

On Dense Subspaces of Generalized Ordered Spaces

by

Harold R. Bennett, Texas Tech University, Lubbock, TX 79409

David J. Lutzer, College of William and Mary, Williamsburg, VA 23187

Steven D. Purisch, Barry University, Miami Shores, FL 33161

Abstract. In this paper we study four properties related to the existence of a dense metrizable subspace of a generalized ordered (GO) space. Three of the properties are classical, and one is recent. We give new characterizations of GO-spaces that have dense metrizable subspaces, investigate which GO-spaces can embed in GO spaces with one of the four properties, and provide examples showing the relationships between the four properties.

AMS Subject codes (1980 scheme): 54F05, 54E35

1. Introduction.

The work of Souslin and Kurepa made clear how important it is to know whether a given linearly ordered space, or a generalized ordered space, has a dense subspace with special metric-related properties [T]. The purpose of this paper is to study the following four such properties:

- (I) X has a dense subspace that is the union of countably many closed, discrete subspaces of X ;
- (II) X has a dense metrizable subspace;
- (III) there are open sets $U(n)$ of X and relatively closed discrete subsets $D(n) \subset U(n)$ such that if G is open and $p \in G$, then for some $n \geq 1$, we have $p \in U(n)$ and $G \cap D(n) \neq \emptyset$ (see [BL]);
- (IV) X has a dense subset that is the union of countably many discrete (but not necessarily closed) subspaces.

These four properties are relevant to the Souslin problem precisely because in any generalized ordered space that has one of the above properties, cellularity = density. The properties have appeared elsewhere in the literature, too. For example, H.E. White showed in [W, Theorem 3.1] and [W2, Theorem 2.6] that they are relevant to the Blumberg problem. Chertanov [C] used properties similar to (I)-(IV) in his study of spaces that are co-absolute with ordered spaces, and in [Q] and [QT], Qiao and Tall used them in their study of non-Archimedean spaces.

Throughout this paper, properties (I), (II), (III), and (IV) will refer to the properties defined above. We will investigate the relationships between those four properties, study which generalized ordered spaces have the properties, investigate the extent to which the properties are hereditary, and determine which generalized ordered spaces embed in a generalized ordered space with one of those four properties. For example, we prove:

- 1.1. A GO-space has Property (I) if and only if it is perfect and has any one of the other properties listed above (see 4.8);
- 1.2. A GO-space has Property (II) if and only if it has a dense subspace with a G_δ -diagonal (see 4.6);
- 1.3. While Properties (II) and (IV) are not equivalent in GO spaces, they are equivalent in any GO space that satisfies the Baire Category Theorem (see 4.5, 4.6);
- 1.4. Properties (I) and (III) are hereditary in GO-spaces, but (II) and (IV) are not (see 5.1, 5.2, 5.3);
- 1.5. Any GO space with one of the properties listed above embeds topologically in a LOTS with the same property. (The proof for (II), (III), and (IV) is in (5.6); the proof for Property (I) is due to W. Shi [S].)

Recall that a generalized ordered space (GO-space) is a triple $(X, \mathcal{T}, <)$ where $<$ is a linear ordering of the set X and where \mathcal{T} is a Hausdorff topology on X having a base whose members are order-convex sets. In case \mathcal{T} is the usual open interval topology of $<$ we say that $(X, \mathcal{T}, <)$ is a linearly ordered topological space (LOTS). It is clear that the class of GO spaces is strictly larger than the class of LOTS, and it is known that the class of GO spaces coincides with the class of subspaces of LOTS. In any linearly ordered set X , we will use interval notations such as $]a, b[= \{x \in X : a < x < b\}$ and $[a, b[= \{x \in X : a \leq x < b\}$ and $[a, \rightarrow [= \{x \in X : a \leq x\}$. To say that S is a convex set means that $[a, b] \subset S$ whenever $a < b$ are points of S . Given any set Y in X , the convex components of Y are the maximal subsets of Y that are convex in X . At several points, we will need to refer to the connected components of a space or a set. To emphasize the distinction, we will use the phrase “connected components” in such situations, even though the phrase involves redundancy.

To say that a topological space X is perfect means that every closed subspace of X is a G_δ -subset of X . Cardinal functions will be as defined in [E]. Throughout this paper, it will be crucial to distinguish between subspaces of X that are relatively discrete (i.e., discrete spaces when endowed with the relative topology from X) and subspaces that are closed and discrete. For example, we will describe the dense sets given by property (I) as

being σ -closed-discrete, and the dense subsets given by property (IV) as being σ -relatively-discrete.

2. Preliminary results and examples.

The basic relationship between the four properties of the introduction are summarized as follows.

2.1 Proposition: *Let X be any GO-space. Then:*

- i) If X is metrizable, then X has property (I);*
- ii) (I) \Rightarrow (II) \Rightarrow (IV);*
- iii) (I) \Rightarrow (III) \Rightarrow (IV).*

Proof: Assertion (i) holds because any metrizable space has property (I). For assertion (ii), suppose X is a GO space with property (I). Let $D = \cup\{D(n) : n \geq 1\}$ be a dense subset of X that is the union of countably many closed discrete subspaces $D(n)$ of X . Then D is a GO space that is semistratifiable in the sense of Creede [Cr], so that D is metrizable [L1]. (See also [F] and [vW].) Thus (I) implies (II). Observe that property (II) implies (IV) in any space, so that (I) \Rightarrow (II) \Rightarrow (IV).

To verify (iii), suppose that X is a GO space having property (I) with respect to the dense subset $D = \cup\{D(n) : n \geq 1\}$. Letting $U(n) = X$ for each $n \geq 1$, we see that X has property (III), and (III) implies (IV) in any space. \square

Examples below will show that without additional hypotheses, (2.1) gives the only valid implications among the four properties. However, before giving those examples, let us place properties (I) through (IV) in a more general context. First, it is easy to see that each of the four properties (I), (II), (III), and (IV) is strictly weaker than metrizability.

2.2 Example. *The lexicographic product space $X = [0, 1] \times \{0, 1\}$ is not metrizable and yet is compact and has properties (I), (II), (III), and (IV). To see that X has property (I), note that X is separable. In the light of (2.1) above, X must have the other three properties as well. \square*

It is no accident that the space of (2.2) is badly disconnected, because van Wouwe [vW] has proved:

2.3 Proposition: *Suppose X is a connected LOTS having property (I). Then X is metrizable. \square*

It is surprisingly difficult to find an example of a linearly ordered topological space that has otherwise reasonable properties but fails to have properties (I), (II), (III), or (IV).

The best example of this type comes from a complicated tree-to-line construction given by Todorčević in [T].

2.4 Example: *There is a first-countable, compact, connected LOTS that does not have a property (IV), i.e., does not have a σ -relatively-discrete dense set [T].* \square

If one does not require first countability, then a more familiar space provides the required example.

2.5 Example: *The space $Y = [0, 1]^{\omega_1}$ with the lexicographic order \prec and the usual open interval topology is a compact and connected LOTS that does not have any of properties (I) through (IV).*

Outline of Proof: That Y is compact and connected follows from results in [Fa]. In the light of (2.1), it will be enough to show that Y does not have property (IV). In Section 4 we will show that properties (II) and (IV) are equivalent for a connected LOTS, so that it will be enough to show that Y does not have a dense metrizable subspace. Lemma 2.5.1 below shows that if Y had (II), then Y would be first-countable at a dense set of points. But, as Faber showed in [Fa, Lemma 1, p 19], Y has no points of countable character.

2.5.1 Lemma: *Suppose that a GO space X has a dense first-countable subspace S . Then X is first countable at each point of S .*

Proof: Fix $x \in S$. If x is isolated in X , there is nothing to prove, so suppose x is not isolated in X . Then x is not isolated in S and there are three easily checked cases to consider, depending upon whether neighborhoods of x in the space X are one-sided, or two-sided. \square

An example constructed in [BL, Example 5.4] and described in (4.5) shows that a LOTS can have property (III) and yet have no points of first countability. Thus, Lemma 2.5.1 cannot be proved for GO-spaces that have only property (III) or property (IV).

3. Characterizations of properties (I) and (II).

We begin with a characterization of property (I), due essentially to Creede [Cr] and van Wouwe [vW]. (See also [L1] and [Fa].)

3.1 Proposition: *A GO-space X has property (I) if and only if X is perfect and has a dense metrizable subspace.* \square

A related characterization of property (I) follows from a result of G.M. Reed. To simplify the statement, let us say that a topological space X has property (I') if there is a sequence of open covers $\mathcal{G}(n)$ of X such that the set $\{x \in X : \{\text{St}(x, \mathcal{G}(n)) : n \geq 1\} \text{ is a neighborhood base at } x\}$ is dense in X .

3.2 Proposition [R, Theorem 1.6]: *Let X be any regular, first countable space. Then X has property (I') if and only if X has property (I). \square*

3.3 Corollary: *For any GO space X , property (I) is equivalent to property (I').*

Proof: That (I') implies (I) follows directly from (3.2), once we prove that a GO space X with property (I') is first countable. To that end, let $\mathcal{G}(n)$ be the sequence of open covers of X that witnesses property (I'), and let D be the dense set such that for each $d \in D$, $\{\text{St}(d, \mathcal{G}(n)) : n \geq 1\}$ is a local base at d in X . We may assume that each $\mathcal{G}(n)$ consists of convex open subsets of X . For each $p \in X$ let $C(p) = \bigcap \{\text{St}(p, \mathcal{G}(n)) : n \geq 1\}$. We claim that each $C(p)$ is finite. If not, we can choose points $a < b$ in $C(p)$, both lying in $] \leftarrow, p[$ or both lying in $]p, \rightarrow [$. We consider only the first case. Then $]a, p[$ is a nonempty open set, so we may choose $d \in D \cap]a, p[$ and then n with $\text{St}(d, \mathcal{G}(n)) \subset]a, p[$. Because $a \in C(p) \subset \text{St}(p, \mathcal{G}(n))$ we may choose $G_0 \in \mathcal{G}(n)$ with $\{a, p\} \subset G_0$. Then convexity of G_0 forces $d \in [a, p] \subset G_0$ so that $\{a, p\} \subset G_0 \subset \text{St}(d, \mathcal{G}(n)) \subset]a, p[$, and that is impossible. Hence $C(p)$ is finite. Because $C(p)$ is a finite G_δ -subset of X that contains p , the point p is also a G_δ -subset of X , so X is first countable at p .

That (I) implies (I') for any GO space follows from (3.2) and the fact that any GO space with property (I) is first-countable. \square

There is a natural necessary and sufficient condition for the existence of a dense metrizable subspace of a GO space (i.e., for property (II)).

3.4 Proposition: *A GO space X has a dense metrizable subspace if and only if X has a dense subspace Y with a G_δ -diagonal in its relative topology.*

Proof: Necessity is obvious. For sufficiency, it will be enough to prove that any GO-space Y with a G_δ -diagonal has a dense metrizable subspace. To that end, let I be the set of isolated points of Y and let $Z = Y - \text{cl}(I)$, where $\text{cl}(I)$ denotes the closure of I in Y . Then Z is open in Y and has no isolated points, and Z also has a G_δ -diagonal.

Let $\{\mathcal{G}(n) : n \geq 1\}$ be a sequence of covers of Z by open, convex subsets of Y such that $\mathcal{G}(n+1)$ refines $\mathcal{G}(n)$ and such that for each $z \in Z$, we have $\bigcap \{\text{St}(z, \mathcal{G}(n)) : n \geq 1\} = \{z\}$. Because Z has a G_δ -diagonal, Z is paracompact [L1, Theorem 4.5] so for each $n \geq 1$ there is a sequence $\mathcal{F}(n, 1), \mathcal{F}(n, 2), \dots$ such that:

- i) each $\mathcal{F}(n, m)$ is a discrete (in Z) collection of relatively closed subsets of Z ;
- ii) $\bigcup \{\mathcal{F}(n, m) : m \geq 1\}$ covers Z and refines $\mathcal{G}(n)$.

For each $F \in \mathcal{F}(n, m)$ choose $p(F) \in F$ and let $D(n, m) = \{p(F) : F \in \mathcal{F}(n, m)\}$. Then $D(n, m)$ is closed and discrete in the space Z . Let $D = \bigcup \{D(n, m) : n, m \geq 1\}$. To show that D is dense in Z , suppose G is open in Z and $q \in G$. Then there is a convex Y -open

neighborhood C of q with $q \in C \subset G$. Because Z contains no isolated points, there are points $u < v < w$ in C . (Note: we cannot assert that $u < q < w$ because q might be an endpoint of C .) Then we can find an n with $\text{St}(v, \mathcal{G}(n)) \subset]u, w[$. Choose $m \geq 1$ so that $v \in \cup \mathcal{F}(n, m)$ and choose $F \in \mathcal{F}(n, m)$ with $v \in F$. Then $p(F) \in G \cap D$ as required to show that D is dense in Z .

Because D is σ -closed discrete in the space Z and is dense in Z , it follows that Z has property (I) and hence (see (3.1)) has a dense metrizable subspace M (indeed, $M = D$). Because I and M are subsets of the disjoint open subsets I and Z of Y , it follows that $I \cup M$ is a dense metrizable subset of Y , as required. \square

Although it is not among the properties listed in the Introduction, it is reasonable to ask about spaces that contain dense subspaces that are completely metrizable. There is a natural characterization of such spaces that resembles (3.4) but is even easier in the light of general theory.

3.5 Proposition: *A GO space X has a dense completely metrizable subset if and only if X contains a dense Čech-complete subspace Y that has a G_δ -diagonal for its relative topology.*

Proof: Necessity is obvious. To prove sufficiency, notice that the GO space Y will be paracompact (because Y is a GO space with a G_δ -diagonal [L1, Theorem 4.5]) and recall that a paracompact, Čech-complete space with a G_δ -diagonal is metrizable. Alternatively, notice that Y embeds in some compact LOTS and that, being Čech-complete, it embeds as a G_δ subset. Now apply the G_δ -diagonal metrization for such GO spaces ([L1, Theorem 5.9]). \square

Clearly, any space that contains a dense completely metrizable subset must be a Baire space (and even more – in the terminology of [AL], such a space is almost Čech-complete). It is natural to ask whether a suitably complete GO-space that contains a dense metrizable subspace must, in fact, contain a dense completely metrizable subspace. The following example provides a negative answer.

3.6 Example: *There is a compact LOTS X that contains a dense metrizable subset (indeed, X has property (I)) but does not contain any dense completely metrizable subset. Let X be the lexicographic product $[0, 1] \times \{0, 1\}$. Clearly $Q \times \{0, 1\}$ is the required dense metrizable subset. A dense completely metrizable subset would necessarily be uncountable, and it is easy to see that no uncountable subspace of X can be metrizable. \square*

However, an idea introduced by Arhangel'skii and Kocinak [AK] does allow us to identify dense completely metrizable subspaces of certain GO spaces.

3.7 Proposition: *A necessary and sufficient condition for a Čech-complete GO space X to have a dense completely metrizable subspace is that there is a G_δ -subset S of $X \times X$ that is a dense subset of the diagonal of X . \square*

Suppose P is one of the properties (I) through (IV). One often encounters (or builds) spaces that are countable unions of well-behaved subspaces, and that raises the question of whether a GO space X will have a property P provided $X = \cup\{X(n) : n \geq 1\}$ where each $X(n)$ has property P . It is easy to prove that such an X will have property P in case P is either property (III) or property (IV), but the space X constructed in [BL, Example 5.4] shows that X can be a countable union of relatively discrete subspaces $X(n)$ and yet not have property (I) or (II).

A very special case of the above situation occurs when the GO space X is the union of countably many dense and metrizable subspaces. For comparison, recall that van Wouwe's proved [vW, Theorem 2.3.4] that if a GO space X is the finite union of dense metrizable subspaces, then X must be metrizable. It is reasonable to ask what would happen if a GO space X is the countable union of such spaces. Unfortunately, such an X can fail to be metrizable and might not have Property (I), as can be seen from the familiar Michael line M . For let P and Q be the sets of irrational and rational numbers, respectively, and index $Q = \{q(n) : n \geq 1\}$. Let $X(n) = P \cup \{q(n)\}$. Each $X(n)$ is a dense completely metrizable subspace of M , and $M = \cup\{X(n) : n \geq 1\}$. However, GO spaces that are the countable union of dense metrizable subspaces are not without interesting structure. One can prove that such a space has a σ -disjoint base for its topology and must have properties (II), (III), and (IV).

4. Relations among the properties.

In Section 2 we noted that (I) \Rightarrow (II) \Rightarrow (IV) and (I) \Rightarrow (III) \Rightarrow (IV). In this section, we will provide examples showing that there are no other relations among properties (I) through (IV) that hold without additional hypotheses. We will also show that among GO spaces that are Baire spaces, properties (II) and (IV) are equivalent, and that all four properties are equivalent for perfect GO spaces.

4.1 Example: *The lexicographic square $L = [0, 1] \times [0, 1]$ has properties (II), (III), and (IV) but not property (I).*

Proof: The set $A = [0, 1] \times \{0, 1\}$ is a closed subset of L that is not a G_δ , so that L does not have property (I) in the light of 3.1. The subset $M = [0, 1] \times]0, 1[$ is a dense open subset that is a topological sum of copies of the open unit interval, so that M is

metrizable. Thus L has (II) and (IV). To see that L has property (III), see [BL, Example 5.1]. \square

As noted in Section 2, the lexicographic product space $[0, 1]^{\omega_1}$ has none of the properties (I) through (IV), and (4.1) shows that the lexicographic product $[0, 1]^2$ is useful in distinguishing among properties (I) through (IV). It is reasonable to ask whether the lexicographic product $[0, 1]^\omega$ might also be of use. Strangely, its properties exactly parallel those of the lexicographic square, as our next example shows.

4.2 Example. *The lexicographic product space $X = [0, 1]^\omega$ is a compact, connected, first countable LOTS that has properties (II), (III), and (IV), but not property (I).*

Proof: That X is compact, connected, and first countable follows from [Fa]. To see that X does not have property (I), apply (3.1) after observing that if X cannot be perfect, because if X were perfect, it would be hereditarily Lindelöf. But the collection $\mathcal{G} = \{G(t) : t \in [0, 1]\}$ is an uncountable collection of pairwise disjoint open sets, where $G(t) = \{f : [0, \omega[\rightarrow [0, 1] : f(0) = t \text{ and } f(1) \in]0, 1[\}$.

To see that X has property (III) we need some special notation. We will think of points of X as infinite sequences of real numbers. Let $\bar{0}$ be the sequence that is constantly equal to zero and let $\bar{1}$ be the sequence that is constantly equal to 1. Given a point $p = (p_1, p_2, \dots, p_n) \in [0, 1]^n$ and a rational number r , we will write

$$p * \bar{0} = (p_1, \dots, p_n, 0, 0, \dots) \text{ and}$$

$$p * r * \bar{0} = (p_1, \dots, p_n, r, 0, 0, \dots).$$

The points $p * \bar{1}$ and $p * r * \bar{1}$ are analogously defined. For each rational number $r \in]0, 1[$ let $D(0, r) = \{r * \bar{0}\}$ and $U(0, r) = X$. For each $n \geq 1$ and each rational number $r \in]0, 1[$, let $D(n, r) = \{p * r * \bar{0} : p \in [0, 1]^n\}$ and $U(n, r) = \cup\{]p * \bar{0}, p * \bar{1}[: p \in [0, 1]^n\}$. Each $U(n, r)$ is open in X , and $D(n, r) \subset U(n, r)$ for each $n \geq 0$.

It is straightforward to verify that $D(n, r)$ is a relatively closed and relatively discrete subset of $U(n, r)$. To complete the proof, suppose that G is open in X and $b \in G$.

Case 1. Suppose that $b \notin \{\bar{0}, \bar{1}\}$. We must find a rational number $r \in]0, 1[$ and an $n \geq 0$ with $b \in U(n, r)$ and $G \cap D(n, r) \neq \emptyset$. First find $a < b < c$ in X with $]a, c[\subset G$. Let n_1 be the first coordinate in which a and b differ, and let n_2 be the first coordinate in which b and c differ. If both n_1 and n_2 are greater than 1, then there is an $n \geq 1$ and a point $p \in [0, 1]^n$ and a rational number $r \in]0, 1[$ with $b \in]p * \bar{0}, p * \bar{1}[\subset U(n, r)$ and $p * r * \bar{0} \in D(n, r) \cap]a, c[\subset D(n, r) \cap G$. If $n_1 = 1$ or $n_2 = 1$ there is a rational $r \in]0, 1[$ with $r * \bar{0} \in]a, c[$ so that $\emptyset \neq D(0, r) \cap]a, c[\subset D(0, r) \cap G$, while $b \in X = U(0, r)$.

Case 2. Suppose $p \in \{\bar{0}, \bar{1}\}$. Let $E = \{\bar{0}, \bar{1}\}$ and $U = X$. Then $p \in U$ and $E \cap G \neq \emptyset$.

The sets $D(n, r), U(n, r), E$, and U satisfy the definition of property (III) for the space $X = [0, 1]^\omega$ as required. Thus X has property (III), and therefore also property (IV). To see that X has a dense metrizable subset, consider $M = \{f : [0, \omega[\rightarrow [0, 1] : \text{for some } n, f(m) = .5 \text{ for each } m \geq n\}$. Alternatively, (4.7) below shows that properties (II) and (IV) are equivalent in a connected LOTS. \square

4.3 Example: *The familiar “closed long line” (i.e., the usual long line with its right endpoint ω_1) is compact, connected, and has properties (II) and (IV), but has neither property (I) nor property (III).* That the closed long line has a dense metrizable subspace is clear. That it has neither property (I) nor property (III) follows from our next lemma.

4.4 Lemma: *If X is a GO space with property (I) or property (III), then X is hereditarily paracompact.*

Proof: For property (III), this follows from [BL, Theorem 4.2] and for property (I) it follows from the fact that any space with (I) is perfect (see (3.1)) and [L1, Theorem 4.8]. \square

4.5 Example: *There is a dense-ordered LOTS X that has properties (III) and (IV), but does not have property (II).* The space X is easy to describe. It is the subset of $[0, \omega_1]^\omega$ consisting of all points $(\alpha_1, \alpha_2, \dots)$ such that for some $n \geq 1$, $\alpha_i < \omega_1$ for each $i < n$ and $\alpha_i = \omega_1$ for each $i \geq n$. Then X is topologized using the open interval topology of the lexicographic order. The space X is constructed in [BL, Example 5.4] where its properties are studied. Because it is not first-countable at any point, (2.5.1) shows that it cannot have property (II). The space X is dense ordered and is the union of countably many relatively discrete subsets. \square

By way of contrast with the example in (4.5), there are types of generalized ordered spaces in which properties (II) and (IV) are equivalent. It follows immediately from a result of H.E. White [W2, Theorem 2.6] that (II) and (IV) are equivalent in first countable generalized ordered spaces, and related results appear in [W, Theorem 3.1]. We prove:

4.6 Proposition: *Let X be a GO space and a Baire space. Then X has property (II) if and only if X has property (IV).*

Proof: For any space, (II) implies (IV). To prove the converse, suppose that X is a GO space and a Baire space that has property (IV). Let $X_0 = \{x \in X : x \text{ is isolated in } X\}$, and let $Y = X - \text{cl}(X_0)$. Then Y is open in X so that Y is a GO space, a Baire space, and has property (IV). We will show that Y contains a dense metrizable subspace M . Then $X_0 \cup M$ will be a dense subspace of X and will be metrizable because X_0 and M are subsets of disjoint open subsets of X .

To prove that Y has the required dense metrizable subspace, we will show that Y

contains a dense subspace that has a G_δ -diagonal for its relative topology, and then we will invoke (3.4). Let $D = \cup\{D(n) : n \geq 1\}$ be a dense subset of Y where each $D(n)$ is relatively discrete. Because Y is open in X and $Y \cap X_0 = \emptyset$, Y has no isolated points. Therefore, each $D(n)$ is nowhere dense in Y . Let $H(n) = Y - \cup\{\text{cl}(D(i)) : i \leq n\}$. Each $H(n)$ is an open dense subset of Y so that $E = \cap\{H(n) : n \geq 1\}$ is a dense subset of the Baire space Y .

For each n let $\mathcal{G}(n) = \{G : G \text{ is a convex component of } H(n) \text{ in } X\}$. Then $\mathcal{G}(n)$ is a pairwise disjoint cover of E and for each $e \in E$ there is a unique member $G(n, e) \in \mathcal{G}(n)$ that contains e . Note that $\text{St}(e, \mathcal{G}(n)) = G(n, e)$. Let $C(e) = \cap\{G(n, e) : n \geq 1\}$. The $C(e)$ is a convex subset of X and $e \in C(e) \subset E$ for each $e \in E$.

We claim that each $C(e)$ is finite. If that is not the case, then we can choose $a < b$ with $a, b \in C(e) \cap] \leftarrow, e[$ or with $a, b \in C(e) \cap]e, \rightarrow [$. The two cases are analogous, so we consider only the first. Because $\emptyset \neq]a, e[\subset C(e) \subset Y$ and D is dense in Y , there is an n and a $d \in D(n)$ such that $d \in]a, e[$. But then convexity forces $d \in [a, e] \subset C(e) \subset G(n, e) \subset H(n)$ and that is impossible because $D(n) \cap H(n) = \emptyset$. Therefore, each $C(e)$ is finite. Because $C(e)$ is convex and Y has no isolated points, we conclude that $|C(e)| \leq 2$ for each $e \in E$.

Next suppose that $e_1 < e_2$ are points of E with $|C(e_i)| = 2$. Suppose $C(e_1) \cap C(e_2) \neq \emptyset$. If $C(e_1) \neq C(e_2)$, then $C(e_1) \cup C(e_2)$ would be a convex subset of X with three points, and that would force Y to contain an isolated point of X , contrary to $Y \cap X_0 = \emptyset$. Therefore, if $C(e_1)$ and $C(e_2)$ each have two points, then the sets $C(e_1), C(e_2)$ are either disjoint or identical.

Let $E_1 = \{e \in E : |C(e)| = 1\}$ and $E_2 = E - E_1$. Define $E_3 = \{\min(C(e)) : e \in E_2\}$ and let $E^* = E_1 \cup E_3$. Then $E_3 \subset E$ and we claim that E_3 is dense in Y . For suppose that $U \neq \emptyset$ is an open, convex subset of Y . Then U must be infinite and we can choose $e_1 < e_2$ in $E \cap U$ because E is dense in Y . If $\{e_1, e_2\} \cap E_1 \neq \emptyset$, or if $e_i \in E_2$ and $e_i = \min(C(e_i))$ for $i = 1$ or 2 , then $U \cap E_3 \neq \emptyset$, so assume that $e_i \in E_2$ and $e_i = \max(C(e_i))$ for $i = 1, 2$. Then we have $C(e_i) = \{e_i^-, e_i\}$ where $e_i^- < e_i$ for $i = 1, 2$. Then $e_1 < e_2$ and $C(e_1) \cap C(e_2) = \emptyset$ so that $e_1 < e_2^- < e_2$. Hence convexity yields $e_2^- \in [e_1, e_2] \subset U$. But then $e_2^- \in U \cap E_3$ as required to show E_3 is dense in Y .

To show that the subspace E_3 has a G_δ -diagonal, let $\mathcal{J}(n) = \{G \cap E_3 : G \in \mathcal{G}(n)\}$. Each $\mathcal{J}(n)$ is a relatively open cover of E_3 . Let $e \in E_3$. If $e \in E_1$, then $\cap\{\text{St}(e, \mathcal{J}(n)) : n \geq 1\} = \{e\}$. If $e \in E_2$, then $\cap\{\text{St}(e, \mathcal{J}(n)) : n \geq 1\} = C(e) \cap E_3 = \{e\}$. Therefore, the subspace E_3 has a G_δ -diagonal, as required. \square

4.7 Corollary: *Let X be a GO space that is either compact or connected. Then X has*

property (II) if and only if X has property (IV). \square

None of the examples used above to distinguish among properties (I) through (IV) is a perfect space. That is no accident, because:

4.8 Proposition: *For a perfect GO space, properties (I), (II), (III), and (IV) are equivalent.*

Proof: In the light of (2.1), it will be enough to show that any GO space X satisfying (IV) must also satisfy (I). That follows from the fact that in any perfect space, any relatively discrete subspace is σ -closed-discrete. \square

The equivalence of properties (I) - (IV) in perfect generalized ordered spaces has been proved several times. For example, [Q, Theorem 4.4.1] establishes the equivalence in (4.8) and links property (I) to the special π -bases studied by White in [W] and [W2].

To close this section, we restate an old question due to Maurice and van Wouwe [vW] and also to Heath. Their question grows out of the fact that any Souslin space is perfect and does not have any one of the properties (I), (II), (III) and (IV), but the existence of Souslin spaces required set theory beyond ZFC.

4.9 Question: Is there a ZFC example of a perfect GO space that does not have property (I)?

For related questions, see also (5.7) and [Q] and [QT].

5. Heredity and embedding problems

Another way to distinguish between properties (I), (II), (III), and (IV) is to examine the extent to which the properties are hereditary. Obviously, each property is inherited by open subspaces, but much more is true for (I) and (III). The first result was proved in [BL, Theorem 4.3].

5.1 Proposition: *If X is a GO space with property (III), then every subspace of X also has property (III).* \square

5.2 Proposition: *If X is a GO space with property (I), then every subspace of X also has property (I).*

Proof: Suppose $Y \subset X$. According to (2.1) and (3.1), X is perfect and has property (III). According to (5.1), Y inherits property (III) from X . Furthermore, Y is perfect. Applying (3.4) to the GO space Y , we conclude that Y has property (I). \square

The situation with properties (II) and (IV) is entirely different, as can be seen from the following example.

5.3 Example: *Properties (II) and (IV) are not hereditary among GO spaces.*

Proof: In [T], Todorčević constructs a compact, connected, first-countable LOTS Y that has no σ -relatively-discrete dense set. Let X be the lexicographic product space $Y \times [0, 1]$. Then X is a compact, connected, first countable LOTS, and $M = Y \times]0, 1[$ is a dense subspace of X that is the union of pairwise disjoint copies of the open interval $]0, 1[$. Thus M is metrizable. Consider the subspace $T = Y \times \{1\}$ of X . That space is homeomorphic to (Y, \mathcal{R}) , where \mathcal{R} is the right Sorgenfrey topology on Y having a base of half-open intervals of the form $[a, b[$. Let \mathcal{I} be the usual open interval topology on Y .

To complete the proof, we show that (Y, \mathcal{R}) cannot have a σ -relatively discrete subset. For suppose $D = \cup\{D(n) : n \geq 1\}$ is a dense subset of (Y, \mathcal{R}) , where each $D(n)$ is relatively \mathcal{R} -discrete. We may assume that the end points of Y belong to $D(1)$. Each $D(n)$ is nowhere dense in (Y, \mathcal{R}) . Let $E(n) = \cup\{\text{cl}_{\mathcal{R}}(D(k)) : 1 \leq k \leq n\}$. Each $E(n)$ is closed and nowhere dense, so the set $G(n) = Y - E(n)$ is open and dense in (Y, \mathcal{R}) . Therefore the set $H(n) = \text{Int}_{\mathcal{I}}(G(n))$ is open and dense in (Y, \mathcal{I}) . Let $Z = \cap\{H(n) : n \geq 1\}$. Because (Y, \mathcal{I}) is a Baire space (being locally compact), Z is dense in (Y, \mathcal{I}) .

Let $\mathcal{H}(n)$ be the family of convex components of the \mathcal{I} -open set $H(n)$. Let $\mathcal{B}(n) = \{H \cap Z : H \in \mathcal{H}(n)\}$. Each $\mathcal{B}(n)$ is a pairwise disjoint relatively open cover of Z . Thus each $\mathcal{B}(n)$ is a discrete collection of open and closed subsets of the space Z . Therefore, we can complete the proof by showing that $\mathcal{B} = \cup\{\mathcal{B}(n) : n \geq 1\}$ is a base for the subspace Z of (Y, \mathcal{I}) , because Z will then be the required dense metrizable subspace of (Y, \mathcal{I}) .

To that end, suppose U is open in (Y, \mathcal{I}) and $p \in U \cap Z$. Because p is not an end point of Y , there are points $a < p < b$ in Y such that $p \in]a, b[\subset U$. Because (Y, \mathcal{I}) is connected, $]a, p[\neq \emptyset \neq]p, b[$. Hence there are integers $m, n \geq 1$ such that $D(m) \cap]a, p[\neq \emptyset$ and $D(n) \cap]p, b[\neq \emptyset$. Let $N = \max(m, n)$, and let H_p be the unique member of $\mathcal{H}(N)$ that contains p . Then $p \in H_p \subset]a, b[\subset U$ so that $H_p \cap Z \in \mathcal{B}(N)$ and $p \in H_p \cap Z \subset U \cap Z$, as required. \square

5.4 Remark: If we do not require first countability, then we can use the much easier example $Y = [0, 1]^{\omega_1}$ in the proof of (5.3) in lieu of Todorčević's space, once we know that $Y = [0, 1]^{\omega_1}$ does not have property (IV). That is proved in (2.5), above.

There are two natural embedding questions associated with properties (I) through (IV). The first is:

First Embedding Question: Let P be one of the four properties (I) through (IV). Which GO spaces X can be embedded in a GO space Y with property P ?

That question is trivial for properties (I) and (III): in the light of (5.1) and (5.2), X can be embedded in a GO space with property (I) or property (III) if and only if X

itself is a GO space with property (I) or property (III). The first Embedding Question is more interesting for properties (II) and (IV), and the following result reduces it to the question “Which connected LOTS have property (II) or property (IV)?” One answer to that question is in (3.4), above.

5.5 Proposition: *Let P be either property (II) or property (IV) and let X be a GO space. Then the following are equivalent:*

- i) X can be embedded in a GO space having property P ;*
- (ii) If $W = \cup\{U \subset X : U \text{ is open and has property } P\}$ and if C is a connected subset of X with $C \cap W = \emptyset$, then $|C| = 1$;*
- (iii) Every non-degenerate connected component of X has property P ;*
- (iv) X embeds in a compact connected LOTS having property P .*

Proof: Because (iv) obviously implies (i), it will be enough to show that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). We will consider the case where $P = (II)$, i.e., where P is the property of having a dense metrizable subspace. The case where $P = (IV)$ is entirely analogous.

To show that (i) implies (ii), suppose X embeds in the GO space Z that has a dense metrizable subspace. Let C be a connected subset of X with $|C| > 1$. Let $C^* = \text{Int}_Z(C)$. Then C^* is a nonempty convex open subset of Z . In addition, C^* is also an open subset of X and the closure of C^* in X is C . Being open in Z , C inherits a dense metrizable subspace. Because C^* is open in X , $\emptyset \neq C^* \subset C \cap W$ as required to prove (ii).

Next suppose that X satisfies (ii) and let C be a nondegenerate connected component of X . Whenever $a < b$ are points of C , the interval $]a, b[$ is an infinite connected subset of X so that $]a, b[\cap W \neq \emptyset$. Therefore, each open subset of C contains an open subset with a dense metrizable subspace. Let \mathcal{V} be a maximal pairwise disjoint subcollection of $\{V \subset C : V \text{ is open in } X \text{ and has a dense metrizable subspace}\}$. Then $\cup \mathcal{V}$ is dense in C and has a dense metrizable subspace, so that C also has such a subspace.

To show that (iii) implies (iv), let \mathcal{C} be the family of non-degenerate connected components of X , and for each $C \in \mathcal{C}$ let $M(C)$ be a dense metrizable subspace of $\text{Int}_X(C)$. Then $M(C)$ is dense in C . Let X^+ be the usual Dedekind completion of the ordered set X , and let $G = X^+ - X$ be the set of gaps of X .

A left-pseudogap of X is a point $x \in X$ such that $[x, \rightarrow [$ is open in X even though $\emptyset \neq] \leftarrow, x[$ does not contain a right endpoint, and a right-pseudogap of X is a point $x \in X$ such that $] \leftarrow, x]$ is open in X even though $\emptyset \neq]x, \rightarrow [$ does not contain a first point. Let L and R be the sets of left and right pseudogaps of X , respectively. Observe that $L \cap R$ might be nonempty. Let $I = \{x \in X - (R \cup L) : \{x\} \text{ is open in } X\}$.

Define $Y = (X \times \{0\}) \cup (L \times]-1, 0[) \cup (R \times]0, 1[) \cup (G \times]-1, 1[)$ and order Y lexicographically. Then Y is a LOTS and X is homeomorphic to the closed subspace $X \times \{0\}$ of Y . Let

$$M = (I \times \{0\}) \cup (R \times]0, 1[) \cup (L \times]-1, 0[) \cup (G \times]-1, 1[) \cup (\cup \{M(C) \times \{0\} : C \in \mathcal{C}\}).$$

Because the sets used to construct M are metrizable subsets of pairwise disjoint open subsets of Y , the subspace M is metrizable. Next we show that M is dense in Y , and for that it is enough to show that $X \times \{0\} \subset \text{cl}_Y(M)$. For contradiction, suppose there is a point $(x, 0) \in (X \times \{0\}) - \text{cl}_Y(M)$. Let U be a convex Y -neighborhood of $(x, 0)$ that is disjoint from $\text{cl}_Y(M)$. Then $U \subset X \times \{0\}$. Observe that $x \notin I \cup L \cup R \cup (\cup \mathcal{C})$ so that U must be an infinite set that contains no left or right pseudogaps. At the same time, U cannot be connected, so that U must contain infinitely many gaps of X , i.e., points of G . But then, convexity of U forces $U \cap (G \times]-1, 1[) \neq \emptyset$, contrary to $U \cap M = \emptyset$. Therefore, M is a dense metrizable subset of the LOTS Y .

There is a compact LOTS Y^+ that contains Y as a dense subspace. Of course, Y^+ might not be connected, because it might have jumps. Insert a copy of the usual open interval $]0, 1[$ into each jump, to form a compact, connected LOTS Z . The space Y^+ contains a dense subspace that is σ -relatively-discrete, and the set $Z - Y^+$ is a disjoint union of copies of $]0, 1[$ so that it also contains a dense subset that is σ -relatively discrete. According to (4.7), it follows that Z has a dense metrizable subset and that completes the proof of (5.5). \square

The second embedding question grows out of the fact that every GO space X can be embedded in a LOTS ([L1]) and that for many topological properties Q , a GO space has property Q if and only if it embeds in a LOTS with property Q . Examples are $Q =$ paracompactness and $Q =$ metrizability.

Second Embedding Question: Suppose that a GO space X has property P , where P is one of the properties (I) through (IV). Can X be embedded in a LOTS with property P ?

For properties (II), (III), and (IV) our next result gives a complete answer to the second embedding question. And affirmative answer for property (I) has been given by Shi [S] (see 5.7, below).

5.6 Proposition: *Let X be a GO space having property P , where P is one of (II), (III), and (IV). Then X embeds as a closed subset of a LOTS having property P .*

Proof: Following the notation of [L1], there is a LOTS X^* that contains X as a closed subspace and has the property that $X^* - X$ consists entirely of isolated points of X^* . We will show that X^* has property P provided X does.

Case 1: If P is property (IV), suppose $D = \cup\{D(n) : n \geq 1\}$ is a dense subspace of X , where each $D(n)$ is relatively discrete. Let $D(0) = X^* - X$. Then $\cup\{D(n) : n \geq 0\}$ is the required dense subset of X^* .

Case 2: If P is property (II), suppose M is a dense metrizable subspace of X . Let $N = X^* - X$ and let $U = X^* - \text{cl}_{X^*}(N)$. Then U is an open subspace of X , so that $(U \cap M)$ is a dense metrizable subspace of U . Then $N \cup (U \cap M)$ is a dense metrizable subset of X^* .

Case 3: In case P is property (III), suppose the open sets $U(n)$ and the relatively closed, relatively discrete subsets $D(n) \subset U(n)$ witness property (III) for $n \geq 1$. Let $V(n) = \cup\{V : V \text{ is open in } X^* \text{ and } |V \cap D(n)| \leq 1\}$. Let $V(0) = X^* - X$ and $D(0) = X^* - X$. Then the sequences $V(n)$ and $D(n)$ for $n \geq 0$ show that X^* has property (III). \square

5.7 The second embedding question for Property (I): In an earlier version of this paper, the authors asked whether a GO space X having property (I) can be embedded in a LOTS Y with property (I). (There is no requirement that X be dense in Y , or closed in Y , or that the ordering of Y must extend the ordering of X .) Wei-Xue Shi answered our question affirmatively in [S]. See [S] for an explanation of how Shi's result is related to the question of Maurice and van Wouwe mentioned in (4.9). For related results, see also [SMG], [Q], and [QT].

Bibliography

- [AK] Arhangel'skii, A. and Kocinek, L., On a dense G_δ -diagonal, Publ. Inst. Math. (Beograd) 47 (1990), 121-126.
- [AL] Aarts, J. and Lutzer, D., Completeness properties designed for recognizing Baire Spaces, Dissertationes Math. 116 (1974), 1-45.
- [BL] Bennett, H. and Lutzer, D., Point-countability on generalized ordered spaces, Topology and its Applications 20 (1996), 1-17.
- [C] Chertanov, G., Compacta that are co-absolute with linearly ordered ones, Siberian Math. J. 26(1985), 765-776.
- [Cr] Creede, G., Concerning semi-stratifiable spaces, Pacific J. Math. 32 (1970), 47-54.
- [E] Engelking, R. General Topology, Heldeman Verlag, Berlin, 1989.
- [Fa] Faber, M., Metrizable in Generalized Ordered Spaces, Math Center Tracts 53, Mathematical Centre, Amsterdam, 1974.
- [L1] Lutzer, D., On generalized ordered spaces, Dissertationes Math. 89 (1971), 1-36.
- [Q] Qiao, Y.Q., On non-Archimedean spaces, Ph.D. Thesis, Univeristy of Toronto, 1992.

- [QT] Qiao, Y.Q. and Tall, F., Perfectly normal non-metrizable non- archimedean spaces are generalized Souslin lines, Proc. Amer. Math. Soc., to appear.
- [R] Reed, G.M., Concerning first countable spaces, Fundamenta Math. 74(1972), 161-169.
- [S] Shi, W-X., Extensions of perfect GO-spaces and σ -discrete dense sets, Proc. Amer. Math. Soc., to appear.
- [SMG] Shi, W., Miwa, T., and Gao, Y., A perfect GO space which cannot densely embed in any perfect orderable space, Topology and its Applications, 66 (1995), 241-249.
- [T] Todorčević, S., Stationary sets, trees, and continuums, Publ. Inst. Math. (Beograd), 29 (1981), 109-122.
- [W] White, H.E., Topological spaces in which Blumberg's theorem holds, II, Illinois J. Math. 23(1979), 464-468.
- [W2] White, H.E., First countable spaces that have special pseudobases, Canad. Math. Bull. 21 (1978), 103-112.
- [vW] van Wouwe, J., GO-spaces and Generalizations of Metrizable, Mathematical Centre Tracts 104, Mathematical Center, Amsterdam, 1979.