Spaces with $<\omega$-Weakly Uniform Bases

by

Harold Bennett, Texas Tech University, Lubbock, TX 79409

and

David Lutzer, College of William and Mary, Williamsburg, VA 23187

Dedicated to the Memory of Zoltan Balogh

Abstract

In this paper we use techniques and examples from ordered space theory to study spaces having various kinds of weakly uniform bases. We show that any generalized ordered space has a $<\omega$-weakly uniform base if and only if it is quasi-developable and has Property N, a generalization of the G_δ-diagonal property. We also show that any linearly ordered topological space with a $<\omega$-weakly uniform base is metrizable. We characterize GO-spaces that have m-weakly uniform bases using quasi-developability and another generalization of the G_δ-diagonal property. We give examples showing that a quasi-developable linearly ordered topological space can fail to have a $<\omega$-weakly uniform base, that a linearly ordered space can have a $<\omega$-weakly uniform base without having any n-weakly uniform base, and that among hereditarily paracompact spaces no two of the countably many properties "X has a $<\omega$-weakly uniform base" and "X has an n-weakly uniform base" are equivalent.

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1 Introduction

In this paper we use techniques and examples from ordered space theory to study spaces having various kinds of weakly uniform bases. In [HL], Heath and Lindgren defined that a collection $\mathcal{B}$ is *weakly uniform* if, given two distinct points x and y of X, there are only a finite number of members of $\mathcal{B}$ that contain $\{x,y\}$. Z. Balogh introduced generalizations of this notion, as follows: the collection $\mathcal{B}$ is n-*weakly uniform* if, given any set F with n points, the collection $\{B\in\mathcal{B}:F\subseteq B\}$ is finite, and $\mathcal{B}$ is $<\omega$-*weakly uniform* if, given any infinite set S, there is some finite $F\subseteq S$ such that $\{B\in\mathcal{B}:F\subseteq B\}$ is finite. Balogh's definitions later appeared in the five-author paper [BDJSS]. Clearly, weakly uniform collections and 2-weakly uniform collections are the same things, and if $m<n$, then m-weakly uniform $\Rightarrow$ n-weakly uniform $\Rightarrow$ $<\omega$-weakly uniform.

Recall that a space X has a G_δ-diagonal if and only if there is a sequence $\langle\mathcal{G}(n)\rangle$ of open covers of X such that for each $p\in X$, $\bigcap_1^\infty \mathrm{St}(p,\mathcal{G}(n))=\{p\}$. In an earlier paper [BL] we characterized GO-spaces having weakly uniform bases as follows:

Theorem 1.1 *: A GO-space has a weakly uniform base if and only if it is quasi-developable and has a G_δ-diagonal.*

In this paper we generalize that result by making use of a property related to the G_δ-diagonal covering condition above. The new property is called *Property N*, where we say that X has Property N if there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that for any infinite set $S \subseteq X$ there is a finite set $F \subseteq S$, a point $p \in F$, and an integer n such that $F \not\subseteq \mathrm{St}(p, \mathcal{G}(n))$. We prove:

Theorem 1.2 *: A GO-space X has a $< \omega$-weakly uniform base if and only if X is quasi-developable and has Property N.*

Heath and Lindgren originally observed [HL] that any LOTS with a weakly uniform base must be metrizable. As a corollary to Theorem 1.2 we will strengthen the Heath-Lindgren result to

Corollary 1.3 *A LOTS X has a $< \omega$-weakly uniform base if and only if X is metrizable.*

Finally, we characterize GO-spaces with m-weakly uniform bases as being those quasi-developable GO-spaces that have open covers $\mathcal{G}(n)$ for $n \geq 1$ such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}(n)) : n \geq 1\}$ has fewer than m points.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, \tau, <)$ where $<$ is a linear ordering of X and where τ is a Hausdorff topology on X that has a base of order-convex sets. In case τ is the open interval topology of the ordering $<$, then we say that $(X, \tau, <)$ is a *linearly ordered topological space* (LOTS). It is well-known that the class of GO-spaces coincides with the class of spaces that embed topologically in a LOTS.

At several points in the next section, we deal with a subset Y of a linearly ordered set X and will need to use *relatively convex subsets of* Y, i.e. subsets $S \subseteq Y$ such that if $y_1 < y_2$ are points of Y that belong to S, then $[y_1, y_2] \cap Y \subseteq S$.

Throughout this paper we reserve the symbols $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{P}$ for the usual sets of real, rational, and irrational numbers, and the set of all integers (positive and negative) is denoted by $\mathbb{Z}$.

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2 GO-spaces with $< \omega$-weakly uniform bases

Lemma 2.1 *Let X be a GO-space with a $< \omega$-weakly uniform base. Then:*

a) X has a $< \omega$-weakly uniform base whose members are convex sets;

b) X is first-countable;

c) X is paracompact.

Proof: To prove a), let $\mathcal{B}$ be any $< \omega$-weakly uniform base for X. Let $\mathcal{C}$ be the family of all convex components of members of $\mathcal{B}$. Then $\mathcal{C}$ is a base for X and is also $< \omega$-weakly uniform. To prove b) we consider any $p \in X$. If p is isolated, there is nothing to prove, so either every neighborhood of p contains points below p, or every neighborhood of p contains points above p (or both). In the first case, compute $\kappa = \mathrm{cf}(\,]\leftarrow, p[\,)$. If $\kappa \geq \omega_1$ then we can find an infinite sequence $C_n \in \mathcal{C}$ of distinct members of $\mathcal{C}$ and a point $q < p$ such that the infinite set $]q, p[$ is a subset of each set C_n, contradicting that $\mathcal{C}$ is a $< \omega$-weakly uniform collection. In the second case, a similar argument shows that the set $]p, \rightarrow[$ has countable coinitiality. Therefore, X is first countable at p.

To prove c), recall that if X is not paracompact then there is a stationary subset S of some regular uncountable cardinal that embeds as a closed subspace of X [EL]. Because the existence of a $<\omega$-weakly uniform base is hereditary, the space S has a $<\omega$-weakly uniform base $\mathcal{C}$ and we may assume that members of $\mathcal{C}$ are convex sets. Let T be the set of non-isolated points of the space S. Then T is also a stationary set, and for each $\lambda \in T$ there is a set $C_\lambda \in \mathcal{C}$ with $\lambda \in C_\lambda \subseteq S \cap [0,\lambda]$. Choose $\alpha(\lambda) < \lambda$ such that $]\alpha(\lambda),\lambda] \cap S \subseteq C_\lambda$. Then $f(\lambda) = \alpha(\lambda)$ is a regressive function, so there is some β and some stationary set $U \subseteq T$ such that $f(\lambda) = \beta$ for each $\lambda \in U$. Let γ be the first limit point of S that is larger than $\beta + \omega$. But then the infinite set $]\beta,\gamma[\ \cap S$ is a subset of each of the uncountably many sets C_λ for $\lambda \in U \cap\ ]\gamma,\rightarrow[\ \cap T$, and that is impossible because $\mathcal{C}$ is $<\omega$-weakly uniform. Hence X must be paracompact. □

Recall that a space is *quasi-developable* if it has a sequence $\langle \mathcal{G}(n) \rangle$ of collections of open sets such that if V is open and $p \in V$ then for some n we have $p \in \mathrm{St}(p,\mathcal{G}(n)) \subseteq V$. It is known that for GO-spaces, quasi-developability is equivalent to the existence of s σ-disjoint base. [B1] [L]. The following lemma is easily proved.

Lemma 2.2 *: Suppose X is a first-countable GO-space. If $X = \bigcup\{X(n) : n \geq 1\}$ where each $X(n)$ is quasi-developable in its subspace topology, then X is quasi-developable.*

Property N, a generalization of the well-known G_δ-diagonal covering property, was introduced in the previous section.

Proposition 2.3 *If a GO-space X has a $<\omega$-weakly uniform base then X is quasi-developable and has Property N.*

Proof: Suppose that X has a $<\omega$-weakly uniform base $\mathcal{B}$. Lemma 2.1 shows that X is paracompact and that we may assume that members of $\mathcal{B}$ are convex.

First we show that X is quasi-developable. Let M be the union of all open, metrizable subspaces of X. Because X is hereditarily paracompact, M is metrizable. Let $Y = X - M$. Observe that it cannot happen that Y contains points $y_1 < y_2$ such that $]y_1,y_2[\ \cap Y$ is finite and non-empty because, X being first-countable, that would force $]y_1,y_2[\ \subseteq M$. However, it might happen that Y has consecutive points, i.e., points $y_1 < y_2$ with $]y_1,y_2[\ \cap Y = \emptyset$. Let L be the set of all points $y_1 \in Y$ such that for some $y_2 \in Y \cap\ ]y_1,\rightarrow[$, the set $]y_1,y_2[\ \cap Y = \emptyset$, and let R be the set of all points $y_2 \in Y$ such that for some $y_1 \in Y \cap\ ]\leftarrow,y_2[$ the set $]y_1,y_2[\ \cap Y = \emptyset$. Let $Y_2 = Y - L$ and $Y_1 = Y - R$. Then $X = M \cup Y_1 \cup Y_2$.

Consider the subspace Y_1. It inherits a $<\omega$-weakly uniform base from X and therefore has a base $\mathcal{D}$ of relatively open, relatively convex subsets that is $<\omega$-weakly uniform. Let a,b be distinct points of Y_1. Without loss of generality, we may suppose $a < b$. The key property of Y_1 is that the set $Y_1 \cap [a,b]$ must be infinite. Choose a finite set $F \subseteq [a,b] \cap Y_1$ such that only a finite number of members of $\mathcal{D}$ contain F. But by relative convexity, any member of $\mathcal{D}$ that contains the points a and b must contain all of F. Hence only a finite number of members of $\mathcal{D}$ can contain both a and b, and therefore $\mathcal{D}$ is a weakly uniform base for Y_1. Then by Theorem 1.1, the subspace Y_1 is quasi-developable. Similarly, Y_2 is also quasi-developable. Because $X = M \cup Y_1 \cup Y_2$, we may apply Lemma 2.2 to conclude that X is quasi-developable.

Second we show that X has Property N. Let $I = \{\{p\} : \{p\} \in \mathcal{B}\}$ and let

$$\mathcal{B}(n) = \{\cap_{i=1}^{n} B_i : \text{ the sets } B_i \text{ are distinct members of } \mathcal{B}\}.$$

Let $\mathcal{G}(1) = \mathcal{B}$. Given the open cover $\mathcal{G}(n)$ for any $n \geq 1$, let $\mathcal{H}(n)$ be a star-refinement of $\mathcal{G}(n)$ and let

$$\mathcal{G}(n+1) = I \cup \{C \in \mathcal{B}(n+1) : C \text{ is a subset of some member of } \mathcal{H}(n)\}.$$

Because any non-isolated point of X belongs to infinitely many members of $\mathcal{B}$, $\mathcal{G}(n+1)$ is an open cover of X.

Now suppose that S is any infinite subset of X. Because $\mathcal{B}$ is $< \omega$-weakly uniform, there is a finite subset $F \subseteq S$ such that $\mathcal{B}_F = \{B \in \mathcal{B} : F \subseteq B\}$ is finite, and we may assume that F has at least two points. Let $k = |\mathcal{B}_F|$ and let $n = k+2$. Let p be any point of F and for contradiction suppose that $F \subseteq \mathrm{St}(p, \mathcal{G}(n))$. Then $F \subseteq \mathrm{St}(p, \mathcal{H}(n))$ so that, because $\mathcal{H}(n)$ star-refines $\mathcal{G}(n-1) = \mathcal{G}(k+1)$ there is a member $G \in \mathcal{G}(k+1)$ with $F \subseteq \mathrm{St}(p, \mathcal{H}(n) \subseteq G$. Because F has at least two points, G is not a singleton and therefore G is the intersection of $k+1$ distinct members of $\mathcal{B}$, and that is impossible because $\mathcal{B}_F$ has only k members. Therefore, X has Property N. $\square$

Proposition 2.4 *If X is a quasi-developable GO-space with Property N, then X has a $< \omega$-weakly uniform base.*

Proof: Being a quasi-developable GO-space, X is paracompact and has a σ-disjoint base $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$. Let $\mathcal{G}(n)$ be the sequence of open covers guaranteed by Property N. Because X is paracompact, we may assume that each $\mathcal{G}(n)$ is a point-finite convex open cover of X and that $\mathcal{G}(n+1)$ refines $\mathcal{G}(n)$ for each $n \geq 1$.

Let $\mathcal{C}(n) = \{B \cap G : B \in \mathcal{B}(n) \text{ and} G \in \mathcal{G}(n)\}$. Then the collection $\mathcal{C} = \bigcup\{\mathcal{C}(n) : n \geq 1\}$ is a base for the space X. To show that $\mathcal{C}$ is $< \omega$ weakly uniform, let S be any infinite subset of X. Use Property N to find a finite set $F \subseteq S$ and an integer n such that for some $p \in F$, $F \not\subseteq \mathrm{St}(p, \mathcal{G}(n))$. Consider $\mathcal{C}_F = \{C \in \mathcal{C} : F \subseteq C\}$. If $i > n$ and there is some member $C \in \mathcal{C}(i)$ with $F \subseteq C$, write $C = G \cap B$ where $G \in \mathcal{G}(n)$ and $B \in \mathcal{B}(n)$. Then

$$F \subseteq G \subseteq \mathrm{St}(p, \mathcal{G}(i)) \subseteq \mathrm{St}(p, \mathcal{G}(n)),$$

contradicting the way n was chosen. Therefore

$$(**) \quad \mathcal{C}_F \subseteq \bigcup\{C \in \bigcup_{i=1}^{n} \mathcal{C}(i) : F \subset C\}.$$

Consider any $i \leq n$. Because $\mathcal{B}(i)$ is a pairwise-disjoint collection, there is at most one member of $\mathcal{B}(i)$ that contains F. Because $\mathcal{G}(i)$ is a point finite cover of X there are at most finitely many members of $\mathcal{G}(i)$ that contain F. Hence there are at most finitely many members of $\mathcal{C}(i)$ that contain F so that in the light of (**), $\mathcal{C}_F$ must be finite. Therefore X has a $< \omega$-weakly uniform base $\square$

Theorem 2.5 *A GO-space X has a a $< \omega$-weakly uniform base if and only if X is quasi-developable and has Property N.*

Proof: Combine Propositions 2.3 and 2.4. $\square$

Our next result generalizes a theorem of Heath and Lindgren [HL] stating that a LOTS with a weakly uniform base must be metrizable.

Theorem 2.6 *If X is a LOTS with a $< \omega$-weakly uniform base, then X is metrizable.*

Proof: From Theorem 2.5, we know that X is quasi-developable, and from Lemma 2.1 we know that X is paracompact. It is known that any paracompact, quasi-developable p-space (in the sense of Arhangelskii) is metrizable [B2].

To show that X is a p-space, begin with any $<\omega$-weakly uniform base $\mathcal{B}$ whose members are convex subsets of X. For each $n \geq 1$, let

$$\mathcal{G}(n) = \{\{p\} : p \text{ is isolated in } X\} \cup \{\cap_{i=1}^{n} B_i : B_1, \cdots, B_n \text{ are distinct members of } \mathcal{B}\}.$$

Because each non-isolated point of X must belong to infinitely many members of $\mathcal{B}$, each $\mathcal{G}(n)$ is an open cover of X.

For each $p \in X$, let $J(p) = \bigcap\{St(p, \mathcal{G}(n)) : n \geq 1\}$. Then $J(p)$ is a convex subset of X containing p. We claim that the set $J(p)$ is finite. If not, then one or both of the sets $J_1 = J(p) \cap \;]\leftarrow, p[$ and $J_2 = J(p) \cap \;]p, \rightarrow[$ is infinite. Without loss of generality, assume J_1 is infinite. Then there is a finite subset $F \subseteq J_1$ such that only a finite number of members, say k, of $\mathcal{B}$ contain F. We may assume that $F \neq \emptyset$. Let q be the least point of the finite set F and consider $\mathcal{G}(n)$ where $n > k$. Because $q \in J(p) \subseteq St(p, \mathcal{G}(n))$ we may choose a set $G \in \mathcal{G}(n)$ that contains both q and p. But then by convexity $F \subseteq [q, p] \subseteq G$. Hence G is not a singleton, so G must be the intersection of n distinct members of $\mathcal{B}$, showing that F is contained in more that k members of $\mathcal{B}$, and that is impossible. Hence $J(p)$ is a finite, convex subset of X.

Because X is a LOTS, for any open set U containing the finite set $J(p)$ there must some interval $]a, b[$ with $J(p) \subseteq \;]a, b[\; \subseteq U$. Because neither a nor b belongs to $J(p)$ there is some n such that neither a nor b belongs to $\mathrm{St}(p, \mathcal{G}(n))$. But then by convexity, $\mathrm{St}(p, \mathcal{G}(n)) \subseteq \;]a, b[\; \subseteq U$. Hence the sets $\mathrm{St}(p, \mathcal{G}(n))$ form an outer base for the compact set $J(p)$. It now follows from a theorem of Burke and Stoltenberg [BuS] that X is a p-space. According to the theorem of Bennett cited above, X is metrizable, as claimed. $\square$

In an earlier version of this paper, we gave a partial characterization of GO-spaces having m-weakly uniform bases. The referee pointed out that our technique could be used to get the desired characterization.

Proposition 2.7 : *Let X be any GO-space and let $m \geq 2$ be an integer. Then X has an m-weakly uniform base if and only if X is quasi-developable and has a sequence $\langle \mathcal{G}(n) \rangle$ of open covers with the property that for each $p \in X$, $\bigcap\{St(p, \mathcal{G}(n)) : n \geq 1\}$ has fewer than m points.*

Proof: First suppose that X has an m-weakly uniform base $\mathcal{B}$. We may assume that members of $\mathcal{B}$ are convex. In the light of Lemma 2.1 we know that X is paracompact. Let I be the set of all isolated points of X.

Let $\mathcal{G}(1) = \mathcal{B}$. Given any $\mathcal{G}(n)$, let $\mathcal{H}(n)$ be an open cover of X that star-refines $\mathcal{G}(n)$ and let $\mathcal{G}(n+1)$ be the collection

$$\{\{x\} : x \in I\} \bigcup \{G = B_1 \cap \cdots \cap B_{n+1} : B_i \in \mathcal{B}, \; B_i \neq B_j \text{ if } i \neq j, \text{ and } G \subseteq \text{ some member of } \mathcal{H}(n)\}.$$

Then $\mathcal{G}(n+1)$ is an open cover of X that also star-refines $\mathcal{G}(n)$.

Fix $p \in X$ and, for contradiction, suppose the set $\bigcap\{St(p, \mathcal{G}(n)) : n \geq 1\}$ has at least m points. Then there is a finite set F with $p \in F \subset \bigcap\{St(p, \mathcal{G}(n)) : n \geq 1\}$ and $|F| = m$. Index F as $F = \{q_1, \cdots, q_m\}$ where $q_1 < q_2 < \cdots < q_m$ and $p = q_i$ for some i. Because $\mathcal{B}$ is an m-weakly uniform base for X, the collection $\{B \in \mathcal{B} : F \subseteq B\}$ has only a finite number of members, say K. We know that $F \subseteq$

$St(p,\mathcal{G}(K+2))$ and that some member $G \in \mathcal{G}(K+1)$ has $F \subseteq St(p,\mathcal{G}(K+2)) \subseteq G$. Because $m \geq 2$ we know that G is not a singleton, so that G is the intersection of $K+1$ distinct members of $\mathcal{B}$. But that is impossible because F is a subset of at most K members of $\mathcal{B}$. Therefore, $\bigcap\{St(p,\mathcal{G}(n)) : n \geq 1\}$ has fewer than m points, as claimed.

Conversely, suppose that X is quasi-developable and has a sequence $\langle \mathcal{G}(n) \rangle$ of open covers as described in the theorem. Being quasi-developable, the GO-space X has a σ-disjoint base $\mathcal{C} = \bigcup\{\mathcal{C}(n) : n \geq 1\}$. We may assume that the members of $\mathcal{C}$ are convex sets. Also because the GO-space X is quasi-developable, we know that X is paracompact. Therefore, we may assume that $\mathcal{G}(n+1)$ refines $\mathcal{G}(n)$ and that each $\mathcal{G}(n)$ is point-finite.

For each $n \geq 1$ define

$$\mathcal{B}(n) = \{C \cap G : C \in \mathcal{C}(n), G \in \mathcal{G}(n)\}.$$

Then $\bigcup\{\mathcal{B}(n) : n \geq 1\}$ is a base for X. We claim that $\mathcal{B}$ is an m-weakly uniform base for X. Suppose F is a finite set with m elements and for contradiction suppose that F is a subset of infinitely many members of $\mathcal{B}$. Each collection $\mathcal{B}(n)$ is point-finite, so that F is a subset of only finitely many members of $\mathcal{B}(n)$ for each n. Hence there is a sequence $n_1 < n_2 < n_3 < \cdots$ such that for some $B_i \in \mathcal{B}(n_i)$ we have $F \subseteq B_i$. Choose $G_i \in \mathcal{G}(n_i)$ with $B_i \subseteq G_i$. But then $F \subseteq \bigcap\{St(p,\mathcal{G}(n_i)) : i \geq 1\} = \bigcap\{St(p,\mathcal{G}(n)) : n \geq 1\}$ and that is impossible because the latter set is known to have fewer than m points. Hence $\mathcal{B}$ is the required m-weakly uniform base. □

3 Examples

Our first example shows that a quasi-developable LOTS may fail to have a $<\omega$-weakly uniform base, even when it is the union of two metrizable subspaces. (Recall that the proof of Proposition 2.3 showed that any GO-space with a $<\omega$-weakly uniform base is the union of three metrizable subspaces.)

Example 3.1 *There is a quasi-developable LOTS that does not have a $<\omega$-weakly uniform base, does not have Property N, and is the union of two metrizable subspaces.*

Proof: Consider the lexicographically ordered set

$$M^* = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z}).$$

Note that, with the relative topology inherited from M^*, both $\mathbb{Q} \times \{0\}$ and $\mathbb{P} \times \mathbb{Z}$ are metrizable and their union is M^*. It is known [BL] that M^* is a quasi-developable LOTS that is not metrizable. One can use Theorem 2.6 to show that M^* does not have a $<\omega$-weakly uniform base. Therefore, Proposition 2.3 shows that X cannot have Property N. □

Next we present a family of examples showing that the various notions of weakly uniform bases defined in the Introduction are distinct.

Example 3.2 *: For each $n \geq 2$ there is a hereditarily paracompact GO-space that has an $(n+1)$-weakly uniform base, but not an n-weakly uniform base.*

Proof: Fix $n \geq 2$ and consider the space

$$Y = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{0, 1, \cdots, (n-1)\})$$

topologized as a subspace of M^* (see Example 3.1). Let $\{I_k : k \geq 1\}$ be a countable base of open intervals with rational endpoints for the usual space of all real numbers, where I_k has length $< \frac{1}{k}$. Let $J_k = \{(x,i) \in Y : x \in I_k\}$. Then

$$\mathcal{B} = \{\{(x,i)\} : x \in \mathbb{P} \text{ and } 0 \leq i \leq (n-1)\} \cup \{J_k : k \geq 1\}$$

is a base for Y that is $(n+1)$-weakly uniform.

For contradiction, suppose Y has an n-weakly uniform base $\mathcal{C}$. We may assume that members of $\mathcal{C}$ are either singletons or are convex subsets of Y that contain some points of $\mathbb{Q} \times \{0\}$. Choose a sequence $C_m \in \mathcal{C}$ such that if $D_m = \{x \in \mathbb{R} :$ there are points $(a,i) < (b,j)$ of C_m with $a < x < b\}$, then the length of D_m is positive and less than $\frac{1}{m}$. Each D_m is an open subset of the usual space of real numbers and each set $U_k = \bigcup\{D_m : m \geq k\}$ is dense in $\mathbb{R}$. Then the Baire Category Theorem yields an irrational number $x \in \bigcap\{U_k : k \geq 1\}$. Then the n-element subset $F = \{(x,0),(x,1),\cdots,(x,n-1)\}$ of Y is a subset of infinitely many of the sets C_m, showing that $\mathcal{C}$ is not n-weakly uniform. □

Example 3.3 *There is a GO-space X that has a $< \omega$-weakly uniform base but that does not have an n-weakly uniform base for any finite n.*

Proof: There are 2^ω G_δ-subsets of $\mathbb{R}$ that contain $\mathbb{Q}$, and every such set has cardinality 2^ω. This allows us to construct (inductively) a sequence P_n of pairwise disjoint subsets of $\mathbb{P}$ such that for each $n \geq 1$, $P_n \cap D \neq \emptyset$ for every G_δ-subset $D \subseteq \mathbb{R}$ that contains $\mathbb{Q}$.

Let $Y = (\mathbb{Q} \times \{0\}) \cup (\bigcup_{i=1}^{\infty} P_n \times \{0,1,\cdots,n\})$, topologized as a subspace of M^* in Example 3.1. Then Y is a GO-space and inherits a $< \omega$-weakly uniform base from M^*.

For contradiction, suppose that $\mathcal{B}$ is an n-weakly uniform base for Y, where $n < \omega$. As in Lemma 2.1, we may assume that members of $\mathcal{B}$ are relatively convex subsets of Y. As in the second half of the proof of Proposition 2.4, let $\mathcal{B}(k)$ be the collection of all sets that are intersections of k distinct members of $\mathcal{B}$ and for each $q \in \mathbb{Q}$ choose $B(q,k) \in \mathcal{B}(k)$ with $q \in B(q,k)$. Let $C(q,k) = \{y \in \mathbb{R} : \exists(x,0),(z,0) \in B(q,k)$ having $(x,0) < (y,0) < (z,0)\}$. Each $C(n,q)$ is open in $\mathbb{R}$ and $\mathbb{Q}$ is contained in the set $O(k) = \bigcup\{C(q,k) : q \in \mathbb{Q}\}$. Then the set $D = \bigcap_{i=1}^{\infty} O(i)$ is a G_δ-subset of $\mathbb{R}$ that contains $\mathbb{Q}$ and so there is a point $y \in P_{n+1} \cap D$. But then the set $F = \{(y,i) : 0 \leq i \leq n+1\}$ is a subset of Y that is contained in infinitely many members of $\mathcal{B}$, and that is impossible. □

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