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A Note on Perfect Generalized Ordered Spaces

by

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1. Introduction.

Recall that a topological space X is perfect if each closed subset of X is a G_δ-subset of X and that X is perfectly normal provided X is both normal and perfect. Experience has shown that many problems in ordered space theory reduce to recognizing when a given ordered space is perfect, and the purpose of this note is to update an earlier paper [BL] on that topic.

In Section 2, we will show that several recent generalizations of perfect normality are, among ordered spaces, equivalent to perfect normality. In Section 3 we use our results to give necessary and sufficient conditions for $L(X)$ and X^*, the two most familiar linearly ordered extensions of a generalized ordered space X, to be perfect. In Section 4 we discuss two interesting generalizations of perfect normality that are *not* equivalent to perfect normality in ordered spaces, namely Kočinac's weakly perfect property (see [K1], [K2], [H], and [BHL]) and the property called $\mathcal{S}$- normality (see [B2]).

A linearly ordered topological space (LOTS) is a linearly ordered set $(X, <)$ equipped with the usual open interval topology of $<$. By a generalized ordered space, or GO-space, we mean a linearly ordered set equipped with a Hausdorff topology that has a base of open, convex subsets, where we say that a set C is convex in X provided for any $a < b < c$ in X, if $\{a, c\} \subset C$ then $b \in C$. It is known that the class of GO-spaces coincides with the class of subspaces of linearly ordered topological spaces. It will be important to distinguish between subsets of a space X that are relatively discrete, i.e., discrete in their subspace topologies, and those subsets that are both closed and discrete. Other notation and terminology will follow [E] and [L1].

2. Properties equivalent to perfect in GO-spaces

Reed [R] defined that a space X is strongly densely normal provided for each open set $U \subset X$, there are open sets $V(1)$, $V(2)$ ... in X such that $\mathrm{cl}(V(n)) \subset U$ and such that $\bigcup\{V(n) : n \geq 1\}$ is dense in U. A weaker concept was introduced by Kočinac [K2] who defined that a space X is almost perfect provided each open subset $U \subset X$ contains a dense subset S that is an F_σ-subset of X. For normal spaces, these two notions coincide.

2.1 **Lemma**: *If X is normal, then X is strongly densely normal if and only if X is almost perfect.* $\square$

Easy examples show that, even for compact Hausdorff spaces, the properties "almost perfect" and "perfect" are quite different.

2.2 **Example**: The usual product space $X = [0,1]^c$ is compact Hausdorff and almost perfect, but not perfect. (To see that X is almost perfect, note that X is separable [E].)

However, in the class of GO-spaces, we have:

2.3 **Proposition**: *For any GO-space X, the following are equivalent:*

a) X is perfect;

b) X is strongly densely normal;

c) X is almost perfect;

d) if E is a relatively discrete subspace of X, then E is an F_σ-subset of X;

e) if S is a closed, nowhere dense subset of X, then S is a G_δ-subset of X;

f) each regularly closed subset T of X (i.e., $T = \mathrm{cl}(\mathrm{Int}(T))$) is a G_δ-subset of X.

Proof: Properties (a) and (e) are equivalent in any space, and the equivalence of (a) and (f) for GO-spaces was proved in [BL]. Clearly a) implies b), and from (2.1), b) and c) are equivalent in any normal space. Therefore, to complete the proof of this proposition it will be enough to show that $c) \Rightarrow d) \Rightarrow a)$.

To prove that c) implies d), suppose that X is almost perfect and that E is a relatively discrete subspace of X. Because X is hereditarily collectionwise normal, there is a collection $\{U(e) : e \in E\}$ of open convex subsets of X such that $U(e) \cap E = \{e\}$ and $U(e) \cap U(e') = \emptyset$ whenever $e, e' \in E$ are distinct. Let $U = \bigcup\{U(e) : e \in E\}$ and choose closed subsets $D(n) \subset X$ such that $\bigcup\{D(n) : n \geq 1\}$ is a dense subset of U. Hence for each $e \in E$ there is an $n \geq 1$ such that $D(n) \cap U(e) \neq \emptyset$. Let $E(n) = \{e \in E : U(e) \cap D(n) \neq \emptyset\}$. Because $E = \bigcup\{E(n) : n \geq 1\}$, it will be enough to show that each $E(n)$ is closed in X

For contradiction, suppose $p \in \mathrm{cl}(E(n)) - E(n)$ and let W be any convex, open neighborhood of p. Then the set $W \cap E(n)$ is infinite. Because $\{U(e) : e \in E\}$ is a pairwise disjoint collection of convex sets, it follows that $U(e) \subset W$ for some $e \in E(n)$. Because $e \in E(n)$, $\emptyset \neq D(n) \cap U(e) \subset W$ so that $W \cap D(n) \neq \emptyset$. Thus, each neighborhood of p meets $D(n)$ so that $p \in \mathrm{cl}(D(n)) = D(n) \subset U$. Now choose $e \in E$ with $p \in U(e)$. But then $U(e)$ is a neighborhood of p that contains at most one point of $E(n)$, contradicting $p \in \mathrm{cl}(E(n)) - E(n)$. Hence $E(n)$ is closed in X as required.

To prove that d) implies a), suppose X is a GO space having property d) and $p \in X$. We first show that the half-line $H = \,]\leftarrow, p[$ is an F_σ-subset of X. If p is not a limit point of H, there is nothing to prove. Hence assume that every neighborhood of p meets H. Let $\kappa = \mathrm{cf}(\,]\leftarrow, p[\,)$ and choose a strictly increasing net $\{x(\alpha) : \alpha < \kappa\}$ whose supremum is p. Let $E = \{x(\alpha) : \alpha < \kappa$ is not a limit ordinal $\}$. Then E is a relatively discrete subspace of X so that, according to d), E is the union of a sequence of closed sets $E(n)$. Let $C(n) = \{x \in X : x \leq e$ for some $e \in E(n)\}$. Then $C(n)$ is closed in X and $H = \bigcup\{C(n) : n \geq 1\}$. An analogous proof shows that $]p, \rightarrow[$ is also an F_σ in X. It follows that each convex subset of X is an F_σ- subset of X.

Now suppose U is an open subset of X. Let $\{U(\alpha) : \alpha \in A\}$ be the family of all convex components of U and for each α choose a point $p(\alpha) \in U(\alpha)$. Let $E = \{p(\alpha) : \alpha \in A\}$. The E is a relatively discrete subset of X so that $E = \bigcup\{E(n) : n \geq 1\}$ where each $E(n)$ is closed in X. Let $A(n) = \{\alpha \in A : p(\alpha) \in E(n)\}$. For each $\alpha \in A(n)$ let $C(\alpha, n, 1) \subset C(\alpha, n, 2) \subset$ ... be closed convex subsets of X having $p(\alpha) \in C(\alpha, n, 1)$ and $\bigcup\{C(\alpha, n, m) : m \geq 1\} = U(\alpha)$. Define $F(n, m) = \bigcup\{C(\alpha, n, m) : \alpha \in A(n)\}$. Then each $F(n, m)$ is closed in X and $U = \bigcup\{F(n, m) : n, m \geq 1\}$ as required. $\square$

2.4 **Remark**: The equivalence of a) and d) in 2.3 is due to Faber [F, Theorem 2.4.5].

3. Perfect ordered extensions of generalized ordered spaces

There are two familiar constructions in ordered space theory for embedding any GO-space into a LOTS. For any GO-space $(X, \mathcal{T})$, let $\mathcal{I}$ denote the usual open interval topology of the given ordering of X. Then $\mathcal{I} \subset \mathcal{T}$. Define subsets of X as follows:

$R = \{x \in X : [x \rightarrow [\, \in \mathcal{T} - \mathcal{I}\}$, and

$L = \{x \in X : \,]\leftarrow, x] \in \mathcal{T} - \mathcal{I}\}$.

Let N be the set of all natural numbers and define

$$L(X) = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\})$$

and

$$X^* = (X \times \{0\}) \cup (\{(x, -n) : x \in R,\ n \in N\}) \cup (\{(x, n) : x \in L,\ n \in N\}).$$

Order both sets lexicographically and endow each with the open interval topology of the lexicographic order. Then X is a dense subspace of the LOTS $L(X)$ and is a closed subspace of the LOTS X^*. It is important to note that $L(X)$ is <u>not</u> a subspace of X^*.

It is of some use in constructing examples to know conditions under which X^* and $L(X)$ must be perfect.

3.1 **Proposition**: *Let X be a GO-space. Then X^* is perfect if and only if X is perfect and the set $R \cup L$ is σ-closed discrete in X.*

Proof: Suppose X^* is perfect. Then so is its subspace X. Let $\mathcal{T}$ be the given topology of X, and let R, and L be as above. Consider $S = \{(x, -1) : x \in R\}$. Then S is a relatively discrete open subset of X^* so that there are closed subsets $S(n)$ of X^* having $S = \bigcup\{S(n) : n \geq 1\}$. Each $S(n)$ is a closed, discrete subspace of X^*. Let $R(n) = \{x \in R : (x, -1) \in S(n)\}$. Because $R = \bigcup\{R(n) : n \geq 1\}$ it will be enough to show that each $R(n)$ is a closed, discrete subset of X. Fix n and fix any point $p \in X$. We will find a neighborhood V of p that contains at most one point of $R(n)$. Consider $(p, 0) \in X^*$. Because $S(n)$ is closed and discrete, there are points $(u, i) < (p, 0) < (v, j)$ of X^* such that $](u, i), (v, j)[\, \cap S(n) = \emptyset$.

If p is an isolated point of X, then $V = \{p\}$ is the required neighborhood of p, so assume p is not isolated. Then at least one of the sets $[p, \rightarrow[$ and $]\leftarrow, p]$ is not open in $(X, \mathcal{T})$. Without loss of generality, assume that $[p, \rightarrow[$ is not open. Then $(p, -1)$ cannot be a point of X^* so that if $(p, n) \in X^*$, then $n \geq 0$. Because $(u, i) < (p, 0)$ we conclude that $u < p$. We claim that $]u, p[\, \cap R(n) = \emptyset$. For suppose that $x \in\,]u, p[\, \cap R(n)$. Then $(x, -1) \in S(n)$. Because $u < x < p$ in X, we have $(u, i) < (x, -1) < (p, 0) < (v, j)$ in X^*. But then $(x, -1) \in\,](u, i), (v, j)[\, \cap S(n) = \emptyset$. Hence $]u, p[\, \cap R(n) = \emptyset$.

If the set $V =]u, p]$ is open in X, then V is the required neighborhood of p with $|V \cap R(n)| \leq 1$, so assume that $]u, p]$ is not open in X. Then $(p, +1) \notin X^*$ so that $(p, n) \in X^*$ implies $n \leq 0$. But then $(p, n) \in X^*$ implies $n = 0$ so that $(p, 0) < (v, j)$ implies $p < v$ in X. As was the case with the interval $]u, p[$, the set $]p, v[\, \cap R(n) = \emptyset$. But then $V =]u, v[$ is a neighborhood of p in X having $|V \cap R(n)| \leq 1$, as required to show that $R(n)$ is a closed discrete subset of X. Therefore, R is σ-closed discrete in X as claimed. Analogously, L is σ-closed discrete in X.

To prove the converse, suppose that $R \cup L$ is σ- closed discrete in X. We first show that the set $X^* - X$ is an F_σ in X^*. Write $R = \bigcup\{R(n) : n \geq 1\}$ where each $R(n)$ is a closed, discrete subset of X. For each $k \leq 0$, define $R(n, k) = \{(x, k) : x \in R(n)\}$. Because $R(n, 0) = R(n)$, the set $R(n, 0)$ is closed and discrete in X^*. Suppose $k < 0$ and let $(p, j) \in X^*$. We will find a neighborhood W of (p, j) in X^* that contains at most one point of $R(n, k)$. If $j \neq 0$ then $\{(p, j)\}$ is the required neighborhood, so assume $j = 0$. Because $R(n)$ is closed and discrete in X, there is a neighborhood U of p in X that is convex in X and has $|U \cap R(n)| \leq 1$. There are four cases to consider, depending upon the shape of U.

<u>Case 1</u>: Suppose $U = \{p\}$. Then $W = \{(p, 0)\}$ is open in X^* and has $|W \cap R(n, k)| \leq 1$ as

required.

<u>Case 2</u>: Suppose that $U = [p, q[$ with $q > p$ and that $\{p\}$ is not open in X. Then $]p, q[\neq \emptyset$ and we may choose $r \in]p, q[$. Because $[p, q[$ is open in X, either p has an immediate predecessor in the ordering of X or else $(p, -1) \in X^*$. In either case, the set $W = [(p, 0), (r, 0)[$ is open in X^*. If $(x, k) \in W \cap R(n, k)$ then $x \in R(n)$ and $(p, 0) \leq (x, k) < (r, 0)$. Then in X, $p \leq x \leq r$ so that $x \in R(n) \cap [p, r] \subset R(n) \cap [p, q[$. By choice of $U = [p, q[$ there is at most one point $x \in R(n) \cap [p, q[$ so that, k being fixed, there is at most one point in $R(n, k) \cap W$.

<u>Case 3</u>: Suppose $U =]q, p]$ with $q < p$ and $\{p\}$ is not open in X. This case parallels Case 2.

<u>Case 4</u>: Suppose $U =]q, r[$ and that neither $[p, \rightarrow [$ nor $] \leftarrow, p]$ is open in X. Then we may choose $q' \in]q, p[$ and $r' \in]p, r[$. If $(x, k) \in R(n, k) \cap](q', 0), (r', 0)[$, then $x \in R(n)$ and $q' \leq x \leq r'$ so that $x \in R(n) \cap [q', r'] \subset R(n) \cap]q, r[$. But there is at most one such point x so that if we let $W =](q', 0), (r', 0)[$ then $(p, 0) \in W$ and (k being fixed) there is at most one point in $W \cap R(n, k)$.

Analogously, the set $\{(x, k) : x \in L \text{ and } k \geq 0\}$ is a σ-closed discrete subset of X^*. Because $x \in R \cup L$ whenever $(x, k) \in X^*$ has $k \neq 0$, we see that $X^* - X$ is a σ-closed discrete subset of X^*.

To complete the proof that X^* is perfect, suppose that U is any open subset of X^*. Then $U \cap X$ is relatively open in X so that, X being perfect and closed in X^*, $U \cap X$ is an F_σ-subset of X^*. Because $U - X \subset X^* - X$ we also know that $U - X$ is an F_σ-subset of X^*. Hence so is U, as required. $\square$

We can also give necessary and sufficient conditions for the space $L(X)$ to be perfect, but the result is less satisfactory than (3.1) because the conditions are not internal to the GO-space X. We prove a slightly more general result, namely:

3.2 **Proposition**: *Suppose $X \subset Y$ are GO-spaces with X dense in Y. Then Y is perfect if and only if each relatively discrete subspace of X is an F_σ-subset of Y. In particular, for a perfect GO-space X, the LOTS $L(X)$ is perfect if and only if each relatively discrete subspace of X is an F_σ-subset of $L(X)$.*

Proof: If Y is perfect, apply (2.3)(d) to conclude that any relatively discrete subspace of X is an F_σ-subset of Y.

To prove the converse, we again apply (2.3). Suppose that D is a relatively discrete subspace of Y. We will show that D is an F_σ-subset of Y. Write $D = (D \cap X) \cup (D - X)$. By hypothesis, $D \cap X$ is an F_σ-subset of Y, so it is enough to show that the set $D - X$ is an F_σ-subset of Y.

Because $D - X$ is a relatively discrete subset of Y, for each $d \in D - X$ there is a convex open subset $U(d)$ of Y such that $U(d) \cap (D - X) = \{d\}$ and because Y is hereditarily collectionwise normal, we may assume that the sets $U(d)$ are pairwise disjoint. For each $d \in D - X$ there is a convex open subset $V(d)$ of Y with $d \in V(d) \subset \mathrm{cl}(V(d)) \subset U(d)$. Because X is dense in Y, we may choose $x(d) \in V(d) \cap X$. Then $P = \{x(d) : d \in D - X\}$ is a relatively discrete subspace of X, so by hypothesis, P is an F_σ-subset of Y, say $P = \bigcup\{P(n) : n \geq 1\}$ where each $P(n)$ is closed and discrete in Y. Let $A(n) = \{d \in D - X : x(d) \in P(n)\}$ and let $\mathcal{V}(n) = \{V(d) : d \in A(n)\}$. It will be enough to show that each set $A(n)$ is closed in Y, because $D - X = \bigcup\{A(n) : n \geq 1\}$. To show that $A(n)$ is closed in Y, it will be enough to prove that $\mathcal{V}(n)$ is a discrete collection in Y. To that end, suppose $q \in Y$. If $q \in \bigcup \mathcal{V}(n)$ choose the unique set $V(d) \in \mathcal{V}(n)$ with $q \in V(d)$. Then $V(d)$ is a neighborhood of q meeting at most one member of $\mathcal{V}(n)$. So suppose $q \notin \bigcup \mathcal{V}(n)$. Because $P(n) \subset \bigcup \mathcal{V}(n)$ we know that $q \notin P(n)$. Hence there is a convex, open neighborhood W of q that is disjoint from $P(n)$. If W meets more than two members of $\mathcal{V}(n)$, then convexity forces W to entirely contain some member $V(d_0) \in \mathcal{V}(n)$. But then $d_0 \in W \cap P(n)$ contrary to our choice of W. Hence W meets at most two members of $\mathcal{V}(n)$. If W meets only one member of $\mathcal{V}(n)$, we are done, so suppose there are two distinct members $V(d_1), V(d_2) \in \mathcal{V}(n)$ that both meet W. Recall that $\mathrm{cl}(V(d_i)) \subset U(d_i)$ so that $\mathrm{cl}(V(d_1)) \cap \mathrm{cl}(V(d_2)) = \emptyset$. We may assume that $q \notin \mathrm{cl}(V(d_1))$. But then $W - \mathrm{cl}(V(d_1))$ is a neighborhood of q that meets at most one member of $\mathcal{V}(n)$. Thus, $\mathcal{V}(n)$ is indeed a discrete collection of open subsets of Y. But then the set $A(n)$ is a closed discrete subset of Y, as required. $\square$

4. Two generalizations of perfect normality in ordered spaces

Recall that a space X is <u>weakly perfect</u> if each closed subset C of X contains a set D having:

a) D is a G_δ-subset of X; and
b) the closure of D in X is C.

This property was introduced in [K1] and [K2] and studied by Heath in [H]. The current authors have studied weakly perfect generalized ordered spaces at length in [BHL]. For this paper, it will be enough to note that a linearly ordered space can be weakly perfect but not perfect: the usual space of countable ordinals is one such example, and [BHL] constructs examples of compact linearly ordered spaces that are hereditarily weakly perfect but not perfect. In addition, [BHL] contains examples showing that if X is a (weakly) perfect GO-space, then the ordered extensions X^* and $L(X)$ might, or might not, be weakly perfect.

Another generalization of perfect normality, called *S-normality*, was studied in [B2]. A topological space X is said to be $\mathcal{S}$-normal if for each closed subset $C \subset X$ there is a countable collection $\mathcal{S}$ of open subsets of X such that if $p \in C$ and $q \in X - C$, then for some $S \in \mathcal{S}$ we have $p \in S$ and $q \notin S$. As will be seen in (4.2), this property is strictly weaker than perfect normality in ordered spaces.

Our next result is easy to verify and gives several sufficient conditions for a space to be $\mathcal{S}$-normal.

4.1 **Proposition**: *A topological space X is $\mathcal{S}$- normal if any one of the following holds:*

a) X is perfectly normal;

b) X has a countable open cover $\mathcal{U}$ that separates points of X, i.e., if x, y are distinct points of X, then some $U \in \mathcal{U}$ has $x \in U \subset X - \{y\}$;

c) X has a weaker topology that is separable and metrizable;

d) X can be embedded in an $\mathcal{S}$-normal space Y. $\square$

4.2) **Example**: There is an $\mathcal{S}$-normal LOTS that is not perfect. Let R, P, Q denote the usual spaces of real, irrational, and rational numbers, respectively, and let Z be the set of all integers. Let X be the lexicographically ordered set $X = (R \times \{0\}) \cup (P \times Z)$. With the usual open interval topology of that ordering, X is a quasi-developable LOTS [B1] that is not perfect (because $Q \times \{0\}$ is a closed subset of X that is not a G_δ-subset of X).

To show that X is $\mathcal{S}$-normal, for each pair of rational numbers $r < s$ define $U(r,s) =](r,0),(s,0)[$ in the ordered set X. Let C be any closed subset of X and let $U_0 = \{(x,i) \in C : (x,i) \text{ is isolated in } X\}$. Let $\mathcal{U} = \{U_0\} \cup \{U(r,s) : r, s \in Q \text{ and } r < s\}$. Then $\mathcal{U}$ is a countable collection of open subsets of X. Now suppose $(x,i) \in C$ and $(y,j) \in X - C$. If $(x,i) \in U_0$, we are done, so suppose $(x,i) \notin U_0$. Then (x,i) is not isolated in X and hence $x \in Q$ and $i = 0$. Because no other point of X has x as its first coordinate, we conclude that $y \neq x$. Hence there are rational numbers r, s with $r < x < s$ and either $y < r$ or $s < y$. In either case, $(x,i) \in U(r,s) \subset X - \{(y,j)\}$ as required. $\square$

$\mathcal{S}$-normal spaces resemble perfect spaces in two important ways. First, each point in such a space must be a G_δ-set, so that an $\mathcal{S}$-normal GO space must be first countable. Second, we have:

4.3 **Proposition**: *If T is a stationary subset of a regular uncountable cardinal κ, then with the topology that it inherits from the usual ordinal space, T is not $\mathcal{S}$- normal. Therefore, any $\mathcal{S}$-normal GO space, and any monotonically normal space that is $\mathcal{S}$-normal, must be hereditarily paracompact*

Proof: Let C be the set of non-isolated points of the set T. Then C is also stationary in κ and is closed in T. Assume that T is $\mathcal{S}$-normal. Then there is a countable collection $\mathcal{U}$ of open subsets of T such that if $x \in C \subset T - \{y\}$ then some $U \in \mathcal{U}$ has $x \in U \subset T - \{y\}$.

Let $\mathcal{U}_1$ be the family of all $U \in \mathcal{U}$ such that some $\lambda(U) \in T$, $[\lambda(U), \rightarrow [\ \cap T \subset U$. Because $\mathcal{U}$ is countable and κ is uncountable and regular, there is an ordinal $\mu \in C$ such that $\lambda(U) < \mu$ for each $U \in \mathcal{U}_1$. Then $[\mu, \rightarrow [\ \cap T \subset U$ for each $U \in \mathcal{U}_1$.

Let $\mathcal{U}_2 = \mathcal{U} - \mathcal{U}_1$ and index $\mathcal{U}_2$ as $\{U(n) : n \geq 1\}$. Let ν be the first element of T that is greater than μ. Then ν is not an element of the set C and for each $\lambda \in C$ with $\lambda > \nu$, we can find some $U \in \mathcal{U}$ with $\lambda \in U \subset T - \{\nu\}$. Because $\nu > \mu$ we conclude $[\mu, \rightarrow [\ \cap T \not\subset U$ so that $U \notin \mathcal{U}_1$. Hence $U \in \mathcal{U}_2$. Therefore, for each $\lambda \in\ ]\nu, \rightarrow [\ \cap C$, there is an integer $n(\lambda) \geq 1$ with $\lambda \in U(n(\lambda)) \subset T - \{\nu\}$. Define $C(m) = \{\lambda \in C \cap\ ]\nu, \rightarrow [\ : n(\lambda) = m\}$. Then $C \cap\ ]\nu, \rightarrow [\ = \bigcup\{C(m) : m \geq 1\}$ so that one of the sets $C(m_0)$ must be stationary. Observe that for each $\lambda \in C(m_0)$, $\lambda \in U(m_0)$ so that there must be an $\alpha(\lambda) < \lambda$ such that $]\alpha(\lambda), \lambda] \cap T \subset U(m_0)$. Apply the Pressing Down Lemma to find some β and a cofinal set $D \subset C(m_0)$ with $\beta = \alpha(\lambda)$ for each $\lambda \in D$. Then $]\beta, \lambda] \cap T \subset U(m_0)$ for each $\lambda \in D$, so that $]\beta, \rightarrow [\ \cap T \subset U(m_0)$ because D is cofinal in T. But then $U(m_0) \in \mathcal{U}_1$, contradicting the fact that $U(m_0) \in \mathcal{U}_2$. That contradiction establishes that the subspace T cannot be $\mathcal{S}$- normal.

Now consider any GO space X, or any monotonically normal space X, that is $\mathcal{S}$-normal. If X is not hereditarily paracompact, then there is an uncountable regular cardinal κ and a stationary subset $T \subset [0, \kappa[$ that embeds topologically in the space X [EL], [BR]. According to (4.1-d), the space T would be $\mathcal{S}$-normal, and that is impossible by the first part of this proof. Thus, X is hereditarily paracompact. $\square$

In [B2], a property related to $\mathcal{S}$-normality was used to characterize quasi-developability in linearly ordered topological spaces. It is natural to ask whether every quasi-developable LOTS must be $\mathcal{S}$-normal. The next example provides a negative answer.

4.4 **Example**: There is a quasi-developable LOTS that is not $\mathcal{S}$-normal. We will begin by constructing a quasi- developable GO space Y that is not $\mathcal{S}$-normal and then, using techniques from [L] we will embed it into a quasi- developable LOTS Y^*. In the light of (4.1-d), Y^* cannot be $\mathcal{S}$-normal.

Let $\{S_\alpha : \alpha < \kappa\}$ be a well-ordering of the collection of all subsets of the set R of real numbers, where $\kappa = 2^c$ and where $S_\alpha \neq S_\beta$ whenever $\alpha < \beta < \kappa$. For each α, let $M(S_\alpha)$ be the Michael line obtained by isolating every point of the set S_α and letting all points of $C_\alpha = R - S_\alpha$ have their usual neighborhoods. Then C_α will be a closed subset of $M(S_\alpha)$.

Let Y be the lexicographically ordered set $[0, \kappa[\ \times R$. In the usual open interval topology $\mathcal{I}$ of Y, each set $Y_\alpha = \{\alpha\} \times R$ is open and is a copy of the usual space of real numbers. Modify the topology $\mathcal{I}$ by isolating every point of $\bigcup\{\{\alpha\} \times S_\alpha : \alpha < \kappa\}$. The resulting GO-space $(Y, \mathcal{T})$ is a topological sum of copies of the Michael lines $M(S_\alpha)$ so that $(Y, \mathcal{T})$ is quasi-developable.

To prove that Y is not $\mathcal{S}$-normal, consider the closed set $C = \bigcup\{\{\alpha\} \times C_\alpha : \alpha < \kappa\}$. Suppose there is a sequence $U(1), U(2), ...$ of open subsets of Y such that if $(\alpha, x) \in C \subset Y - \{(\beta, y)\}$, then for some n we have $(\alpha, x) \in U(n) \subset Y - \{(\beta, y)\}$. Let $V(n) = \mathrm{Int}_{\mathcal{I}}(U(n))$ for each $n \geq 1$ and observe that for each $\alpha < \kappa$, $V(n) \cap Y_\alpha$ is an open subset of R in the usual topology. Further, because $\mathcal{I}$-neighborhoods and $\mathcal{T}$-neighborhoods are the same for points of the closed set C, we know that if $(\alpha, x) \in C \subset Y - \{(\beta, y)\}$, then for some n we have $(\alpha, x) \in V(n) \subset Y - \{(\beta, y)\}$.

The usual space of real numbers has only c many open sets. Hence the set $\Sigma = \{(W_1,\ W_2,\ ...) : \text{each } W_i \text{ is open in the usual topology of } R\}$ also has $|\Sigma| = c$. Using the fixed sequence $< V(n) >$ found above, define a function $\sigma : [0, \kappa[\ \to \Sigma$ by the rule $\sigma(\alpha) = (V(1) \cap Y_\alpha,\ V(2) \cap Y_\alpha,\ ...)$. (In the definition of σ we are identifying the subset $V(n) \cap Y_\alpha$ of Y_α with the subset $\pi_\alpha[V(n) \cap Y_\alpha]$, where $\pi_\alpha : Y_\alpha \to R$ is second coordinate projection.) Because $\kappa = 2^c$, we know that $\sigma(\alpha) = \sigma(\beta)$ for some $\alpha \neq \beta$ in $[0, \kappa[$.

Because $\alpha \neq \beta$ we know that $S_\alpha \neq S_\beta$. Without loss of generality, choose $x \in S_\beta - S_\alpha$. Then $(\alpha, x) \in C_\alpha \subset C$ while $(\beta, x) \notin C$ so that for some n, $(\alpha, x) \in V(n) \subset Y - \{(\beta, x)\}$. However, from $\sigma(\alpha) = \sigma(\beta)$ we conclude that $V(n) \cap Y_\alpha = V(n) \cap Y_\beta$ so that $(\alpha, x) \in V(n) \cap Y_\alpha$ forces $(\beta, x) \in V(n) \cap Y_\beta \subset V(n)$, contrary to our choice of $V(n)$. That contradiction shows that Y is not $\mathcal{S}$-normal, as required. □

4.5 **Example**: Let X be the usual Sorgenfrey line. Then X is perfect but its ordered extension X^* is not $\mathcal{S}$-normal. Consider the closed subset $C = X \times \{0\}$ of X^*, and suppose $\mathcal{S}$ is a countable collection of open subsets of X^* as in the definition of $\mathcal{S}$-normality. We may assume that if $S \in \mathcal{S}$, then every convex component of S meets C. Consequently, we may assume that every member of $\mathcal{S}$ is convex. Choose $(x, 0) \in C$ that is not one of the countably many end-points of members of $\mathcal{S}$. Then the points $(x, 0) \in C$ and $(x, -1) \in X^* - C$ cannot be separated by any member of $\mathcal{S}$. □

4.6 **Example**: Let X be the GO-space obtained by isolating every point of $[0, \omega_1[$. Then X is perfect, but because the ordered extension $L(X)$ contains a topological copy of the usual space $[0, \omega_1[$, (4.3) shows that $L(X)$ is not $\mathcal{S}$-normal. □

In closing, let us note that the classes of weakly perfect GO-spaces and $\mathcal{S}$-normal

GO-spaces are quite distinct. The usual space of countable ordinals is weakly perfect but not $\mathcal{S}$-normal, while the Michael line is $\mathcal{S}$-normal (by (4.1-c)) but not weakly perfect.

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