Lines, Trees, and Branch Spaces

by

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Abstract: In this paper we examine the interactions between the topology of certain linearly ordered topological spaces (LOTS) and the properties of trees in whose branch spaces they embed. As one example of the interaction between ordered spaces and trees, we characterize hereditary ultraparacompactness in a LOTS (or GO-space) X in terms of the possibility of embedding the space X in the branch space of a certain kind of tree.

Key words and phrases: linearly ordered topological space, LOTS, generalized ordered space, GO-space, tree, branch space, ultraparacompact, Souslin space.

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1. Introduction.

By the word "line" in the title of this paper we mean a linearly ordered topological space, or LOTS. Ever since Kurepa's early work on the Souslin problem [K], topologists have used trees to study lines. Starting with a line one can construct a tree whose members are convex subsets of the original line and that sometimes reflects crucial properties of the line. Alternatively, one can start with a tree from some other source and construct its branch spaces, often obtaining lines with interesting properties. For more details, see Section 2 below, and for recent examples of these constructions, see [HJNR], [Sh], [To].

The goal of this paper is to explore certain aspects of the interaction between lines, trees, and branch spaces, and to obtain some results related to the problem "Which lines can be realized as the branch spaces of nice trees?" In Section 2 we give relevant definitions and describe the background and motivation for our study. In Section 3 we characterize those situations in which a LOTS X embeds in the branch space of a tree constructed using convex subsets of X. In Section 4 we give examples of "nice" properties that a tree might have and list the topological consequences for their branch spaces. In Section 5 we give an example of the interaction between trees and lines by characterizing hereditarily ultraparacompact LOTS in terms of the kinds of branch spaces in which they embed. Specifically, we prove that a LOTS X is hereditarily ultraparacompact if and only if X embeds in the branch space of a tree $(\mathcal{T}, \sqsubseteq)$ having the properties that:

1) if N is a node of $\mathcal{T}$ and if $t \in N$ is not an endpoint of N, then $[t]$ is a clopen subset of the branch space; and

2) if t is an endpoint of the node N, then $\{s \in \mathcal{T} : s \sqsubseteq t\}$ is a branch of $\mathcal{T}$; and

3) if B is a branch of $\mathcal{T}$, then both sets $\{\alpha :$ the node of $\mathcal{T}$ to which $B(\alpha)$ belongs has a left (resp. right) endpoint $\}$ are hereditarily paracompact subsets of the ordinal space $[0, \kappa)$ where κ is the height of the tree $\mathcal{T}$.

This result is not intended as a working characterization of ultraparacompactness in ordered spaces but rather as an example of the kind of theorems about line-tree-branch space interactions that are possible. We also give examples showing that the conditions above cannot be replaced by others considered in Section 4.

2. Background and statement of the problem

By a *linearly ordered topological space* (LOTS) we mean a triple $(X, <, \mathcal{I})$ where $(X, <)$ is a linearly ordered set and $\mathcal{I}$ is the usual open interval topology of the order $<$. Easy examples show that there can be subsets $Y \subseteq X$ with the property that the subspace topology on Y is not the same as the open interval topology induced on Y by the restriction to Y of the linear ordering of X. That led Čech to introduce the notion of a *generalized ordered space* (GO-space), i.e. a triple $(X, <, \mathcal{T})$ where $<$ is a linear ordering of X and where $\mathcal{T}$ is a Hausdorff topology on X having a base consisting of convex subsets of X. (Recall that a subset $C \subseteq X$ is *convex* provided $x \in C$ whenever x lies between two points of C.) It is known that the GO-spaces are precisely the spaces that embed topologically in some LOTS.

By a *tree* we mean a partially ordered set $(\mathcal{T}, \sqsubseteq)$ with the property that for each $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} : s \sqsubseteq t\}$ is well-ordered by $\sqsubseteq$. By $\mathcal{T}_\alpha$ we mean the set of all elements $t \in \mathcal{T}$ with the property that the set of all predecessors of t in $(\mathcal{T}, \sqsubseteq)$ is order-isomorphic to the set $[0, \alpha)$ of ordinals. The set $\mathcal{T}_\alpha$ is called the α-th level of $\mathcal{T}$ and for any $t \in \mathcal{T}_\alpha$ we define the *level of* t by $lv(t) = \alpha$. Clearly, no tree $\mathcal{T}$ can have $\mathcal{T}_\alpha \neq \emptyset$ for $\alpha > |\mathcal{T}|$ so that for sufficiently large $\alpha,\ \mathcal{T}_\alpha = \emptyset$ and the height of $\mathcal{T}$ is the least ordinal κ such that $\mathcal{T}_\kappa = \emptyset$. The ordinal κ might or might not be a limit ordinal. As explained below, the elements of a tree might be convex subsets of some linearly ordered set, but they might also be some completely different kind of object. See for example the construction of an Aronszajn tree given in [J]. [R2], or [To]. To emphasize this fact, we will often speak of an *abstract* tree when we want to emphasize that the tree is not necessarily made up of convex subsets of some known linearly ordered set.

For any $t \in \mathcal{T}$ the *node containing* t is the set of all $s \in \mathcal{T}$ having exactly the same predecessors as does t. If s and t belong to the same node of $\mathcal{T}$ then s and t belong to the same level of $\mathcal{T}$, and s and t are incomparable using the partial order of the tree. By a *branch* of $\mathcal{T}$ we mean a maximal linearly ordered subset of $(\mathcal{T}, \sqsubseteq)$. If B is a branch of the tree, then for each α, either $B \cap \mathcal{T}_\alpha = \emptyset$ or the set $B \cap \mathcal{T}_\alpha$ has exactly one element which we denote by B_α or $B(\alpha)$. Observe that a branch might or might not have a last element.

Suppose each node N of $\mathcal{T}$ is given a linear ordering $<_N$. (There is no assumption made about how the linear orders of different nodes are related to each other.) Given these orderings of the nodes, we can linearly order the set of all branches of $\mathcal{T}$ as follows. Let B and C be distinct branches of $\mathcal{T}$. Let $\alpha = fd(B, C)$ by which we mean the first ordinal where B and C differ. Then $\beta < \alpha$ implies $B_\beta = C_\beta$ so that B_α and C_α belong to the same node N of $\mathcal{T}$. We define $B <_{\mathcal{B}} C$ if and only if $B_\alpha <_N C_\alpha$ where $\alpha = fd(B, C)$. With the usual open interval topology of the order $<_{\mathcal{B}}$, the set $\mathcal{B}$ of all branches of $\mathcal{T}$ is a LOTS called the *branch space of* $\mathcal{T}$. To some extent, that is a misnomer because the orderings chosen for the nodes of $\mathcal{T}$ determine the branch space just as surely as does the tree itself. However, one always has:

2.1 Lemma: For each $t \in \mathcal{T}$, the set $[t] = \{B \in \mathcal{B} : t \in B\}$ is a convex subset of the branch space, and so is $[N] = \bigcup\{[t] : t \in N\}$ for any node N of $\mathcal{T}$.

Remark: In general one cannot say whether the sets $[t]$ and $[N]$ will be open or closed in the branch space, and those simple topological properties of $[t]$ and $[N]$ are crucial as will be seen from later sections.

The process described above goes from a tree to a line. There is a reverse process that constructs trees starting with linearly ordered sets. This second process has been described in different ways by different authors, and not all descriptions are equivalent. What we describe below is a very general construction that seems to cover most cases.

The levels of the tree are constructed recursively, and depend upon the choice of a process that breaks up the convex subsets of the linearly ordered set $(X, <)$. For any convex subset $C \subseteq X$, let $\mathcal{N}(C) = \emptyset$ if $|C| \le 1$ and if $|C| \ge 2$, then let $\mathcal{N}(C)$ be a pairwise disjoint collection of convex proper subsets that cover C. Define $\mathcal{T}_0 = \{X\}$. If $\mathcal{T}_\alpha$ is defined, then let $\mathcal{T}_{\alpha+1} = \bigcup\{\mathcal{N}(C) : C \in \mathcal{T}_\alpha\}$. If λ is a limit ordinal and $\mathcal{T}_\alpha$ is defined for all $\alpha < \lambda$, then let $\mathcal{T}_\lambda = \bigcup\{\mathcal{N}(C) : C = \bigcap\{t_\alpha : \alpha < \lambda\},\ t_\alpha \in \mathcal{T}_\alpha \text{ and } |C| > 1\}$. For large enough ordinals $\alpha,\ \mathcal{T}_\alpha = \emptyset$. (This is guaranteed to happen for $\alpha > 2^{|X|}$, for example.) Define $\mathcal{T}$ to be the union of all nonempty $\mathcal{T}_\alpha$. The partial ordering of $\mathcal{T}$ is reverse inclusion, i.e., $s \sqsubseteq t$ if and only if as convex subsets of X we have $t \subseteq s$. Then $\mathcal{T}$ is a tree. (Note that we do not really need to define $\mathcal{N}(C)$ for every convex subset $C \subseteq X$, but only for those convex sets that appear somewhere in the recursion.) Any tree constructed in this fashion is called a *partition tree* for $(X, <)$.

Unlike the situation for abstract trees, there is a natural linear ordering of each node of the partition tree $\mathcal{T}$; indeed the same natural linear ordering applies to each level $\mathcal{T}_\alpha$ of $\mathcal{T}$. Members of each node N of $\mathcal{T}$ are pairwise disjoint convex subsets of $(X, <)$ and we define that $s <_N t$ if each point of s precedes each point of t in the ordering of $(X, <)$. This ordering of nodes will be called *the precedence order from* X. Using that natural linear ordering of the nodes of $\mathcal{T}$, we can define a branch space just as we did for abstract trees. If one uses the precedence order from X to impose a linear ordering on each node of $\mathcal{T}$, it is reasonable to ask about the relation between the original LOTS X and the branch space of the partition tree $\mathcal{T}$. For any point $x \in X$ the set $B(x) = \{t \in \mathcal{T} : x \in t\}$ is a branch of $\mathcal{T}$ and the function $e : X \to \mathcal{B}$ given by $e(x) = B(x)$ is 1-1 and increasing. However, there is no guarantee that e is continuous or onto. We begin with two easy examples.

2.2 Example Let $[0, 1)$ be a subspace of the usual set of real numbers. For each convex set $[a, b) \subseteq [0, 1)$ choose a strictly increasing sequence $a = c_0 < c_1 < \cdots$ that has $\lim_{n\to\infty} c_n = b$ and $|c_{n+1} - c_n| \le \frac{b-a}{2}$ for each $n \ge 0$. We may assume that if a and b are both rational, then so is each c_n. (This extra assumption is not needed in the current example, but will be useful later, in (5.9).) Define $\mathcal{N}([a, b)) = \{[c_n, c_{n+1}) : n \ge 0\}$. Now define $\mathcal{T}_0 = \{[0, 1)\}$ and for $n \ge 0$ let $\mathcal{T}_{n+1} = \bigcup\{\mathcal{N}(t) : t \in \mathcal{T}_n\}$. The height of $\mathcal{T}$ is ω and $\mathcal{T} = \bigcup\{\mathcal{T}_n : n < \omega\}$. Each node of $\mathcal{T}$ beyond level 0 is countably infinite and its linear ordering is the precedence order induced from $[0, 1)$ as described above. We obtain a linearly ordered branch space $\mathcal{B}$ and, as will be seen in Section 3, the original space $[0, 1)$ embeds in $\mathcal{B}$ under the mapping $e(x) = \{t \in \mathcal{T} : x \in t\}$. In fact, because $[0, 1)$ is connected and $\mathcal{B}$ has no right endpoint, one can show that e is a homeomorphism from $[0, 1)$ onto $\mathcal{B}$. $\Box$

Our next example shows that it can happen that a LOTS X does not embed in the branch space of one of its partition trees and that the function $e : X \to \mathcal{B}$ may fail to be continuous.

2.3 Example Let $X = [0, 1)$ be a subspace of the real line. For any convex set $[a, b) \subseteq X$ define $\mathcal{N}([a, b)) = \{[a, \frac{a+b}{2}), [\frac{a+b}{2}, b)\}$. Let $\mathcal{T}_0 = \{X\}$ and for $n \ge 0$ define $\mathcal{T}_{n+1} = \bigcup\{\mathcal{N}(C) : C \in \mathcal{T}_n\}$ and $\mathcal{T} = \bigcup\{\mathcal{T}_n : n \ge 0\}$. Each node of $\mathcal{T}$ has exactly two members. A result of Todorčevic (see 4.1 below) shows that because each node of our tree is finite, the branch space must be compact. Consequently, $X = [0, 1)$ is not homeomorphic to the branch space in this example. But even more is true: the function $e(x) = B(x)$ is not continuous because if it were, then it would be an embedding of $X = [0, 1)$ into $\mathcal{B}$, and the latter space is totally disconnected so that the initial space $X = [0, 1)$ cannot be embedded in the branch space at all. $\Box$

Examples such as the two mentioned above led us to wonder when a line-to-tree-to-branch-space construction would result in a branch space that contained the original line. This problem is solved in Section 3, at least for the natural mapping from X to the branch space. Examples also led us to ask which LOTS could be realized as the branch space of a tree.

That second question is easily answered. In his article [To], Todorčevic points out an extreme example of the line-to-tree construction.

2.4 Example: Every linearly ordered set $(X, <)$ is the branch space of some tree. For any convex set C in any linearly ordered set X, define $\mathcal{N}(C) = \{\{x\} : x \in C\}$ if $|C| \geq 2$ and $\mathcal{N}(C) = \emptyset$ otherwise. The resulting partition tree has exactly two levels $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_1 = \{\{x\} : x \in X\}$, and its only non-trivial node is $\mathcal{T}_1$ which we linearly order to make it a copy of the original X. The branch space of this tree is exactly the original linearly ordered set $(X, <)$. Thus every LOTS is the branch space of some tree, and every GO-space embeds in the branch space of some tree. $\square$

Some might see Tordočevic's Example 2.4 as showing that there is no topological utility in trying to embed LOTS in the branch spaces of trees, since they already *are* branch spaces of trees. Others might conclude that the world of branch spaces is exactly as pathological as the world of lines. We reached a different conclusion, namely that it would be interesting to investigate further restrictions on trees so that their branch spaces would have interesting topological properties. In Section 4 we give examples of special properties of trees that have significant ramifications for their branch spaces. In Section 5 we give a characterization of hereditarily ultraparacompact GO-spaces based on the existence of certain kinds of trees whose branch spaces contain the given GO-space.

3. Continuity of the natural injection

As noted in Section 2, if we begin with a linearly ordered set $(X, <)$, we can build a partition tree $\mathcal{T}$ whose members are convex subsets of X, Furthermore, each node, and indeed each level, of $\mathcal{T}$ inherits a natural linear ordering from X that we call the *precedence order from* X, namely that a convex set $s \in \mathcal{T}_\alpha$ precedes another convex set $t \in \mathcal{T}_\alpha$ if and only if each point x of the set s has $x < y$ in X for every $y \in t$. Using that natural ordering for each node, we construct the branch space $\mathcal{B}$ of the tree $\mathcal{T}$ and endow it with the usual open interval topology of its ordering.

As noted in Section 2, there is a natural function $e : X \to \mathcal{B}$ given by $e(x) = \{t \in \mathcal{T} : x \in t\}$, and this function is always 1-1 and increasing. We will call e the *natural injection* of X into the branch space of $\mathcal{T}$. Examples 2.2 and 2.3 show that e might or might not be continuous. Because any 1-1, increasing function from one LOTS into another is always an open mapping from its domain onto its image set, we see that the natural injection $e : X \to \mathcal{T}$ is continuous if and only if it is a topological embedding of X into the branch space $\mathcal{B}$ of $\mathcal{T}$. In this section we study the set $e[X]$ and give necessary and sufficient conditions for continuity of the natural injection e.

3.1 Proposition: The set $e[X]$ is always dense in $\mathcal{B}$.

Proof: Suppose $B, D \in \mathcal{B}$ with $B <_\mathcal{B} D$ and $(B, D) \neq \emptyset$, where $<_\mathcal{B}$ denotes the linear order on the branch space $\mathcal{B}$. Compute $\alpha = fd(B, D)$. Then B_α, the unique member of $B \cap \mathcal{T}_\alpha$, precedes D_α, the unique member of $D \cap \mathcal{T}_\alpha$ while $B_\beta = D_\beta$ for each $\beta < \alpha$. Furthermore, B_α and D_α belong to the same node N_α of $\mathcal{T}$. Let $<_\alpha$ be the linear ordering of N_α. If there is some $t \in N_\alpha$ with $B_\alpha <_\alpha t <_\alpha D_\alpha$, choose any $x \in t$, Then $e(x) \in (B, D)$. If B_α and D_α are adjacent members of N_α, choose any $C \in (B, D)$ and note that $B_\beta = C_\beta = D_\beta$ for each $\beta < \alpha$, while either $B_\alpha = C_\alpha <_\alpha D_\alpha$ or else $B_\alpha <_\alpha C_\alpha = D_\alpha$. Consider the case where $B_\alpha = C_\alpha <_\alpha D_\alpha$, the other case being analogous. Compute $\gamma = fd(B, C)$. Then $\gamma > \alpha$ and in the node N_γ that contains both B_γ and C_γ we have $B_\gamma <_\gamma C_\gamma$. Choose any $x \in C_\gamma$. Then $e(x) \in (B, D)$. Therefore the set $e[X]$ is dense in $\mathcal{B}$. $\square$

Our next lemma records the facts that we will need about the interactions between $(X, <)$, the convex subsets of X that belong to the partition tree $(\mathcal{T}, \sqsubseteq)$, and the branch space $(\mathcal{B}, <_\mathcal{B})$.

3.2 Lemma: Let $\mathcal{B}$ be the branch space of a partition tree $\mathcal{T}$ of the linearly ordered set $(X, <)$. Then, with notation as above:

a) if $B \in \mathcal{B}$ and if $\emptyset \neq B_\beta \subseteq (\leftarrow, z)$ for some β and some $z \in X$, then $B <_{\mathcal{B}} B(z)$;

b) if B is a branch of $\mathcal{T}$, then $|\bigcap\{t : t \in B\}| \leq 1$;

c) if $\mathcal{D}$ is a non-empty subcollection of $\mathcal{T}$ that is linearly ordered by reverse inclusion (i.e., by the partial ordering $\sqsubseteq$ of $\mathcal{T}$) and has $\bigcap \mathcal{D} = \emptyset$, then the collection $B = \{t \in \mathcal{T} : t$ contains some member of $\mathcal{D}\}$ is a branch of $\mathcal{T}$.

Proof: To prove (c) observe that if $t_i \in B$ for $i = 1, 2$, then there exist $d_i \in \mathcal{D}$ with $d_i \subseteq t_i$. Hence $t_1 \cap t_2 \neq \emptyset$ so that t_1 and t_2 are comparable in $\mathcal{T}$. Hence B is a linearly ordered subset of $\mathcal{T}$. Suppose B is not a branch of $\mathcal{T}$. Then there is a branch C of $\mathcal{T}$ with $B \subset C$. Choose $s \in C - B$. Then s is comparable to each $d \in \mathcal{D}$, and $d \subseteq s$ is false for each $d \in \mathcal{D}$ (otherwise $s \in \mathcal{B}$). Hence $s \subseteq d$ for each $d \in \mathcal{D}$, showing that $\bigcap \mathcal{D}$ contains the non-empty set s, contrary to $\bigcap \mathcal{D} = \emptyset$. Therefore, B is a branch of $\mathcal{T}$ as claimed. $\square$

3.3 Proposition: Let $\mathcal{B}$ be the branch space of a partition tree $\mathcal{T}$ of the LOTS $(X, <, \mathcal{I})$. With notation as above, the following are equivalent:

a) the function $e : X \to \mathcal{B}$ is *not* continuous at the point $x \in X$;

b) either x is a limit point in X of the set $(\leftarrow, x)$ and for some α the set $B_\alpha(x)$ is defined and has $x \in B_\alpha(x) \subseteq [x, \rightarrow)$ and the collection $\mathcal{D} = \{t \in \mathcal{T} : t$ is a cofinal subset of $(\leftarrow, x)\}$ is nonempty and has $\bigcap \mathcal{D} = \emptyset$, or else x is a limit point in X of the set $(x, \rightarrow)$ and for some α the set $B_\alpha(x)$ is defined and has $x \in B_\alpha(x) \subseteq (\leftarrow, x]$ and the collection $\mathcal{E} = \{t \in \mathcal{T} : t$ is a coinitial subset of $(x, \rightarrow)\}$ is non-empty and has $\bigcap \mathcal{E} = \emptyset$.

Proof: First we show that a) implies b). Suppose that e is not continuous at x. Without loss of generality we may assume that x is a limit point of $(\leftarrow, x)$ and that

(*) some $B \in \mathcal{B}$ has the property that $B(y) <_{\mathcal{B}} B <_{\mathcal{B}} B(x)$ for every $y \in (\leftarrow, x)$.

Let $\alpha = fd(B, B(x))$. Then B_α precedes $B_\alpha(x)$ in X and $B_\beta = B_\beta(x)$ for every $\beta < \alpha$. We claim that $x \in B_\alpha(x) \subseteq [x, \rightarrow)$. To verify that set inclusion, suppose some $y \in B_\alpha(x)$ has $y < x$ in X. Then for $\beta \leq \alpha$, $B_\beta(y) = B_\beta(x) = B_\beta$ while $B_\alpha(y) = B_\alpha(x)$ showing that B_α precedes $B_\alpha(y)$ in X. Hence $B <_{\mathcal{B}} B(y) <_{\mathcal{B}} B(x)$, contradicting (*). Thus $B_\alpha(x) \subseteq [x, \rightarrow)$.

We claim that the set B_α is a cofinal subset of $(\leftarrow, x)$. That $B_\alpha \subseteq (\leftarrow, x)$ follows from the fact that B_α precedes $B_\alpha(x)$ in X. To see that B_α is cofinal in $(\leftarrow, x)$, suppose not. Then there is some $y \in X$ with $y < x$ and $B_\alpha \subseteq (\leftarrow, y]$. Because x is a limit point of $(\leftarrow, x)$ there is a $z \in (y, x)$. Apply Lemma 3.2(a) to conclude that $B <_{\mathcal{B}} B(z) <_{\mathcal{B}} B(x)$ contrary to (*). Therefore B_α is a cofinal subset of $(\leftarrow, x)$ and the collection $\mathcal{D}$ is non-empty.

Next, we show that if $\beta > \alpha$ and B_β is defined, then the set B_β is a cofinal subset of $(\leftarrow, x)$. Because $B_\beta \subseteq B_\alpha \subseteq (\leftarrow, x)$ it is enough to check cofinality. The proof of cofinality uses (3.2-a) again. Therefore $\{B_\beta : \beta \geq \alpha$ and B_β is defined $\} \subseteq \mathcal{D}$, so that $\bigcap \mathcal{D} \subseteq \bigcap\{B_\beta : \beta \geq \alpha$ and B_β is defined $\} = \bigcap\{t : t \in B\}$. Hence, to complete the proof of (b), it is enough to show that $\bigcap\{t : t \in B\} = \emptyset$.

Because B is a branch of $\mathcal{T}$ we know that $|\bigcap\{t : t \in B\}| \leq 1$ by (3.2-b). For contradiction, suppose $\bigcap\{t : t \in B\} = \{w\}$ for some $w \in X$. Then $B = B(w)$ and $w \in B_\alpha$ forces $w < x$. But x is a limit point of

$(\leftarrow, x)$ so there is some $z \in (w, x)$ and then we have $B = B(w) <_{\mathcal{B}} B(z) <_{\mathcal{B}} B(x)$ contrary to (*). Therefore, $\bigcap \mathcal{D} = \emptyset$ as claimed in (b).

We now prove that b) implies a). Assume that we have a point x that is a limit point in X of the set $(\leftarrow, x)$ and an α so that $x \in B_\alpha(x) \subseteq [x, \rightarrow)$ and that the collection $\mathcal{D} = \{t \in \mathcal{T} : t \text{ is a cofinal subset of } (\leftarrow, x)\}$ is nonempty and has $\bigcap \mathcal{D} = \emptyset$. Because any two members of $\mathcal{D}$ have non-empty intersection, any two members of $\mathcal{D}$ are comparable in $\mathcal{T}$. Then the collection $B = \{t \in \mathcal{T} : t \text{ contains some member of } \mathcal{D}\}$ is a branch of $\mathcal{T}$ in the light of (3.2-c). By hypothesis on $\mathcal{D}$, some $B_\gamma \in B$ has $B_\gamma \subseteq (\leftarrow, x)$ and therefore $B <_{\mathcal{B}} B(x)$ by (3.2-a). Consider any $y < x$ in X. There is some $d \in \mathcal{D} \subseteq B$ with $y \notin d$. Because $y < x$ and d is cofinal in $(\leftarrow, x)$, we conclude that $d \subseteq (y, \rightarrow)$. But then $B(y) <_{\mathcal{B}} B$ and therefore $e(x)$ is not a limit point in $\mathcal{B}$ of $\{e(y) : y < x\}$ even though x is a limit point in X of $(\leftarrow, x)$. Therefore e is not continuous at x, as claimed in (a). $\square$

4. Examples of “nice” tree properties

In this section, we return to the study of abstract trees, i.e., trees that do not necessarily come from convex sets in a given linearly ordered set. What kinds of properties of such trees might lead to interesting topological properties of their branch spaces? The following result, used in Example 2.3 above, is due to Todorčevic [To].

4.1 Proposition: Suppose that each node of a tree $\mathcal{T}$ is linearly ordered in such a way that it is order-complete. Then the branch space of $\mathcal{T}$ is compact.

Another property of trees that has very strong topological consequences for branch spaces (and for anything that embeds in one of the branch spaces) is that for each $t \in \mathcal{T}$, the set $[t] = \{B \in \mathcal{B} : t \in B\}$ is open in $\mathcal{B}$. What would force each $[t]$ to be open in $\mathcal{B}$? One sufficient condition is that for each $t \in \mathcal{T}$ the node $N = \{u \in \mathcal{T} : u \text{ is an immediate successor of } t\}$ is nonempty and the linearly ordered set $(N, <_N)$ has no endpoints. The next result describes properties of the branch space of such a tree.

4.2 Proposition: Let $\mathcal{T}$ be a tree with the property that $[t]$ is open in the branch space $\mathcal{B}$ for each $t \in \mathcal{T}$. Then:

a) each set $[t]$ is clopen in $\mathcal{B}$;

b) the branch space $\mathcal{B}$ is zero-dimensional;

c) $\{[t] : t \in \mathcal{T}\}$ is a base for $\mathcal{B}$ that is a tree under reverse inclusion, i.e., $\mathcal{B}$ is non-archimedean;

d) any space that embeds in $\mathcal{B}$ has a continuous separating family in the sense of [BL] [St]; and

e) any space that embeds in $\mathcal{B}$ is hereditarily paracompact.

Proof: Fix $t \in \mathcal{T}$ and compute $\alpha = lv(t)$. For each $\beta < \alpha$ there is a unique $t_\beta \in \mathcal{T}$ with $t_\beta \sqsubseteq t$. Then

$$\mathcal{B} - [t] = \bigcup \{[s] : s \in \mathcal{T}_\alpha - \{t\}\} \cup \bigcup \{[s] : \beta < \alpha \text{ and } s \in \mathcal{T}_\beta - \{t_\beta\}\}.$$

Hence each $[t]$ is clopen in $\mathcal{B}$. For each branch $B \in \mathcal{B}$, $\{[t] : t \in B\}$ is a local base at B. Therefore $\mathcal{B}$ is zero dimensional. The collection $\{[t] : t \in \mathcal{T}\}$ is a base for $\mathcal{B}$, and is a tree under reverse inclusion. Thus c) holds.

The proof of d) is due to Gary Gruenhage. The notion of a continuous separating family was introduced in [St] and we verify that $\mathcal{B}$ satisfies the definition given there. For each pair (B, C) of distinct branches of $\mathcal{T}$ find $\alpha = fd(B, C)$ and let $D = \min(B, C)$. Define $\Psi(B, C)$ to be the characteristic function of the clopen set $[D(\alpha)]$. Then $\Psi(B, C)$ is a continuous function from X to $\mathbb{R}$, and $\Psi : \mathcal{B}^2 - \Delta \rightarrow C_u(\mathcal{B})$ is continuous, where $C_u(\mathcal{B})$ is the space of all continuous real-valued functions on $\mathcal{B}$ topologized by the topology of uniform convergence. Furthermore, $\Psi(B, C)(B) \neq \Psi(B, C)(C)$. Hence d) holds.

Assertion e) now follows from a result of [BL] that any GO-space with a continuous separating family must be hereditarily paracompact. Direct proofs of hereditary paracompactness of $\mathcal{B}$ are also possible. $\square$

Let us give another example of how embedding in branch spaces of "nice" trees can have topological consequences. Recall that a Souslin tree is a tree $(\mathcal{S}, \sqsubseteq)$ of height ω_1 that contains no uncountable chains (= linearly ordered subsets) or anti-chains (= sets whose elements are pairwise incompatible in the partial ordering of the tree). Whether a Souslin tree exists is undecidable in ZFC. Starting with any Souslin tree, one can obtain another Souslin tree $\mathcal{T}$ with the property that each node is a countably infinite set. Linearly order the nodes of that second tree to make each a copy of the set $\mathbb{Z}$ of all integers. Then $\mathcal{T}$ has the property that $[t]$ is open in the branch space for each $t \in \mathcal{T}$. Gruenhage proved:

4.3 Proposition: Suppose $(\mathcal{T}, \sqsubseteq)$ is a Souslin tree with the property that each $[t]$ is open in the branch space $\mathcal{B}$. If X is a space that can be embedded in $\mathcal{B}$, then $X \times X$ is hereditarily paracompact. In particular, $\mathcal{B}$ is a non-separable, hereditarily Lindelöf LOTS for which $\mathcal{B}^2$ is hereditarily paracompact.

Proof: Let $\mathcal{W}$ be any collection of open subsets of $\mathcal{B}^2$ and let $Z = \bigcup \mathcal{W}$. It will be enough to show that there is an open cover of Z that is star-countable and refines $\mathcal{W}$.

We say that an ordered pair (s,t) of elements of $\mathcal{T}$ is minimal if the open set $[s] \times [t]$ is a subset of some member of $\mathcal{W}$ while no other ordered pair (s', t') with $s' \sqsubseteq s$ and $t' \sqsubseteq t$ has that same property. Then $\mathcal{U} = \{[s] \times [t] : (s,t) \text{ is minimal }\}$ is an open refinement of $\mathcal{W}$ that covers $\bigcup \mathcal{W}$. We claim that $\mathcal{U}$ is star-countable. For contradiction, suppose that (s_0, t_0) is minimal and that there are distinct minimal pairs (s_α, t_α) with the property that $([s_0] \times [t_0]) \cap ([s_\alpha] \times [t_\alpha]) \neq \emptyset$ for each α in the uncountable set A. It follows that $[s_0] \cap [s_\alpha] \neq \emptyset$ so that s_0 and s_α must be comparable in $(\mathcal{T}, \sqsubseteq)$ for each $\alpha \in A$. The element s_0 has only countably many predecessors in $\mathcal{T}$. Suppose that there is one predecessor u of s_0 with the property that the set $B = \{\alpha \in A : s_\alpha = u\}$ is uncountable. For each $\alpha \in B$ choose $W_\alpha \in \mathcal{W}$ with $[u] \times [t_\alpha] \subseteq W_\alpha$. The set $\{t_\alpha : \alpha \in B\}$ cannot be an anti-chain, so there exist distinct $\alpha_1, \alpha_2 \in B$ with $t_{\alpha_1} \sqsubseteq t_{\alpha_2}$. But then

$$[s_{\alpha_2}] \times [t_{\alpha_2}] = [u] \times [t_{\alpha_2}] \subseteq [u] \times [t_{\alpha_1}] = [s_{\alpha_1}] \times [t_{\alpha_1}] \subseteq W_{\alpha_1}$$

and that is impossible because $(s_{\alpha_2}, t_{\alpha_2})$ is minimal. Therefore, only countably many of the pairs (s_α, t_α) have $s_\alpha \sqsubseteq s_0$. Let $C = A - \{\alpha \in A : s_\alpha \sqsubseteq s_0\}$. Then C is uncountable.

Let $\alpha \in C$. Because $([s_0] \times [t_0]) \cap ([s_\alpha] \times [t_\alpha]) \neq \emptyset$, we conclude that $[t_\alpha] \cap [t_0] \neq \emptyset$ and hence t_α is comparable to t_0 in $\mathcal{T}$. Since t_0 has only countably many predecessors in $\mathcal{T}$, it follows that the set $D = \{\alpha \in C : t_0 \sqsubseteq t_\alpha\}$ is uncountable. Choose any $\alpha_1 \in D$ with $(s_0, t_0) \neq (s_{\alpha_1}, t_{\alpha_1})$. But then the existence of (s_0, t_0) contradicts minimality of $(s_{\alpha_1}, t_{\alpha_1})$. Therefore, the collection $\mathcal{U}$ is a star-countable open cover of $\bigcup \mathcal{W}$ that refines $\mathcal{W}$, as required. $\square$

Gruenhage's Proposition 4.3 contrasts sharply with an earlier result about certain Souslin lines that is due to Rudin [R1]:

4.4 Proposition: Suppose X is a compact Souslin space, i.e., a compact non-separable, hereditarily Lindelöf LOTS. Then X^2 is not hereditarily normal.

The hypothesis that each $[t]$ is open in the branch space is very restrictive. An example of a weaker, but still "nice," condition that a tree might satisfy is that for each $t \in T$, the set $[t]$ has non-empty interior. We thank the referee for pointing out that our original proof that any such branch space is a Baire space actually proves more, namely that any such branch space is α-favorable. Consequently, the product of any such branch space with another Baire space is a Baire space.

To define α-favorability, recall the Banach-Mazur game, a topological game played by two players α and β using non-empty open sets. Player β chooses any non-empty open set V_1 and player α responds by choosing a non-empty open subset $U_1 \subseteq V_1$. Then player β chooses a non-void open set $V_2 \subseteq U_1$. For any $n \geq 1$, once β has chosen the open set $V_n \subseteq U_{n-1}$, player α responds by choosing a non-empty open set $U_n \subseteq V_n$. Player α wins the game if $\bigcap\{U_n : n \geq 1\} \neq \emptyset$, and a topological space is said to be *α-favorable* if player α has a winning strategy in the Banach-Mazur game [Ox], [MN].

4.5 Proposition: Suppose that $\mathcal{T}$ is a tree with the property that for each $t \in \mathcal{T},\ int_{\mathcal{B}}([t]) \neq \emptyset$. Then the branch space $\mathcal{B}$ is α-favorable.

Proof: Suppose we are at stage n of the Banach-Mazur game and that nonempty open sets $V_1 \supseteq U_1 \supseteq \cdots \supseteq U_{n-1} \supseteq V_n$ have been chosen by the two players. Then player α should find the first ordinal δ_n such that some $t \in \mathcal{T}_{\delta_n}$ has $[t] \subseteq V_n$. Player α should choose any such t and respond with $U_n = Int([t])$. Any play of this game will then produce a sequence $\delta_1 \leq \delta_2 \leq \cdots$ of ordinals and points $t_n \in \mathcal{T}_{\delta_n}$ with $t_1 \leq_{\mathcal{T}} t_2 \leq_{\mathcal{T}} \cdots$. There is a branch B of $\mathcal{T}$ that contains each t_n, and this branch has $B \in \bigcap\{U_n : n \geq 1\}$ as required. $\square$

It is much easier for a tree to satisfy the hypothesis of (4.5) than to satisfy the more restrictive hypothesis that each set $[t]$ is open in the branch space. One sufficient condition, in the light of (2.1), is that at least three branches run through each $t \in \mathcal{T}$.

5. Characterization of ultraparacompactness via embeddings in branch spaces

Recall that a topological space is *ultraparacompact* if each open cover of X has a disjoint open refinement. In this section, we give what could be viewed as a characterization of ultraparacompactness in a LOTS or GO-space. However, that is not our real goal. Instead, our goal is to illustrate the possible interaction between properties of an ordered space X and the kinds of trees whose branch spaces contain X as a subspace.

Faber [F] gave a particularly useful version of an earlier characterization of paracompactness in a LOTS or GO-space that is due to Gillman and Henriksen [GH].

5.1 Proposition: A GO space X is paracompact if and only if whenever $X = G \cup H$ where G and H are convex open subsets of X with the property that $x < y$ for each $x \in G$ and each $y \in H$, there are closed discrete subsets $D \subset G$ and $E \subset H$ such that D is cofinal in G and E is coinitial in H. (Note that one of the sets G, H might be empty.)

Because a topological space X is hereditarily paracompact if and only each open subspace is paracompact, and because each open subspace of a GO-space is the topological sum of its convex components, we have

5.2 Proposition: A GO-space X is hereditarily paracompact if and only if each open convex subspace Y of X contains a relatively closed discrete subset that is both cofinal and coinitial in Y.

Remark: One way to obtain a relatively closed discrete cofinal (respectively coinitial) subset of Y in (5.2) is to show that there is a cofinal (resp. coinitial) convex subset $Z \subseteq Y$ that admits a pairwise disjoint open cover by sets that are not cofinal (resp. not coinitial) in Y. Then choosing one point from each member of the cover of Z gives the required set.

Our next result must be well-known. It can be proved using the "method of coherent collections" described in [L]. We thank Jerry Vaughan for pointing out that it also follows from the fact that "ultranormal plus paracompact implies ultraparacompact" which is proved in [El].

5.3 Proposition: A GO-space X is (hereditarily) ultraparacompact if and only if X is (hereditarily) paracompact and zero-dimensional. $\square$

Our goal in this section is to characterize GO-spaces that are hereditarily ultraparacompact in terms of certain kinds of trees in whose branch spaces they embed. We will consider three properties of a tree $(\mathcal{T} \sqsubseteq)$:

R-1) if N is a node of $\mathcal{T}$ and if $t \in N$ is not an endpoint of the linearly ordered set $(N, <_N)$, then $[t]$ is a clopen subset of the branch space;

R-2) if t is an endpoint of the node N, then $\{s \in \mathcal{T} : s \sqsubseteq t\}$ is a branch of $\mathcal{T}$;

R-3) if B is a branch of $\mathcal{T}$, then both sets $\{\alpha :$ the node of $\mathcal{T}$ to which $B(\alpha)$ belongs has a left (resp. right) endpoint $\}$ are hereditarily paracompact subspaces of the ordinal space $[0, \kappa)$ where κ is the height of $\mathcal{T}$.

We will prove:

5.4 Theorem: A GO-space X is hereditarily ultraparacompact if and only if X embeds in the branch space of a tree having properties R-1, R-2, and R-3.

Outline of Proof: We will prove a sequence of lemmas that, when combined, establish (5.4). In (5.5) we will show that the branch space of any tree satisfying R-1, R-2, and R-3 must be zero-dimensional. In (5.6) we will show that the branch space of a tree satisfying R-1, R-2, and R-3 must be hereditarily paracompact. Combining (5.6) and (5.5) with (5.3) will show that a branch space of a tree satisfying R-1, R-2, and R-3 must be hereditarily ultraparacompact. That establishes half of (5.4). For the converse, in (5.8) we will start with any GO-space that is hereditarily ultraparacompact and embed it into a LOTS with the same property. Then we will then show how such a LOTS can be used to construct a tree satisfying R-1, R-2, and R-3, and will invoke results from Section 3 to insure that the LOTS embeds in the branch space of the tree. That will complete the proof of (5.4) □

5.5 Lemma: Suppose the tree $(\mathcal{T}, \sqsubseteq)$ has properties R-1, R-2, and R-3. Then its branch space $\mathcal{B}$ is zero dimensional.

Proof: We show that if $B <_{\mathcal{B}} D$ in $\mathcal{B}$, then $[B, D]$ is not connected. Compute $\alpha = fd(B, D)$. Then B_α and D_α belong to the same node N_α of $\mathcal{T}$ and $B_\alpha <_\alpha D_\alpha$ where $<_\alpha$ denotes the linear ordering of N_α. If there is some $t \in N_\alpha$ with $B_\alpha <_\alpha t <_\alpha D_\alpha$ then by R-1 the set $[t]$ is clopen in $\mathcal{B}$ and the sets $\mathcal{G} = \{C \in \mathcal{B} : C \leq_{\mathcal{B}} E$ for some $E \in [t]\}$ and $\mathcal{H} = \mathcal{B} - \mathcal{G}$ are clopen sets in $\mathcal{B}$ that separate B and D. In case no such t exists, then B and D are the left and right endpoints of N_α so that by R-2, $B = \{B_\beta : \beta \leq \alpha\}$ and $D = \{D_\beta; \beta \leq \alpha\}$ and we see that B and D are adjacent points of $\mathcal{B}$ so that $[B, D]$ is not connected. □

5.6 Lemma: Suppose the tree $(\mathcal{T}, \sqsubseteq)$ has properties R-1, R-2, and R-3. Then its branch space $\mathcal{B}$ is hereditarily paracompact.

Proof: In the light of (5.2) it will be enough to show that each convex open subspace $\mathcal{Y}$ of $\mathcal{B}$ has a relatively closed discrete subset that is cofinal and coinitial in $\mathcal{Y}$. We will construct a cofinal relatively closed discrete subset, the coinitial set construction being analogous. Note that if $\mathcal{Y}$ has a right endpoint, or if $\mathcal{Y}$ has cofinality ω, there is nothing to prove, so we will assume that $\mathrm{cf}(\mathcal{Y}) \geq \omega_1$.

For each α there is at most one set $t_\alpha \in \mathcal{T}_\alpha$ so that $[t_\alpha] \cap \mathcal{Y}$ is a cofinal subset of $\mathcal{Y}$. Let $\sigma = \min\{\alpha : t_\alpha$ does not exist $\}$. Then σ is less than or equal to the height of the tree $\mathcal{T}$.

Observe that the set $\{t_\alpha : \alpha < \sigma\}$ is linearly ordered by $\sqsubseteq$ and that in $\mathcal{B}$ the corresponding collection $\{[t_\alpha] \cap \mathcal{Y} : \alpha < \sigma\}$ is well-ordered by reverse inclusion. There are several cases to consider, based on the nature of the set $\mathcal{S} = \bigcap\{[t_\alpha] \cap \mathcal{Y} : \alpha < \sigma\}$ and the ordinal σ.

Case 1: Where $\mathcal{S} \neq \emptyset$. Then $\mathcal{S}$ is a cofinal subset of $\mathcal{Y}$ so that $\mathcal{S}$ must be infinite, because $\mathrm{cf}(\mathcal{Y}) \geq \omega_1$. For each $B \in \mathcal{S}$ we have $B \in [t_\alpha]$ for each $\alpha < \sigma$ so that $\{t_\alpha : \alpha < \sigma\} \subseteq B$. Because there is more than one branch B with this property, the set $\{t_\alpha : \alpha < \sigma\}$ cannot be a branch of $\mathcal{T}$. Therefore B_σ is defined for each $B \in \mathcal{S}$ and all B_σ for $B \in \mathcal{S}$ belong to the same node N_σ of $\mathcal{T}$. Furthermore, no set $[B_\sigma] \cap \mathcal{Y}$ can be cofinal in $\mathcal{Y}$ because of the definition of σ. Hence, no B_σ is the right endpoint of N_σ and at most one is the left endpoint of N_σ. Therefore, using all but at most one of the sets $[B_\sigma]$ we obtain a pairwise disjoint cover of a cofinal convex subset of $\mathcal{Y}$ by clopen sets, none of which is cofinal in $\mathcal{Y}$. Choosing one point from each of the clopen sets, we obtain a relatively closed discrete cofinal subset of $\mathcal{Y}$ as required.

Case 2: $\mathcal{S} = \emptyset$ and σ is not a limit ordinal. Write $\sigma = \gamma + 1$. Then $\mathcal{S} = [t_\gamma] \cap \mathcal{Y} \neq \emptyset$ is a cofinal convex subset of $\mathcal{Y}$, and that is impossible in Case 2.

Case 3: where $\mathcal{S} = \emptyset$ and σ is a limit ordinal. If $\mathrm{cf}(\sigma) = \omega$ then the fact that $\mathcal{S} = \emptyset$ would force $\mathcal{Y}$ to have $\mathrm{cf}(\mathcal{Y}) = \omega$ contrary to $\mathrm{cf}(\mathcal{Y}) \geq \omega_1$. Hence assume that $\mathrm{cf}(\sigma) \geq \omega_1$. Let $L = \{\alpha < \sigma : \text{the node } N_\alpha \text{ of } \mathcal{T} \text{ to which } t_\alpha$ belongs has a left endpoint in its given linear order $<_\alpha\}$. According to R-3, L is hereditarily paracompact when viewed as a subspace of the ordinal space $[0, \sigma)$. Because $\mathrm{cf}(\sigma) > \omega$ there is a closed unbounded set $C \subseteq [0, \sigma)$ with $C \cap L = \emptyset$. We next establish a sequence of claims.

Claim 1: $\bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha < \sigma\} = \emptyset$ where N_α is the node of $\mathcal{T}$ to which t_α belongs and $[N_\alpha]$ is as defined in (2.1). For suppose $B \in \bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha < \sigma\}$. Fix any $\alpha < \sigma$. Because σ is a limit ordinal, $\alpha + 1 < \sigma$. But $[N_{\alpha+1}] \subseteq [t_\alpha]$ so that $B \in \mathcal{S}$. Hence $\bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha < \sigma\} \subseteq \mathcal{S}$. But in case 3, $\mathcal{S} = \emptyset$ so that Claim 1 holds.

Claim 2: If $\lambda < \sigma$ is a limit ordinal, then $\bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha < \lambda\} \subseteq [N_\lambda]$. For suppose $B \in \bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha < \lambda\}$. Then for each $\alpha < \lambda,\ \alpha + 1 < \lambda$ so that $B \in [N_{\alpha+1}] \subseteq [t_\alpha]$. Therefore $\{t_\alpha : \alpha < \lambda\} \subseteq B$. Because $\lambda < \sigma$ the set $\{t_\alpha : \alpha < \lambda\}$ is not a maximal linearly ordered subset of $\mathcal{T}$ so that $\{t_\alpha : \alpha < \sigma\} \subseteq B$ forces B_λ to be defined. Because B_λ has exactly the same predecessors in $(\mathcal{T}, \sqsubseteq)$ as does t_λ we see that $B_\lambda \in N_\lambda$ and therefore $B \in [N_\lambda]$ as required.

Claim 3: For each $\alpha \in C$ the set $[N_\alpha] \cap \mathcal{Y}$ is a clopen cofinal convex subset of $\mathcal{Y}$. The set is always convex, and is cofinal in $\mathcal{Y}$ because $t_\alpha \in N_\alpha$. It remains to verify that the set is clopen. The only possible problem is that $[N_\alpha] \cap \mathcal{Y}$ might have a limit point $B \in \mathcal{Y}$ such that $[N_\alpha] \cap \mathcal{Y} \subseteq (B, \rightarrow)$. Because $B \notin [N_\alpha],\ B$ does not have exactly the same predecessors as t_α. Choose the first $\gamma < \alpha$ such that $B_\gamma \neq t_\gamma$. Then $B_\gamma \in N_\gamma$. Because $[t_\gamma] \cap \mathcal{Y}$ is cofinal in $\mathcal{Y}$, it cannot be that $B_\gamma >_\gamma t_\gamma$ so that so that $B_\gamma <_\gamma t_\gamma$ where $<_\gamma$ is the given linear ordering of the node N_γ. Because B_α is defined and $\gamma < \alpha$, it cannot be true that B_γ is the left endpoint of N_γ, in the light of R-2. Because $B_\gamma <_\gamma t_\gamma,\ B_\gamma$ is not the right endpoint of N_γ. By R-1, $[B_\gamma]$ is a clopen subset of $\mathcal{B}$ that contains B and is disjoint from $[t_\gamma]$. But $\gamma < \alpha$ so that $[N_\alpha] \subseteq [t_\gamma]$ showing that B is not a limit point of $[N_\alpha]$. Therefore $[N_\alpha]$ is closed in $\mathcal{Y}$.

For each $\alpha \in C$, let α^+ be the first element of C that is larger than α. Such an ordinal exists because $\mathrm{cf}(C) > \omega$. Define $\mathcal{E}(\alpha) = \mathcal{Y} \cap ([N_\alpha] - [N_{\alpha^+}])$. Then each $\mathcal{E}(\alpha)$ is clopen in $\mathcal{Y}$ and no set $\mathcal{E}(\alpha)$ is cofinal in $\mathcal{Y}$, for $\alpha \in C$. Let α_0 be the first member of C and fix $B_0 \in \mathcal{Y} \cap [N_{\alpha_0}]$.

Claim 4: $[B_0, \rightarrow) \cap \mathcal{Y} \subseteq \bigcup\{\mathcal{E}(\alpha) : \alpha \in C\}$. For suppose $B \in \mathcal{Y}$ and $B \geq B_0$. Because C is a cofinal subset of $[0, \sigma)$, Claim 1 yields $\bigcap\{[N_\alpha] \cap \mathcal{Y} : \alpha \in C\} = \emptyset$. Choose the first $\alpha \in C$ with $B \notin [N_\alpha]$. Then $B \in [N_\beta]$ for every $\beta \in C$ with $\beta < \alpha$. If α were a limit point of C, then Claim 2 would apply to show that $B \in \bigcap\{[N_\beta] \cap \mathcal{Y} : \beta \in C \cap [0, \alpha)\} = [N_\alpha]$ contrary to $B \notin [N_\alpha]$. Because C is a closed subset of $[0, \sigma)$, there is some $\beta \in C$ with $\alpha = \beta^+$, and then $B \in \mathcal{E}(\beta)$. That proves Claim 4.

Now choose one point from each set $\mathcal{E}(\alpha)$. As noted in the Remark after Proposition 5.2, we obtain a relatively closed discrete cofinal subset of $\mathcal{Y}$. That completes the proof of Lemma 5.6. □

5.7 Corollary: Any GO-space that embeds in the branch space of a tree with properties R-1, R-2, and R-3 must be hereditarily ultraparacompact.

Proof: Combine (5.3) and (5.6). $\square$

5.8 Lemma: Any hereditarily ultraparacompact GO-space embeds in the branch space of a tree having properties R-1, R-2, and R-3.

Proof: Let Y be any hereditarily ultraparacompact GO-space. We first show that Y embeds in some LOTS X that is hereditarily ultraparacompact and dense ordered in the sense that $(x, y) \neq \emptyset$ whenever $x < y$ are points of X. Denote the given topology of Y by $\mathcal{S}$ and the usual open interval topology of Y by $\mathcal{I}$. The first step is to construct the lexicographically ordered set X_1 given by

$$(Y \times \{0\}) \cup \{(y, q) : y \in Y,\ [y \rightarrow) \in \mathcal{S} - \mathcal{I},\ q \in \mathbb{Q},\ q \leq 0\} \cup \{(y, q) : y \in Y,\ (\leftarrow, y] \in \mathcal{S} - \mathcal{I},\ q \in \mathbb{Q},\ q \geq 0\},$$

where $\mathbb{Q}$ is the set of rational numbers. Then X_1 is a hereditarily ultraparacompact LOTS that contains Y as a closed subspace. However X_1 might not be dense ordered, so we fill in any jumps of X_1 with copies of $\mathbb{Q}$. Let $J_0 = \{x \in X :$ for some $y \in X,\ x < y$ and $(x, y) = \emptyset\}$ and $J_1 = \{y \in X :$ for some $x < y,\ (x, y) = \emptyset\}$. Now let X be the lexicographically ordered set

$$X = (X_1 \times \{0\}) \cup \{(z, q) : q \in \mathbb{Q}, q > 0, z \in J_0\} \cup \{(z, q) : q \in \mathbb{Q},\ q < 0, z \in J_1\}.$$

Then X is a hereditarily ultraparacompact LOTS that is densely ordered and contains the original GO-space Y as a closed subspace. It will be enough to show that X embeds in the branch space of a partition tree $\mathcal{T}$ that satisfies R-1, R-2, and R-3.

For any convex subset $C \subseteq X$, let $\mathcal{N}(C) = \emptyset$ if $|C| \leq 1$. If $|C| > 1$, then let $EP(C)$ be the set of all endpoints of C, if any. Because X is hereditarily ultraparacompact, there is a pairwise disjoint collection $\mathcal{P}(C)$ that covers $C - EP(C)$ and has the properties that

a) each member of $\mathcal{P}(C)$ is a clopen convex subset of X and is a subset of $C - EP(C)$;

b) the collection $\mathcal{P}(C)$ has no first or last members in terms of the precedence order from X.

We note that it is possible to obtain (b) because the set X is dense-ordered. Now define $\mathcal{N}(C) = \mathcal{P}(C) \cup \{\{x\} : x \in EP(C)\}$ and linearly order $\mathcal{N}(C)$ using the precedence ordering from X. Observe that for each convex set $C \subseteq X$ with more than one point, every member of $\mathcal{P}(C)$ is an infinite convex set.

Now define the partition tree $\mathcal{T}$ as described in Sections 2 and 3 and consider its branch space $\mathcal{B}$. Each node and each level of $\mathcal{T}$ is linearly ordered by a precedence ordering inherited from X: we say that a convex set s precedes a convex set t in the precedence order from X provided $x < y$ for each $x \in s$ and $y \in t$. Observe that

(*) if N is a node of $\mathcal{T}$ at a non-limit level, then $(N, <_N)$ has no end members.

Proposition 3.3 guarantees that the natural injection embeds X into $\mathcal{B}$. It remains to prove that the partition tree $(\mathcal{T}, \sqsubseteq)$ – where $\sqsubseteq$ is reverse inclusion – satisfies properties R-1, R-2, and R-3.

First we verify R-1. Fix an ordinal α and a node N_α at level α of the tree. Let $E(N_\alpha)$ be the set of end members of N_α in the precedence order from X and let $t \in N_\alpha - E(N_\alpha)$. Then, viewed as a subset of X, the convex set t has no endpoints, so that the node N' of all immediate successors of t is exactly the collection $\mathcal{P}(t)$

and, by construction, that collection has no end members with respect to the precedence order from X. That is enough to force $[t]$ to be an open subset of the branch space $\mathcal{B}$. Next we show that $[t]$ is also closed in the branch space $\mathcal{B}$. Because t is not an end member of the node N_α we may choose members $s, u \in N_\alpha$ such that the convex subset s of X precedes t and t precedes u in $(X, <)$. Therefore, $[t]$ is not a cofinal or coinitial subset of $[N_\alpha]$. Because the interior of $[N_\alpha]$ in the branch space $\mathcal{B}$ is covered by the pairwise disjoint open convex sets $[v]$ for $v \in N_\alpha - E(N_\alpha)$, we see that $[t]$ is also a closed subset of $\mathcal{B}$.

Next we verify R-2. Suppose that N is a node of $\mathcal{T}$ and that t is an end member of N is the precedence ordering inherited from X. Then by (*) above, N must be a node at a limit level λ and each member of N has exactly the same set of predecessors in $(\mathcal{T}, \sqsubseteq)$, namely $\{t_\gamma : \gamma < \lambda\}$ where t_γ is the unique member of T_γ that lies below t in the ordering of $(\mathcal{T}, \sqsubseteq)$. Viewing each t_γ as a convex sunset of X, let $C = \bigcap\{t_\gamma : \gamma < \lambda\}$. Then $t = \{x\}$ for one of the endpoints $x \in C$, and therefore t has no successor in $(\mathcal{T}, \sqsubseteq)$ at level $\lambda + 1$ or higher. Therefore, $\{s \in \mathcal{T} : s \sqsubseteq t\}$ is a branch of $\mathcal{T}$ as required by R-2.

Finally, we verify R-3. Let B be any branch of $\mathcal{T}$. Then for some ordinal σ we have $B = \{t_\alpha : \alpha < \sigma\}$ where $t_\alpha \in \mathcal{T}_\alpha$. First consider the case where σ is a limit ordinal, i.e., where B has no final element in $(\mathcal{T}, \sqsubseteq)$. For each $\alpha < \sigma$ let N_α be the node of $\mathcal{T}$ to which t_α belongs. Because B continues beyond level $\alpha + 1$, t_α cannot be an end member of N_α is the precedence ordering from X. Therefore, if $\alpha < \beta < \sigma$ and if we view t_α, t_β as convex subsets of X, then $t_\alpha \supseteq t_\beta$. Even more is true:

(**) if $\gamma < \delta < \sigma$ then there are points $a, b \in X$ such that, as subsets of X, $t_\delta \subseteq (a, b) \subseteq [a, b] \subseteq t_\gamma$.

Let $L = \{\alpha < \sigma : N_\alpha$ has a left end member in the precedence order inherited from $X\}$. Then $L \subseteq [0, \sigma)$. We show that L is hereditarily paracompact by proving that L embeds into the hereditarily paracompact space X.

Let $\alpha \in L$. Then by (*) above, α must be a limit ordinal and the left end member of the node N_α has the form $\{p_\alpha\}$ where p_α is the left endpoint in X of the convex set $C_\alpha = \bigcap\{t_\gamma : \gamma < \alpha\}$. For each $\alpha \in L$ define $f(\alpha) = p_\alpha$.

The function f is strictly increasing. For suppose $\alpha < \beta < \sigma$ are in L. Then both α and β are limit ordinals and $\alpha + 1 < \alpha + 2 < \beta$ so that in the space X we have $C_\beta = \bigcap\{t_\gamma : \gamma < \beta\} \subseteq t_{\alpha+2} \subseteq t_{\alpha+1} \subseteq C_\alpha$. As noted in (**) above, we can find points $a, b \in t_{\alpha+1}$ such that in X the convex set $t_{\alpha+2}$ is a subset of (a, b). Therefore $p_\alpha \le a < p_\beta$, i.e., $f(\alpha) < f(\beta)$.

To show that $f : L \to X$ is continuous, suppose $\alpha \in L$ is a limit point of L. We must show that in X, $f(\alpha) = \sup\{f(\beta) : \beta \in L, \beta < \alpha\}$. For each limit ordinal $\beta < \alpha$, C_β is the convex subset of X given by $C_\beta = \bigcap\{t_\gamma : \gamma < \beta\}$. Because α is a limit ordinal and $\beta < \alpha$, $t_{\beta+1}$ is defined and $t_{\beta+1} \subseteq C_\beta$ so that $|C_\beta| \ge \omega$. Furthermore, if $\beta \in L$, then $f(\beta) = p_\beta$ is the left endpoint of C_β in X. Observe that because α is a limit point of L we have

$$\bigcap\{C_\beta : \beta < \alpha, \beta \in L\} = \bigcap\{\bigcap\{t_\gamma : \gamma < \beta\} : \beta \in L, \beta < \alpha\} = \bigcap\{t_\gamma : \gamma < \alpha\} = C_\alpha$$

and that forces the endpoints p_β to converge upwards to p_α in the LOTS X. Hence $f(\alpha) = \sup\{f(\beta) : \beta < \alpha, \beta \in L\}$ as required to prove continuity of f.

Finally we show that f is a closed mapping from L onto $f[L] \subseteq X$. (Note that this is not automatic because the domain L is not known to be a LOTS in the order that it inherits from $[0, \sigma)$.) It will be enough to show that if $\alpha \in L$ is *not* a limit point of $\{\beta \in L : \beta < \alpha\}$, then $f(\alpha)$ is not a limit point of $\{f(\beta) : \beta \in L,\ \beta < \alpha\}$ in X.

Because α is not a limit point of $[0,\alpha)\cap L$, there is some $\gamma<\alpha$ with $[0,\alpha)\cap L\subseteq[0,\gamma]$. Because $\alpha\in L,\ \alpha$ is a limit ordinal so that $\gamma+1<\alpha$. Consider any $\beta\in L$ with $\beta<\alpha$. Then $\beta\leq\gamma$ so that in the LOTS $(X,<,\mathcal{I})$ we have

$$C_\beta=\bigcap\{t_\delta:\delta<\beta\}\supseteq t_\gamma\supseteq t_{\gamma+1}\supseteq t_{\gamma+2}\supseteq\bigcap\{t_\delta:\delta<\alpha\}=C_\alpha.$$

As noted in (**) above, we may choose points $a,b\in X$ with $t_{\gamma+2}\subseteq(a,b)\subseteq[a,b]\subseteq t_{\gamma+1}$. But then $C_\alpha\subseteq(a,b)\subseteq[a,b]\subseteq C_\beta$ and that forces us to conclude that the left endpoints $p_\alpha,\ p_\beta$ of C_α and C_β respectively must satisfy $p_\beta\leq a<p_\alpha$. Observe that the choice of a depended only on γ and therefore we have shown that $f(\beta)=p_\beta\leq a<p_\alpha=f(\alpha)$ whenever $\beta\in L\cap[0,\alpha)$. Therefore, $f(\alpha)$ is not a limit point in X of $\{f(\beta):\beta\in L\cap[0,\alpha)\}$. Therefore, $f:L\to X$ is a closed mapping onto $f[L]$.

At this stage we know that f is a continuous, 1-1, closed mapping from L onto the subspace $f[L]$ of the hereditarily paracompact space X. Therefore, L is also hereditarily paracompact, as required for R-3.

It remains to consider the case where the branch B has a last member. Then $\sigma=\lambda+1$ for some limit ordinal λ. We apply the first part of the proof to the linearly ordered set $\{t_\alpha:\alpha<\lambda\}$ to conclude that the set $L_0=\{\alpha<\lambda:$ the node N_α has a left end member $\}$ is hereditarily paracompact as a subspace of $[0,\sigma)$. But then so is the set $L=L_0\cup\{\lambda\}$ as required by R-3. □

It is natural to ask whether R-1 in Theorem 5.4 could be strengthened to require that $[t]$ is open in the branch space for each $t\in\mathcal{T}$. Then answer is "No" as the next example shows.

5.9 Example: Let X be the lexicographically ordered LOTS $X=\mathbb{R}\times\{n\in\mathbb{Z}:n\leq 0\}$. Then X is hereditarily ultraparacompact and cannot be embedded in the branch space $\mathcal{B}$ of any tree $\mathcal{T}$ where $[t]$ is open in $\mathcal{B}$ for each $t\in\mathcal{T}$.

Proof: For contradiction, suppose X embeds in the branch space $\mathcal{B}$ of a tree $\mathcal{T}$ with the property that $[t]$ is open in $\mathcal{B}$ for each $t\in\mathcal{T}$. It follows from Proposition 4.2 that $\mathcal{B}$ has a base for its topology that is a tree under reverse inclusion. Hence so does each subspace of $\mathcal{B}$. However, the usual Sorgenfrey line is the subspace $S=\{(x,0):x\in\mathbb{R}\}$ of X and hence of $\mathcal{B}$, and it is well-known that S does not have any base of open sets that is a tree under reverse inclusion. □

Remark: Notice that Theorem 5.4 does *not* say that whenever a hereditarily ultraparacompact LOTS embeds in the branch space of one of its partition trees, then that partition tree must satisfy R-1, R-2, and R-3. For example, return to (2.2) and replace the interval $[0,1)$ by the LOTS $X=\mathbb{Q}\cap[0,1)$ where $\mathbb{Q}$ is the set of rational numbers. Replace every member t of the partition tree in (2.2) by the set $t\cap\mathbb{Q}$. The resulting partition tree for X is isomorphic as a partially ordered set to the partition tree in (2.2). Therefore its branch space $\mathcal{B}$ is the same as the branch space found in (2.2), and thus $\mathcal{B}$ is homeomorphic to $[0,1)$. Clearly the partition tree for X does not satisfy R-1, R-2, and R-3.

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