact Hausdorff space, but not a LOTS, that was constructed by Heath in

[H]. Our construction establishes a close linkage between such spaces and special subsets of the unit interval.

3.1. **Lemma**: *For any subset $P \subset [0,1]$ define the set*

$$X(P) = ([0,1] \times \{0\}) \cup (P \times \{-1,1\})$$

and order $X(P)$ lexicographically. Then $X(P)$ is a compact LOTS and is perfect if and only if the set P is countable.

Proof: That $X(P)$ is a compact LOTS is clear. If the set P is countable, then $X(P)$ is separable and hence is perfect. Conversely, if $X(P)$ is perfect, then it is hereditarily Lindelöf so that each relatively discrete subset must be countable. Hence $P \times \{0\}$ is a countable set. $\square$

3.2 **Definition**: A subset $P \subset [0,1]$ is *perfectly meager* provided that for each closed, dense-in-itself subset $C \subset [0,1]$, the set $P \cap C$ is a first category subset of C.

Note: It is crucial to understand that $P \cap C$ must be a first category subset of C with its relative topology, and not merely a first category subset of $[0,1]$. It is often the case that the set C in definition (3.2) will be nowhere dense in $[0,1]$, so that $P \cap C$ would automatically be of first category in $[0,1]$.

3.3 **Lemma**: *Perfectly meager sets of cardinality ω_1 exist in ZFC.* [MI] $\square$

An earlier version of this paper showed that if P is an uncountable subset of $[0,1]$ then $X(P)$ is weakly perfect if and only if P is a perfectly meager subset of $[0,1]$. Gary Gruenhage pointed out a theorem concerning first category subsets of the unit interval whose proof can be modified to show that the compact LOTS $X(P)$ is *hereditarily* weakly perfect and not perfect. Gruenhage observed that if $Y \subset [0,1]$ is of the first category in itself, then Y must contain a countable dense subset D that is a relative G_δ-subset of Y. The techniques in his proof allow us to shorten and sharpen our earlier results, and details appear in Propositions 3.7 and 3.8, below.

Our proof requires three technical lemmas. The first two must be well known and we omit the proofs.

3.4 **Lemma**: *Let (Y,d) be a totally bounded metric space and D a dense subset of Y. Suppose $\epsilon > 0$ and $F_1 \subset Y$ has the property that if $p \neq q$ belong to F_1, then $d(p,q) \geq \epsilon$. Then there is a subset $F_2 \subset D - F_1$ such that:*

a) the set F_2 is finite and if $p,q \in F_1 \cup F_2$ are distinct, then $d(p,q) \geq \frac{\epsilon}{2}$;

b) for each point $y \in Y$, some $p \in F_1 \cup F_2$ has $d(y,p) \leq \frac{\epsilon}{2}$. $\square$

3.5 **Lemma**: *Let $\{W_\alpha : \alpha \in A\}$ be any collection of pairwise disjoint open subsets of a space Z, and suppose that H_α is a G_δ-subset of Z with $H_\alpha \subset W_\alpha$ for each α. Then $H = \bigcup\{H_\alpha : \alpha \in A\}$ is a G_δ-subset of Z. In particular, if each point of Z is a G_δ-subset of Z and if Z is hereditarily collectionwise Hausdorff, then any relatively discrete subset of Z is a G_δ in Z.* $\square$

3.6 **Lemma**: *Let P be a perfectly meager subset of $I = [0, 1]$ and let Z be any subspace of $X(P)$. Let C be a relatively closed subset of Z that is dense-in-itself. Then there is a set $M \subset C$ that is dense in C and is a relative G_δ-subset of the subspace Z.*

Proof: Let $X = X(P)$. Let K be the closure of C in X. Then K is compact and dense in itself, and $K \cap Z = C$. Let $\pi : X \to I$ be first coordinate projection. Then $\pi[K]$ is a closed, dense-in-itself subset of I and $\pi[C]$ is a dense subset of $\pi[K]$. Because P is perfectly meager, there are closed nowhere dense subsets $T(1) \subset T(2) \subset \ ...$ of $\pi[K]$ such that $P \cap \pi[K] \subset \bigcup\{T(n) : n \geq 1\} \subset \pi[K]$.

Let $V(0) = \pi[K]$. Applying (3.4) to the totally bounded subspace $\pi[K]$ of the unit interval, we recursively define pairwise disjoint finite sets $F(n)$, dense, relatively open subsets $V(n)$ of $\pi[K]$, and relatively open subsets $U(p, n) \subset \pi[K]$ for each $p \in S(n) = \bigcup\{F(j) : j \leq n\}$ such that the following seven conditions are satisfied for each $n \geq 1$:

(1) $F(n) \subset \pi[C] \cap [\pi[K] - T(n)] \cap V(0) \cap \ ... \ \cap V(n-1)$;

(2) for every point $x \in \pi[K]$, $|x - p| \leq 2^{-n}$ for some point $p \in S(n)$;

(3) the collection $\{U(p, n) : p \in S(n)\}$ is pairwise disjoint, each of its members has diameter less than 2^{-n}, and $p \in U(p, n)$ for each $p \in S(n)$;

(4) if $q \in S(n)$ has $q \in U(p, n-1)$ for some $p \in S(n-1)$, then $U(q, n) \subset U(p, n-1)$;

(5) if $p \in F(n)$, then $U(p, n) \subset [\pi[K] - T(n)] \cap V(0) \cap \ ... \ \cap V(n-1)$;

(6) if $V'(n) = [\pi[K] - T(n)] - (\bigcup\{\mathrm{cl}_{\pi[K]}(U(p, n)) : p \in S(n)\})$, then $V(n) = V'(n) \cup (\bigcup\{U(p, n) : p \in S(n)\}$ is a dense, relatively open subset of $\pi[K]$;

(7) $S(n) \subset \bigcap\{V(j) : j \leq n\}$.

Given those seven conditions, let $S = \bigcup\{S(n) : n \geq 1\}$ and observe that S is a countable subset of $\pi[C]$. To complete the proof, we verify the following four claims.

Claim 1: $\bigcap\{V(n) : n \geq 1\} \subset S \cup (\pi[K] - P)$. For suppose $q \in V(n)$ for each n. This can happen in several ways. If there is a sequence $n(1) < n(2) < \ ...$ such that $q \in V'(n(k))$ for each k, then $q \in \pi[K] - T(n(k))$ for each k so that $q \notin \bigcup\{T(n(k)) : k \geq 1\} = \bigcup\{T(n) : n \geq 1\}$. Then $q \notin \pi[K] \cap P$ so that $q \in \pi[K] - P$ as required. Hence we may assume there is an n_0 such that $q \notin V'(n)$ whenever $n \geq n_0$. For each $n \geq n_0$ choose $p(n) \in S(n)$ with $q \in U(p(n), n)$. It is possible that a single point $p \in S$ is repeated infinitely often in the sequence $p(n)$, and in that case recursion condition (3) forces $q = p \in S$. The only remaining case is where no point occurs infinitely often in the sequence $p(n)$. In that case, let $m_1 = n_0$ and recursively choose integers $m_1 < m_2 < m_3 <$ ... such that m_{k+1} is the first index greater than m_k such that $p(m_{k+1}) \neq p(m_k)$. Consider $p(m_2)$ and $p(m_1)$. We claim that $p(m_2) \notin S(m_2 - 1)$. For if $p(m_2) \in S(m_2 - 1)$ then recursion condition (4) would guarantee that $q \in U(p(m_2), m_2) \subset U(p(m_2), m_2 - 1)$. At the same time, $p(m_2 - 1) = p(m_1)$ so that we would have $U(p(m_2 - 1), m_2 - 1) = U(p(m_1), m_2 - 1)$ which would yield $q \in U(p(m_2), m_2 - 1) \cap U(p(m_1), m_2 - 1)$ and that contradicts the pairwise disjointness assumption in recursion condition (3). Therefore, $p(m_2) \in F(m_2)$ so that recursion condition (5) yields $q \in U(p(m_2), m_2) \subset \pi[K] - T(m_2)$. An analogous argument shows that $q \in \pi[K] - T(m_k)$ for each k. Because $T(n) \subset T(n+1)$ for all n, we conclude that $q \notin \bigcup\{T(n) : n \geq 1\}$ and hence $q \notin \pi[K] \cap P$. Therefore $q \in \pi[K] - P$ as claimed.

Claim 2: If $L = \bigcap\{\pi^{-1}[V(n)] : n \geq 1\}$ and $E = \{(x, i) \in \pi^{-1}[S] : (x, i) \notin C\}$, then $(L - E) \cap Z \subset C$. For suppose $(q, j) \in (L - E) \cap Z$. For each n, $q \in V(n)$ so that by Claim 1, $q \in S \cup [\pi[K] - P]$. If $q \in S$, then $(q, j) \in \pi^{-1}[S]$. Because $(q, j) \in (L - E)$ we know that $(q, j) \in C$ as required. Therefore, it will be enough to consider the case where $q \in \pi[K] - P$. Then $q \notin P$ so that the definition of $X(P)$ forces $j = 0$. Then $q \in \pi[K]$ guarantees $(q, j) = (q, 0) \in K$. Because C is relatively closed in Z, we know that $K \cap Z = C$. Hence $(q, j) = (q, 0) \in K \cap Z = C$ as claimed.

Claim 3: $(L - E) \cap Z$ is dense in C. Fix any point $(x_0, j_0) \in C$. Because C is dense in itself, at least one of the sets $C \cap\,]\leftarrow, (x_0, j_0)[$ and $C \cap\,](x_0, j_0), \rightarrow[$ has (x_0, j_0) as a limit point. Without loss of generality, assume it is $C \cap\,]\leftarrow, (x_0, j_0)[$. Then (x_0, j_0) has no immediate predecessor in X. Let W be any open neighborhood of (x_0, j_0) in X. Then there is a point $(u, i) \in X$ with $(u, i) < (x_0, j_0)$ such that $](u, i), (x_0, j_0)[\, \subset W$. Because (x_0, j_0) has no immediate predecessor in X,we must have $u < x_0$ in the unit interval $I = [0, 1]$, and because $](u, i), (x_0, j_0)[\, \cap C$ must be infinite, the set $]u, x_0[\, \cap \pi[C]$ is also infinite and has x_0 as a limit point. In the light of recursion conditions (1) and (2), the set S is dense in $\pi[C]$ so we may choose a point $r \in S \cap\,]u, x_0[\, \cap \pi[C]$. Because $r \in \pi[C]$, there is an integer k with $(r, k) \in C$. However $(r, k) \in \pi^{-1}[S]$ and $(r, k) \notin E$ so that $(r, k) \in (L - E) \cap Z$. But also $(r, k) \in\,](u, i), (x_0, j_0)[\, \subset W$ so that $W \cap (L - E) \cap Z \neq \emptyset$ whenver W is open in X and contains a point of C. Therefore, $(L - E) \cap Z$ is dense in C, as claimed.

Claim 4: The set $M = (L - E) \cap Z$ is a G_δ-subset of Z. Each $V(n)$ is relatively open in the compact set $\pi[K]$ (see (7)) which is a G_δ-subset of $[0, 1]$, so each $V(n)$ is a G_δ-set in $[0, 1]$. Hence L is a G_δ-set in X. Because S is countable, so is E. Hence $L - E$ is also a G_δ-subset of X. Therefore, $M = (L - E) \cap Z$ is a G_δ-subset of the subspace Z, as claimed.

$\square$

3.7 **Proposition**: *Suppose P is an uncountable perfectly meager subset of $[0,1]$. Then $X(P)$ is a compact LOTS that is hereditarily weakly perfect but $X(P)$ is not perfect.*

Proof: By (3.1), the space $X = X(P)$ is not perfect. Let $Z \subset X$ and suppose C is a relatively closed subset of Z. We will find two subsets of C, each a G_δ-subset of Z, whose union is dense in C, and that will complete the proof.

Let $I(C) = \{x \in C :$ for some open set U in X, $x \in U$ and $U \cap C$ is finite $\}$. Because X is first countable, it follows from (3.5) that $I(C)$ is a G_δ-subset of X. The set $I(C)$ is one of the two G_δ-sets promised in the first paragraph.

Let $K_0 = \mathrm{cl}_X(I(C))$ and $W = X - K_0$. Let $\{W(\alpha) : \alpha \in A\}$ be the family of all convex components in X of the open set W. For each $\alpha \in A$, let $D(\alpha) = C \cap W(\alpha)$. If $D(\alpha) = \emptyset$, let $H(\alpha) = \emptyset$. If $D(\alpha) \neq \emptyset$ consider any $p \in D(\alpha)$ and let U be any neighborhood of p in X. The $U \cap W(\alpha)$ is an open neighborhood of p, so that $p \in X - K_0$ forces $C \cap (W(\alpha) \cap U)$ to be infinite (for otherwise $p \in I(C) \subset K_0$). Thus, $D(\alpha)$ is dense-in-itself. Let $C(\alpha) = \mathrm{cl}_Z(D(\alpha))$. Then because $W(\alpha)$ is an open convex subset of X and C is relatively closed in Z, the set $C(\alpha) - D(\alpha)$ has at most two points, and those two points are the end points of $W(\alpha)$ in X. Apply Lemma 3.6 to each set $C(\alpha)$ to find a G_δ-subset $H^{'}(\alpha)$ of Z that is contained in, and dense in, $C(\alpha)$. Because $W(\alpha)$ is open in X, the set $H(\alpha) = H^{'}(\alpha) \cap W(\alpha)$ is also a G_δ-subset of Z and is dense in $D(\alpha)$. From (3.5) we know that the set $H = \bigcup\{H(\alpha) : \alpha \in A\}$ is a G_δ-subset of Z that is dense in $\bigcup\{D(\alpha) : \alpha \in A\} = \bigcup\{C \cap W(\alpha) : \alpha \in A\} = C \cap W = C - \mathrm{cl}_X(I(C))$. Hence $I(C) \cup H$ is a G_δ-subset of Z that is contained in, and dense in, the set C. Therefore, the subspace Z is weakly perfect. $\square$

In the remainder of this section, we prove that uncountable perfectly meager sets are exactly what ones needs in order to construct spaces $X(P)$ that are weakly perfect but not perfect.

3.8 **Proposition**: *Suppose that $P \subset [0,1]$, and that the space $X(P)$ of (3.1) is weakly perfect and not perfect. Then P is an uncountable perfectly meager subset of $[0,1]$.*

Proof: Write $X = X(P)$ and let $\pi : X \to [0,1]$ be first coordinate projection. Suppose X is weakly perfect but not perfect. Then P is uncountable, by (3.1). To show that P is perfectly meager, we need some special notation and a lemma.

For each $x \in [0,1]$, let $i(x) = 0 = j(x)$ if $x \notin P$ and if $x \in P$ then define $i(x) = +1$ and $j(x) = -1$. Suppose $C \subset [0,1]$ is closed and dense-in-itself. Define $R(C) = \{x \in C \cap P :]x, y[\cap C = \emptyset$ for some $y > x\}$ and let $L(C) = \{x \in C \cap P :]y, x[\cap C = \emptyset$ for some

$y < x\}$. Observe that $R(C) \cap L(C) = \emptyset$, that $(x, 1)$ is an isolated point of $\pi^{-1}[C]$ if and only if $x \in R(C)$, and that $(x, -1)$ is an isolated point of $\pi^{-1}[C]$ if and only if $x \in L(C)$. In addition, note that $P \times \{0\}$ consists entirely of isolated points of X. Therefore, if we define

$$C^* = \pi^{-1}[C] - (((C \cap P) \times \{0\}) \cup (R(C) \times \{1\}) \cup (L(C) \times \{-1\})),$$

then C^* is a closed subset of X.

Because X is weakly perfect, there are open subsets $G(n)$ of X such that the set $S = \bigcap\{G(n) : n \geq 1\}$ is a dense subset of C^*. Apply Lemma 3.9, below, to the sets $G(n)$ to produce sets $H(n) = H(G(n), C)$, each of which is a relatively open, dense subset of C. Because C satisfies the Baire Category Theorem in its relative topology, the set $H = \bigcap\{H(n) : n \geq 1\}$ must be a dense subset of C.

We claim that $H \cap P = \emptyset$. For suppose there is a point $a \in H \cap P$. Because C has no relatively isolated points, it follows from (3.9) below that for each n we can find points $s_n < a < t_n$ such that $](s_n, i(s_n)), (t_n, j(t_n))[\subset G(n)$. Then $(a, 0) \in](s_n, i(s_n)), (t_n, j(t_n))[\subset G(n)$ for every n so that $(a, 0) \in \bigcap\{G(n) : n \geq 1\} \subset C^*$. But because $a \in P$ and $a \in H \subset C$ we see from the definition of C^* that $(a, 0) \notin C^*$ and that contradiction shows that $H \cap P = \emptyset$.

Therefore, we have proved that if C is closed in $[0, 1]$ and dense-in-itself, then there are relatively open, dense subsets $H(n)$ of C such that $\bigcap\{H(n) : n \geq 1\} \cap P = \emptyset$. Then $P \cap C \subset \bigcup\{C - H(n) : n \geq 1\}$ so that $P \cap C$ is indeed a first category subset of C. Hence the set P must be perfectly meager, as claimed. $\square$

We now prove the lemma needed in 3.8, above. The proof is straightforward, but somewhat technical.

3.9 **Lemma**: *Let C be a closed, dense-in-itself subset of $[0, 1]$ and let C^* be as in (3.8). Suppose G is an open subset of X such that $G \cap C^*$ is dense in C^*. Let $H(G, C) = \{x \in C :$ there are points $s < x < t$ in $[0, 1]$ with $](s, i(s)), (t, j(t))[\subset G\}$. Then $H(G, C)$ is relatively open in C and is dense in C.*

Proof: Notation is as in (3.8). Write $H = H(G, C)$. To show that H is relatively open in C, let $x \in H$. Find numbers $s < x < t$ with $](s, i(s)), (t, j(t))[\subset G$. Then $U =]s, t[\cap C$ is the required relative neighborhood of x in C that is contained in H.

Obviously $H \subset C$. To show that H is dense in C, suppose $s < t$ are real numbers with $]s, t[\cap C \neq \emptyset$. We will verify that $]s, t[\cap H \neq \emptyset$.

We claim that $](s, i(s)), (t, j(t))[\cap C^* \neq \emptyset$. For choose $c_0 \in]s, t[\cap C$. Then $(c_0, k) \in](s, i(s)), (t, j(t))[$ whenever $(c_0, k) \in X$. If $c_0 \notin P$, then we know that $(c_0, 0) \in \pi^{-1}[C]$

and $(c_0, 0) \notin ((C \cap P) \times \{0\}) \cup (R(C) \times \{1\}) \cup (L(C) \times \{-1\})$ so that we have $(c_0, 0) \in C^* \cap\,](s, i(s)), (t, j(t))[$ as required. If $c_0 \in P$, consider the points $(c_0, -1)$ and $(c_0, +1)$ of $\pi^{-1}[C]$. If $c_0 \notin R(C)$ then $(c_0, 1) \in C^* \cap\,](s, i(s)), (t, j(t))[$, as required. If $c_0 \in R(C)$, then because $R(C) \cap L(C) = \emptyset$ we have $c_0 \notin L(C)$ so that $(c_0, -1) \in C^* \cap\,](s, i(s)), (t, j(t))[$. In any case, therefore, $](s, i(s)), (t, j(t))[\, \cap C^* \neq \emptyset$

Because $G \cap C^*$ is dense in C^*, and because $C^* \cap\,](s, i(s)), (t, j(t))[$ is a non-void relatively open subset of C^*, there must be some point $(x_1, k_1) \in G \cap C^* \cap\,](s, i(s)), (t, j(t))[$. The special definitions of $i(s), j(t)$ in (3.8) yield

(a) $s < x_1 < t$.

Because $(x_1, k_1) \in C^*$ we conclude

(b) if $x_1 \in P$, then $k_1 \neq 0$ and

(c) if $k_1 = 1$ then $x_1 \notin R(C)$ and if $k_1 = -1$ then $x_1 \notin L(C)$.

Because $G \cap\,](s, i(s)), (t, j(t))[$ is open, we may choose points $(u_i, m_i) \in X$ with

(d) $(x_1, k_1) \in\,](u_1, m_1), (u_2, m_2)[\, \subset G \cap\,](s, i(s)), (t, j(t))[$.

Then we must have

(e) $u_1 \leq x_1 \leq u_2$

and the special definitions of $i(u_1), j(u_2)$ yield

(f) $](u_1, i(u_1)), (u_2, j(u_2))[\, \subset\,](u_1, m_1), (u_2, m_2)[\, \subset G$.

Consider the inequalities in (e). If $u_1 = x_1 = u_2$, the (d) yields $k_1 = 0$ and $x_1 \in P$, contradicting (b). Hence at least one of the inequalities in (e) must be strict. If both are strict, i.e., if $u_1 < x_1 < u_2$, then (f) yields $x_1 \in H$ so that $x_1 \in H \cap\,]s, t[$ as required. Consider the case where $u_1 = x_1 < u_2$, the case where $u_1 < x_1 = u_2$ being analogous. Because $u_1 = x_1$ we have $k_1 \geq 0$ so that (d) yields $x_1 \in P$ (because two distinct points of X have x_1 as their first coordinate). Then (b) gives $k_1 = 1$. Because $(x_1, 1) = (x_1, k_1) \in C^*$, we conclude that $x_1 \notin R(C)$. Because $x_1 \in P \cap C$, it follows that $]x_1, y[\, \cap C \neq \emptyset$ whenever $y > x_1$. That allows us to choose a point $x_2 \in C$ with $u_1 = x_1 < x_2 < u_2$. Then (f) gives $](x_1, i(x_1)), (x_2, j(u_2)[\, \subset G$ so that $x_2 \in H$.

To complete the proof, note that $u_1 = x_1 < x_2 < u_2$ so that (d) yields $s < x_2 < t$. Hence $x_2 \in H \cap\,]s, t[$ as required to show that H is dense in C. $\square$

Combining (3.7) and (3.8) gives a proof of the following theorem:

3.10 **Theorem**: *Suppose* $P \subset [0, 1]$ *and* $X = X(P)$. *Then* X *is a compact LOTS and* X *is weakly perfect but not perfect if and only if* P *is an uncountable perfectly meager subset of* $[0, 1]$. $\square$

4. Special subspaces of $X(P)$.

In this section, we will start with *any* subset $P \subset [0,1]$. One can still construct the compact LOTS $X(P)$ and it is conceivable that certain interesting subspaces of $X(P)$ are weakly perfect even though $X(P)$ is not. We will begin by examining one such subspace, namely, $Y(P) = \{(x,i) \in X(P) : x \in P,\ i \in \{0,1\}\}$. Our results (see (4.1)) show that, in fact, it is just as difficult for the subspace $Y(P)$ to be weakly perfect as it is for the compact space $X(P)$ to be weakly perfect. Other interesting subspaces of $X(P)$ will be studied in Section 5, below.

An equivalent way to describe $Y(P)$ is to begin with the lexicographically ordered LOTS $Y = P \times \{0,1\}$ and modify the topology to isolate every point of $P \times \{0\}$. We will give necessary and sufficient conditions for the space $Y(P)$ to be weakly perfect. Some of our conditions do not involve P with its usual topology, but rather P topologized as a subspace of the Sorgenfrey line in which basic open sets have the form $[a,b[$ for real numbers $a < b$. We will denote that Sorgenfrey topology on $[0,1]$ by $\mathcal{S}$, and for a subset $D \subset [0,1]$ we will write $(D, \mathcal{S}_D)$ for D topologized as a subspace of $([0,1], \mathcal{S})$. Analogously, $\mathcal{E}$ will denote the usual Euclidean topology on $[0,1]$. Clearly $(P, \mathcal{S}_P)$ is homeomorphic to the subspace $\{(x,1) : x \in P\}$ of $Y(P)$. Our characterization is as follows:

4.1 **Proposition**: *Let P be any subset of $[0,1]$. The following are equivalent:*

a) $Y(P)$ is weakly perfect.

b) for any closed subset C of the space $(P, \mathcal{S}_P)$ there is a set T satisfying

- *i) T is countable subset of C;*
- *ii) T is a G_δ-subset of $(P, \mathcal{S}_P)$;*
- *iii) T is a dense subset of the space $(C, \mathcal{S}_C)$.*

c) P is a perfectly meager subset of $[0,1]$.

d) $X(P)$ is hereditarily weakly perfect.

Proof: From (3.7), and (3.8) we know that c) and d) are equivalent, and because $Y(P)$ is a subspace of $X(P)$ we know that d) implies a). Thus it remains only to prove that a) implies b) and b) implies c). The proof that a) implies b) is contained in Lemma 4.4 below, and the proof that b) implies c) appears in Lemma 4.5. Those two lemmas rely on technical results that appear in Lemmas 4.2 and 4.3.

4.2 **Lemma**: *Suppose that $T \subset P \times \{1\}$ is a G_δ-subset of $Y(P)$. Then T is countable.*

Proof: For $A \subset P$, the set $T = A \times \{1\}$ is a G_δ-subset of the subspace $A \times \{0,1\}$ of $Y(P)$ if and only if $A \times \{0\}$ is an F_σ-subset of $A \times \{0,1\}$. Because A is hereditarily Lindelöf,

$A \times \{0\}$ does not contain any uncountable closed subsets of of $A \times \{0, 1\}$, and so $A \times \{0\}$ is not an F_σ-subset unless A is countable. $\square$

To understand the point of our next lemma, recall the easy fact that the closed subspace $P \times \{1\}$ of $Y(P)$ may have G_δ-subsets in its relative topology that are not G_δ-subsets of the space $Y(P)$. Indeed, the set $P \times \{1\}$ is one such set, provided P is uncountable. However, when it comes to *countable* sets, we have:

4.3 **Lemma**: *Suppose T is a countable subset of $P \times \{1\}$ that is a G_δ-subset of the subspace $P \times \{1\}$ of $Y(P)$. Then T is a G_δ-subset of $Y(P)$.*

Proof: If $T = A \times \{1\} = \bigcap\{G_n \times \{1\} : n \geq 1\}$ where each G_n is open in $(P, \mathcal{S}_P)$, then the set $S = A \times \{0, 1\} = \bigcap\{G_n \times \{0, 1\} : n \geq 1\}$ is a G_δ-subset of $Y(P)$. But $A \times \{0\}$ is countable, and we remove one point of $A \times \{0\}$ at a time to show that T is a G_δ-subset of $Y(P)$. $\square$

4.4 **Lemma**: *In Proposition 4.1, a) implies b).*

Proof: Suppose $Y(P)$ is weakly perfect. If $C \subset P$ is closed in the space $(P, \mathcal{S}_P)$, then $C \times \{1\}$ is closed in $Y(P)$, so there is a subset $T \times \{1\} \subset C \times \{1\}$ that is a G_δ-set in $Y(P)$ and is dense in $C \times \{1\}$. By Lemma 4.2, T must be countable. Clearly T is a G_δ-subset of $(P, \mathcal{S}_P)$ and is dense in C, as required to verify (b).$\square$

4.5 **Lemma**: *In Proposition 4.1, (b) implies (c).*

Proof: Suppose (b) holds. Let K be any nonempty dense-in-itself $\mathcal{E}$-closed subset of $[0, 1]$. Our goal is to show that the set $F = P \cap K$ is of the first category in K. Consider the set W of all $x \in F$ having a neighborhood in $(F, \mathcal{S}_F)$ that has countable intersection with F. This set is countable and open in the hereditarily Lindelöf space $(F, \mathcal{S}_F)$. Replacing K by $K - W$ if necessary, we may assume that $W = \emptyset$ and that each $\mathcal{S}_F$ neighborhood of each point of F meets F in an uncountable set.

Because $F = P \cap K$ is closed in $(P, \mathcal{S}_P)$, there is a countable $T \subset F$ such that T is dense in $(F, \mathcal{S}_F)$ and is a G_δ-subset of $(P, \mathcal{S}_P)$. Write $T = \bigcap\{U_n : n \geq 1\}$ where each U_n is an open subset of $(P, \mathcal{S}_P)$. Being a countable subset of the dense-in-iteslf set K, T is clearly first category in $(K, \mathcal{E}_K)$. Hence, to show that F is of first category in $(K, \mathcal{E}_K)$, it will be enough to show that each set $F - U_n$ is nowhere dense in $(K, \mathcal{E}_K)$. For contradiction, suppose there exist $a < b \in K$ such that

$$(*)\ \emptyset \neq\ ]a, b[\ \cap K \subset \mathrm{cl}_{\mathcal{E}_K}(F - U_n) \subset \mathrm{cl}_{\mathcal{E}_K}(F) - int_{\mathcal{E}_K}(U_n).$$

Because T is dense in $(S, \mathcal{S}_F)$, we have $T\cap\ ]a,b[\ \neq \emptyset$, so that there exists $t \in T\cap\ ]a,b[$ and there must be some $u \in [0,1]$ with $u > t$ such that $[t,u[\ \cap P \subset U_n$. It follows that the uncountable set $]t,u[\ \cap F \subset int_{\mathcal{E}_K}(U_n)$. But (*) yields $]a,b[\ \cap K \cap int_{\mathcal{E}_K}(U_n) = \emptyset$, and that is impossible.$\square$

5. Michael lines in $X(P)$

In this section P is any subspace of $I = [0,1]$. Besides the space $Y(P)$ studied in Section 4, there is another natural subspace of $X(P)$, namely $M(P) = \{(x,0) : 0 \leq x \leq 1\}$. Clearly, $M(P)$ is homeomorphic to the Michael line space obtained by isolating all points of P and letting all other points have their usual Euclidean neighborhoods. An argument reminiscent of the one used in (3.7) and (3.8) to characterize weak perfectness in $X(P)$ allows us to prove:

5.1 **Proposition**: *Let $P \subset I = [0,1]$. Then the following are equivalent:*
a) $M(P)$ is hereditarily weakly perfect;
b) $M(P)$ is weakly perfect;
c) for each dense-in-itself closed subset $K \subset I$ such that $K - P$ is dense in K, the set $K \cap P$ is a first category subset of K. $\square$

From (3.8) or (5.1) we obtain:

5.2 **Corollary**: *If P is a perfectly meager subset of* $[0,1]$*, then $M(P)$ is hereditarily weakly perfect.* $\square$

However, in contrast to the situation for the spaces $X(P)$ and $Y(P)$ described above, to say that $M(P)$ is weakly perfect is strictly weaker than the assertion "P is perfectly meager," as can be seen from:

5.3 **Example**: *There is a subspace $P \subset I$ such that $M(P)$ is weakly perfect but not perfect, and yet P is not perfectly meager.* Let $P_1 = [0, \frac{1}{2}[$ and let P_2 be an uncountable perfectly meager subset of $[\frac{1}{2}, 1]$ with $\frac{1}{2} \in P_2$. Let $P = P_1 \cup P_2$. Clearly, P is not perfectly meager. However, $M(P)$ is the disjoint union (or topological sum) of the subspaces $X_1 = M(P) \cap [0, \frac{1}{2}[$ and $X_2 = M(P) \cap [\frac{1}{2}, 1]$. Every point of the subspace X_1 is isolated, so that X_1 is weakly perfect. The proof of (3.7) shows that X_2 is weakly perfect but not perfect. Hence $M(P)$ is weakly perfect but not perfect. $\square$

Spaces of the type $M(P)$ can be used to answer a natural question about the role of the property "weakly perfect" in certain parts of metrization theory. Recall that a topological space Z is *quasi-developable* provided there is a sequence of collections $\mathcal{G}(n)$ of open subsets of Z such that for each $z \in Z$, the collection $\{\mathrm{St}(z, \mathcal{G}(n)) : n \geq 1\}$ contains a neighborhood base at z. (Some of the sets $\mathrm{St}(z, \mathcal{G}(n))$ might be empty.) Also recall that a quasi-developable space Z is developable if and only if it is perfect, so that if Z is a LOTS or GO-space, then Z is metrizable if and only if Z is quasi-developable and perfect.

In [H], R.W. Heath showed that one can construct quasi-developable spaces that are weakly perfect but not perfect, and asserted that similar examples could be found inside of the lexicographic square. A space of the type described in (5.4) below is probably what he had in mind. Observe that the following example does not require any special set theoretic assumptions.

5.4 **Proposition**: *In ZFC, there is a quasi-developable, hereditarily weakly perfect GO-space Z that is not perfect.*

Proof: Let P be an uncountable perfectly meager subset of $[0, 1]$ as in Section 3. Construct $M(P)$ as above. The resulting space is a quasi-developable GO-space. However, $M(P)$ is not perfect because the set P cannot be an F_σ-subset of $M(P)$ since P is a perfectly meager subset of $[0, 1]$. In addition, $M(P)$ embeds as a subspace of the space $X(P)$ constructed in Section 3. In the light of (3.7), Z is hereditarily weakly perfect. $\square$

To find a generalized ordered space that is Lindelöf, quasi-developable, and weakly perfect but not perfect, we need to assume the existence of an ω_1-scale.[BD]

5.5 **Proposition**: *If ${}^\omega\omega$ has an ω_1-scale, then there is a GO-space that is Lindelöf, quasi-developable, weakly perfect, and not perfect.*

Proof: Let $\mathcal{E}$ denote the usual topology on the set $I = [0, 1]$. We know that in any model of set theory, $[0, 1]$ contains a set that is a λ-set but not a λ'-set [MI, Theorem 4.6]. Starting with such a set L, we may add countably many points to L in such a way that the enlarged set L^* contains a countable subset C that is not a G_δ-subset of L^* when L^* is topologized as a subspace of $[0, 1]$. Because any λ-set is perfectly meager [Theorem 4.2, MI], and because adding countably many points to a perfectly meager set produces another perfectly meager set, we know that L^* is perfectly meager.

Because C is not a G_δ-subset of L^*, the existence of an ω_1 scale allows us to construct an uncountable set E with $C \subset E \subset L^*$ such that if U is an $\mathcal{E}$-open subset of $[0, 1]$ that contains C, then $E - U$ must be countable. See [BD, Lemma 2.2]. Let $P = E - C$. Then P is an uncountable perfectly meager set, so that the space $M(P)$ is hereditarilty weakly perfect (see (5.2)) and not perfect. Let $\mathcal{M}$ denote the topology of $M(P)$.

The subspace $(E, \mathcal{M}_E)$ of $M(P)$ is hereditarily weakly perfect and quasi-developable. We claim that $(E, \mathcal{M}_E)$ is Lindelöf. For suppose $\mathcal{U}$ is a collection of $\mathcal{M}_E$-open sets that covers E. Choose a countable subcollection $\mathcal{U}_0$ that covers the countable set C. Because

$C \subset M(P) - P$, each $\mathcal{M}$ neighborhood of a point $x \in C$ is an $\mathcal{E}$ neighborhood of x. Let $V_0 = \bigcup \mathcal{U}_0$. Then $C \subset V_0$ so that $E - V_0$ must be countable. Choose a countable subcollection $\mathcal{U}_1 \subset \mathcal{U}$ that covers $E - V_0$. Then $\mathcal{U}_0 \cup \mathcal{U}_1$ covers all of E, showing that $(E, \mathcal{M}_E)$ is Lindelöf.

Finally, note that $(E, \mathcal{M}_E)$ is not metrizable and not perfect, because $(E, \mathcal{M}_E)$ is Lindelöf and yet contains an uncountable set of isolated points. $\square$

6. A characterization of perfectly meager sets.

Combining the results of Sections 3 and 5 of this paper yields new internal topological characterizations of perfectly meager subsets of the unit interval. A shorter, more direct proof will appear in [GL].

6.1 **Proposition**: *The following properties of a topological space X are equivalent:*

a) X is a perfectly meager subset of the unit interval $I = [0, 1]$;

b) $X \subset I$ and for every $A \subset X$ there is a countable set $B \subset A$ that is dense in A and is a relative G_δ-subset of X;

c) X is a zero-dimensional, separable metrizable space and for every $A \subset X$ there is a countable set $B \subset A$ that is dense in A and is a G_δ-subset of X.

Proof: By [E, 6.3.2], every zero-dimensional separable metrizable space can be embedded in $[0, 1]$ so that (c) and (b) are equivalent. We will verify the equivalence of (a) and (b).

$(a) \Rightarrow (b)$. In this section of the proof, we will need to use two separate topologies on I, namely the Euclidean topology $\mathcal{E}$ and a certain Michael line topology $\mathcal{M}$. To avoid ambiguity, we will use terms such as "$\mathcal{E}$ dense-in-itself" and "relatively $\mathcal{M}$-closed."

Suppose X is a perfectly meager subset of $(I, \mathcal{E})$. Fix a subset $A \subset X$ and a countable dense subset $A_0 \subset A$. Let $P = X - A_0$. Then P is also perfectly meager, so that $X(P)$ is hereditarily weakly perfect (see (3.7)). Hence $M(P)$ is hereditarily weakly perfect. Denote the topology of $M(P)$ by $\mathcal{M}$. As sets, $X \subset M(P) = I$ so that X inherits two different topologies, namely $\mathcal{E}_X$ and $\mathcal{M}_X$.

The space $(X, \mathcal{M}_X)$ is weakly perfect and A_0 is a relatively closed subset of $(X, \mathcal{M}_X)$. Hence there is a subset $B \subset A_0$ such that B is a G_δ-subset of $(X, \mathcal{M}_X)$ and such that B is dense in A_0 with respect to the $\mathcal{M}_X$ topology. Observe that in the space $M(P)$, the $\mathcal{M}$ neighborhoods of points of A_0 are the same as the $\mathcal{E}$-neighborhoods of those points, and that the same is true for points of B, because $B \subset A_0 \subset I - P$. Therefore, B is also dense in A_0 with respect to the relative Euclidean topology. Because A_0 is $\mathcal{E}$-dense in A , B is also dense in A with respect to the $\mathcal{E}$-topology. Next, because B is a relative G_δ-subset of $(X, \mathcal{M}_X)$, there are $\mathcal{M}$-open sets $G(n) \subset M(P)$ such that $B = \bigcap\{G(n) \cap X : n \geq 1\}$.

Once again because the $\mathcal{E}$ and $\mathcal{M}$ neighborhoods of points of B are the same, we may assume that each $G(n)$ is $\mathcal{E}$-open. Thus B is as described in (b).

$(b) \Rightarrow (a)$: In this part of the proof, all topologies are Euclidean topologies (and their subspace topologies). Given X as in (b), let K be a closed, dense-in-itself subspace of $I = [0,1]$. Let $A = K \cap X$. We must show that A is a first category subset of K. Use (b) to find a countable subset $B \subset A$ that is dense in A and is a relative G_δ-subset of X. A standard argument shows that A is of the first category in K.$\square$

6.2 **Remark**: Proposition 4.1 allows us to add another equivalent statement to the list in (6.1) that resembles condition (b), except that it involves non-metrizable subsets of the the Sorgenfrey line rather than subsets of the line with its usual Euclidean topology. We did not include it in order to simplify the statement of (6.1).

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