Point Countability in Generalized Ordered Spaces

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1. Introduction.

In general spaces, there is a wide spectrum of base axioms that generalize, in different ways, the notion of metrizability. But in the more restrictive category of ordered spaces, these different base axioms fall into three broad classes – those equivalent to metrizability, those equivalent to quasi- developability, and those equivalent to the existence of a point-countable base, as can be seen from the following well-known theorems.

(1.1) Theorem: *The following properties of a generalized ordered space are equivalent:*

a) *X is metrizable;*
b) *X is developable;*
c) *X has a σ-discrete base of open sets;*
d) *X has a σ-locally finite base of open sets;*
e) *X has a σ-locally countable base of open sets* [*Fe*].

(1.2) Theorem: The following properties of a generalized ordered space are equivalent:

a) *X is quasi-developable;*
b) *X has a σ-disjoint base of open sets;*
c) *X has a σ-point-finite base of open sets.*

(1.3) Theorem: *The following properties of a generalized ordered space are equivalent:*

a) *X has a point-countable base;*
b) *For each $x \in X$ there is a countable collection $\mathcal{B}(x)$ of open sets such that if a sequence $x(n)$ in X converges to x, then $\{\mathcal{B}(x(n)) : n \geq 1\}$ contains a base at x.* [*G*]

It is easy to see that for generalized ordered spaces, any property from (1.1) implies any property from (1.2), and any property from (1.2) implies any property from (1.3). Well known examples show that none of these implications can be reversed. The space of Example 2.11 of [Be2] has a point-countable base, but not a σ-disjoint base, and the Michael line [M] has a σ-disjoint base but is not metrizable.

There is a simple topological property that serves as a bridge between the properties in (1.1) and (1.2), namely the property "X is perfect (i.e., each closed subset of X is a G_δ-set)." By that statement we mean that a generalized ordered space X with any one of the properties in (1.2) will have any one of the properties in (1.1) if and only if X is perfect. The purpose of this paper is to introduce and study a topological property that is the bridge, for a generalized ordered space X, between the property "X has a point-countable base" and any one of the properties in (1.2). The relevant property is defined as follows:

(1.4) Definition: A topological space X has Property III provided there are sequences $\{U(n) : n \geq 1\}$ and $\{D(n) : n \geq 1\}$ satisfying:

a) each $U(n)$ is an open subset of X;
b) each $D(n)$ is a relatively closed subset of $U(n)$ and is discrete-in-itself;
c) if G is open in X and $x \in G$, then there is an $n \geq 1$ such that $x \in U(n)$ and $G \cap D(n) \neq \phi$.

Property III is an interesting combination of discreteness and denseness, and is much stronger than the statement "X has a dense, σ-discrete-in-itself subset," as can be seen from Example 5.3-c below. Property III allows us to prove the main result of our paper, namely:

(1.5) Theorem: *The following properties of a generalized ordered space X are equivalent:*

a) *X is quasi-developable;*
b) *X has a σ-disjoint base of open sets;*
c) *X has a σ-point finite base of open sets;*
d) *X has a point countable base and has Property III.*

The equivalence of (d) with the other parts of Theorem 1.5 has several applications to familiar classes of spaces. To the best of our knowledge, these applications are new.

(1.6) Proposition: *A generalized ordered space is quasi- developable if and only if it satisfies any one of the following:*

a) *X has a point-countable base and a σ-minimal base*
b) *X has a point-countable base and a quasi-G_δ- diagonal.*

The proofs of (1.5) and of its applications in (1.6) are given in Section 3, below. The proofs rely heavily on the paracompactness of the spaces being considered. In Section 4 we examine Property III in the context of ordered spaces. For example, we show that if X is a generalized ordered space having Property III, then every subspace of X also has Property III. It is interesting that we must first prove that any generalized ordered space with Property III is hereditarily paracompact. Section 5 presents examples relevant to Property III in generalized ordered spaces and Section 6 studies Property III in general topological spaces. It is interesting to note that any topological space X is a closed subspace of a space $B(X)$ that has Property III and mimics many properties of X.

2. Definitions and basic results on paracompactness.

Suppose Y is a subset of a space X. We will use the phrase "Y is discrete-in-itself" to mean that when endowed with the relative topology from X, Y is a discrete topological space. There is no guarantee that Y is also closed in X. This distinction is crucial at several points in this paper. To say that a set Z is "σ-discrete-in-itself" means that Z can be written as a countable union of subspaces, each of which is discrete-in-itself.

A linearly ordered topological space ($LOTS$) is a triple consisting of a set X, a linear order $<$ of X, and the usual open interval topology induced on X by $<$. A generalized ordered space ($GO-space$) consists of a set X, a linear ordering $<$ of X, and a topology on X that contains the usual order topology of $<$ and has a base consisting of order-convex sets. It is known that the class of GO-spaces is precisely the class of subspaces of $LOTS$. See [L1] for a general discussion of these spaces.

If p and q are points of a linearly ordered set, then $[p, q[$ will denote the interval $\{x \in X : p \leq x < q\}$. Intervals of the form $]p, q[$ and $[p, q]$ are defined analogously. The notation $]\leftarrow, p]$ will mean $\{x \in X : x \leq p\}$, and the notation $[p, \rightarrow[$ is defined analogously. A subset C of X is convex if $[p, q] \subset C$ whenever $p < q$ are both points of C.

To say that a space X is quasi-developable [Be1] means that there is a sequence $\{\mathcal{G}(n) : n \geq 1\}$ of collections of open subsets of X such that if U is open in X and if $x \in U$, then for some $n \geq 1$, $St(x, \mathcal{G}(n))$ is a non-empty subset of U. If it happens that each of the collections $\mathcal{G}(n)$ covers X, then X is said to be developable. Analogously, to say that X has a quasi-G_δ-diagonal means that there is a sequence $\{\mathcal{G}(n) : n \geq 1\}$ of collections of open sets in X such that if x and y are distinct points of X, then for some n, $St(x, \mathcal{G}(n))$ is non-empty and does not contain y.

To say that a space has a σ-minimal base [Au, BB] means that there is a base $\mathcal{B} = \{\mathcal{B}(n) : n \geq 1\}$ for the open sets in X such that for each set B in each collection $\mathcal{B}(n)$, some point of B belongs to no other member of $\mathcal{B}(n)$.

The proofs in Sections 3 and 4 below rely heavily on paracompactness. There are two different approaches to paracompactness in generalized ordered spaces, and we need both. The first, given in [EnL], is

(2.1) Proposition: *A generalized ordered space X is not paracompact if and only if there is a stationary set S in a regular uncountable cardinal such that S embeds in X as a closed* *subspace (by an increasing or decreasing homeomorphism). A generalized ordered space X fails to be hereditarily paracompact if and only if some stationary subset of a regular uncountable cardinal is homeomorphic to a subspace of X.*

The second approach to paracompactness in ordered spaces depends upon the theory of Q-gaps developed by Gillman and Henricksen [GH]. Their theory is easily extended to include generalized ordered spaces (see [L2]).

(2.2) Proposition: *A generalized ordered space X is paracompact if and only if for each non-empty closed subset C of X, there is a set $E \subset C$ that is relatively closed in C, discrete-in-itself, and both cofinal and coinitial in C. Furthermore, X is hereditarily paracompact if and only if for every non-empty subset C of X, there is a relatively closed, discrete-in-itself subset E of C that is both cofinal and coinitial in C. In particular, if C is convex in X, then for each convex component F of $C - E$, there are points e' and e'' in E with $F \subset [e', e'']$.*

Proof: Consider the case where C is a closed subset of the paracompact space X, or the case where C is an arbitrary subspace of the hereditarily paracompact space X. In either case, C is a paracompact generalized ordered space in its relative topology. We will construct two relatively closed, discrete in themselves subsets E' and E'' of C that are, respectively, coinitial and cofinal in C, and then let $E = E' \cup E''$.

If C contains a right hand end point e'' of itself, let $E'' = \{e''\}$. Now suppose C does not contain a right hand endpoint of itself. Then it follows from [GH, L2] that there is a regular cardinal κ and a transfinite increasing sequence $\{e''(\alpha) : \alpha < \kappa\}$ of points of C that is cofinal in C and has the property that for every limit ordinal $\lambda < \kappa$, the set $\{e''(\alpha) : \alpha < \kappa\}$ is closed in the space C. (That is, the transfinite sequence $e''(\alpha)$ is a Q-sequence in the sense of [GH], reinterpreted in the class of generalized ordered spaces – see [L2].) Then let $E'' = \{e''(\alpha) : \alpha < \kappa\}$.

The coinitial set E' is constructed analogously, and we make sure that $e'(0) < e''(0)$. Now let $E = E' \cup E''$. $\square$

3. Proof and applications of Theorem 1.5.

We will begin this section by proving the following result, stated in the Introduction:

Theorem: *The following properties of a generalized ordered space are equivalent:*

a) *X is quasi-developable;*
b) *X has a σ-disjoint base of open sets;*
c) *X has a σ-point finite base of open sets;*

d) *X has a point countable base and has Property III.*

Outline of the proof. The equivalence of (a), (b), and (c) is Theorem (1.2) and proofs appear in [Be1]. It is easy to check that (b) implies (d), so that the proof will be complete once we establish the converse. That requires a sequence of lemmas.

(3.1) Lemma: *Let D be a discrete-in-itself subspace of a first countable generalized ordered space X. Then there is a collection $\mathcal{B} = \cup\{\mathcal{B}(n) : n \geq 1\}$ of open sets in X such that*

a) *each $\mathcal{B}(n)$ is a pairwise disjoint collection;*

b) *if G is open in X and $x \in D \subset G$, then some $B \in \mathcal{B}$ has $x \in B \subset G$.*

Proof: Let $U = \cup\{V : V \textit{ is open in } X \textit{ and } V \cap D \textit{ has at most one point}\}$. Then D is a discrete subspace of U. Because X is hereditarily collectionwise normal [L1] there is a collection $\{W(d) : d \in D\}$ of open subsets of U that is discrete in U and satisfies $d \in W(d)$. Since X is first countable, for each $d \in D$ there is a base of open sets {W(n,d): $n \geq 1$} at the point d that satisfies $W(n,d) \subset W(d)$. Then let $\mathcal{B}(n) = \{W(n,d) : d \in D\}$ to define the required collections. $\square$

(3.2) Proposition: *If X is a generalized ordered space with Property III and a point-countable base, then X has a σ- disjoint base.*

Proof: Let $D(n)$ and $U(n)$ be as in the definition of Property III. For each n, let $\{C(n,\alpha) : \alpha \in A(n)\}$ be the collection of all convex components in X of the open set $U(n) - D(n)$. Because X has a point countable base, or alternatively because X has Property III, X is hereditarily paracompact ([$Be2$] or (4.2), below). Therefore, for each set $C(n,\alpha)$, we may use (2.2) to find a discrete-in-itself, relatively closed subset $E(n,\alpha)$ of $C(n,\alpha)$ with the property that for each convex component F of $C(n,\alpha) - E(n,\alpha)$, there are points e' and e'' in $E(n,\alpha)$ with $F \subset [e', e'']$. Let $\mathcal{F}(n,\alpha)$ be the set of all convex components of $C(n,\alpha) - E(n,\alpha)$. Then each $\mathcal{F}(n,\alpha)$ is a pairwise disjoint collection of open sets.

For each n, the set $D(n)$ is discrete-in-itself, as is the set $E(n) = \cup\{E(n,\alpha) : \alpha \in A(n)\}$. Since X has a point countable base, X is certainly first countable, so that (3.4) yields a σ-disjoint collection $\mathcal{B}$ that contains a local base at each point of the set $Y = \cup\{D(n) \cup E(n) : n \geq 1\}$. It remains to construct a σ-disjoint collection that contains a base at each point of $X - Y$.

Let $\mathcal{P}$ be a point-countable base for the space X. Without loss of generality, we may assume that the members of $\mathcal{P}$ are convex. Let $\mathcal{P}(n) = \{P \in \mathcal{P} : P \cap D(n) \neq \phi\}$. For each $\alpha \in A(n)$, let $\mathcal{Q}(n,\alpha) = \{P \in \mathcal{P}(n) : P \cap C(n,\alpha) \textit{ is cofinal in } C(n,\alpha)\}$. For each $F \in \mathcal{F}(n,\alpha)$, let $\mathcal{R}(n,\alpha,F) = \{P \cap F : P \in \mathcal{Q}(n,\alpha)\}$. We claim that $\mathcal{R}(n,\alpha,F)$ is countable. To verify that claim, choose e' and e'' in $E(n,\alpha)$ such that $F \subset [e', e'']$. For each $P \cap F \in \mathcal{R}(n,\alpha,F)$, the set $P \cap C(n,\alpha)$ is cofinal in $C(n,\alpha)$ and is convex in X, so that because $e'' \in E(n,\alpha) \subset C(n,\alpha)$, we know that $e'' \in P$. But $\mathcal{P}$ is point countable in X, so that there can be at most countably many sets P with this property. Knowing that $\mathcal{R}(n,\alpha,F)$ is countable, choose an indexing of the form $\mathcal{R}(n,\alpha,F) = \{R(n,\alpha,F,k) : k \geq 1\}$. Observe that, for a fixed n, if α and β are distinct elements of $A(n)$, then for any choices of F and F', $R(n,\alpha,F,k)$ and $R(n,\beta,F',k)$ are subsets of distinct convex components $C(n,\alpha)$ and $C(n,\beta)$ and are therefore disjoint. Also observe that for a fixed $\alpha \in A(n)$, if F and F' are distinct members of $\mathcal{F}(n,\alpha)$, then $R(n,\alpha,F,k)$ and $R(n,\alpha,F',k)$ are subsets of the disjoint convex components F and F' respectively. It follows that the collection defined by

$$\mathcal{R}(n,k) = \{R(n,\alpha,F,k) : \alpha \in A(n) \textit{ and } F \in \mathcal{F}(n,\alpha)\}$$

is pairwise disjoint, so that $\mathcal{R} = \cup\{\mathcal{R}(n,k) : n,k \geq 1\}$ is a σ-disjoint collection of open subsets of X.

Next let $\mathcal{S}(n,\alpha) = \{P \in \mathcal{P}(n) : P \cap C(n,\alpha) \textit{ is coinitial in } C(n,\alpha)\}$ and for each $F \in \mathcal{F}(n,\alpha)$, let $\mathcal{T}(n,\alpha,F) = \{P \cap F : P \in \mathcal{S}(n,\alpha)\}$. Each collection $\mathcal{T}(n,\alpha,F)$ is countable and we let $\mathcal{T}(n,\alpha,F) = \{T(n,\alpha,F,k) : k \geq 1\}$. Just as above, the collection $\mathcal{T}(n,k) = \{T(n,\alpha,F,k) : \alpha \in A(n) \textit{ and } F \in \mathcal{F}(n,\alpha)\}$ is pairwise disjoint, so the collection $\mathcal{T}$, defined to be $\cup\{\mathcal{T}(n,k) : n,k \geq 1\}$, is a σ-disjoint collection of open subsets of X.

To complete the proof, suppose that $x \in X - Y$ is a point of an open set G. Choose $P \in \mathcal{P}$ with $x \in P \subset G$. According to Property III there is an n with $x \in U(n)$ and $P \cap D(n) \neq \phi$. Choose $d \in P \cap D(n)$. Because $x \notin Y$, we have $x \notin D(n)$ and hence $x \neq d$.

Consider the case where $x < d$. Since $x \in U(n) - D(n)$, there is a unique α with $x \in C(n,\alpha)$. Because $d \in C(n,\alpha)$ and $C(n,\alpha)$ is convex, it follows that $C(n,\alpha) \subset]\leftarrow, d[$. Because both x and d belong to the convex set P, $P \cap C(n,\alpha)$ must be cofinal in $C(n,\alpha)$, so that P is a member of $\mathcal{Q}(n,\alpha)$. Because x is not in $E(n)$ we know that x is not in $E(n,\alpha)$ so that there is a unique convex component F of $C(n,\alpha) - E(n,\alpha)$ that contains x. Then $P \cap F$ is in $\mathcal{R}(n,\alpha,F)$ so that for some k, we have $P \cap F = R(n,\alpha,F,k) \subset \mathcal{R}(n,k) \subset \mathcal{R}$. Because $x \in P \cap F \subset P \subset G$, we have shown that $\mathcal{R}$ contains a local base at x.

In case $d < x$, an analogous argument shows that $\mathcal{T}$ contains a local base at x. Therefore the σ-disjoint collection $\mathcal{B} \cup \mathcal{R} \cup \mathcal{T}$ is a base of open sets for the space X, as required. $\square$

As an application of (3.2), we show that certain more familiar types of generalized ordered spaces have σ-disjoint bases.

(3.3) <u>Proposition</u>: *Let X be a generalized ordered space. Then X has a σ-disjoint base (equivalently, X is quasi- developable) if any one of the following holds:*

a) *X has a point-countable base and a σ-minimal base;*

b) *X has a point-countable base and a quasi-G_δ-diagonal;*

c) *X has a point-countable base and is the union of countably many (not necessarily closed) subspaces, each of which is discrete-in-itself.*

Proof: In Lemmas (3.4), (3.5), (3.6) and (3.7) we will show that each of the spaces described in (3.3) has Property III, and then apply (3.2) to obtain the desired conclusion.

(3.4) <u>Lemma</u>: *If X is a generalized ordered space with a σ-minimal base, then X has Property III.*

Proof: Let $\mathcal{B} = \cup\{\mathcal{B}(n) : n \geq 1\}$ be a σ-minimal base for X. For each n and for each $B \in \mathcal{B}(n)$, choose a point $p(n, B) \in B$ that belongs to no other member of $\mathcal{B}(n)$. Let $D(n) = \{p(n, B) : B \in \mathcal{B}(n)\}$. Clearly $D(n)$ is a discrete-in-itself subspace of X. Let $U(n) = \cup\mathcal{B}(n)$. To show that $D(n)$ is relatively closed in $U(n)$, suppose $q \in U(n) - D(n)$ is a limit point of $D(n)$. Choose $C \in \mathcal{B}(n)$ that contains the point q. Then C must contain at least two distinct points of $D(n)$, say r and s. Because r and s belong to $D(n)$, each belongs to one and only one member of $\mathcal{B}(n)$, so that unique member must be the set C. Hence $r = p(n, C)$ and $s = p(n, C)$, contrary to $r \neq s$. Therefore $D(n)$ is relatively closed in $U(n)$.

Now suppose that x is a point of an open set G in X. Because $\mathcal{B}$ is a base for X, there is an n and a set $B \in \mathcal{B}(n)$ having $x \in B \subset G$. Then $x \in U(n)$ and $p(n, B) \in G \cap D(n)$, as required to show that X has Property III. $\square$

(3.5) <u>Lemma</u>: *If X is a generalized ordered space with a quasi-G_δ-diagonal, then X is hereditarily paracompact.*

Proof: If not, by (2.2) there is a stationary set S in a regular uncountable cardinal κ such that S is homeomorphic to a subspace of X. But then S also has a quasi-G_δ-diagonal. Let $\{\mathcal{G}(n) : n \geq 1\}$ be a sequence of open collections in the space S with the property that if x and y are distinct points of S, then there is an n such that $x \in St(x, \mathcal{G}(n)) \subset S - \{y\}$. We may assume that members of each $\mathcal{G}(n)$ are convex subsets of S.

For each $x \in S$, let x' be the first point of S that is larger than x. Define $S(n) = \{x \in S : x \in St(x, \mathcal{G}(n)) \subset S - \{x'\}\}$. Then $S = \cup\{S(n) : n \geq 1\}$, so that there in an n such that $S(n)$ is also a stationary set. Then so is the set T consisting of all points of $S(n)$ that are not isolated in $S(n)$. For each $t \in T$, $St(t, \mathcal{G}(n))$ is an open neighborhood of t. Because t is not isolated in S, there is a point $p(t)$ in the set S with $p(t) < t$ and with

$[p(t), t] \subset St(t, \mathcal{G}(n))$. Apply the Pressing Down Lemma to the regressive function $p(t)$ to conclude that there are points $r < s$ in the set T with $p(r) = p(s)$. Write $q = p(r) = p(s)$. Because $[q, s] \subset St(s, \mathcal{G}(n))$ we choose $G'' \in \mathcal{G}(n)$ with $\{q, s\} \subset G''$. Because G'' is convex in S, we then have $r \in [q, s] \subset G''$, and so $G'' \subset St(r, \mathcal{G}(n))$. But $r < s$ so that r', the successor of r in S, satisfies $r < r' \leq s$ and hence $r' \in [q, s] \subset G'' \subset St(r, \mathcal{G}(n))$. That is impossible because $r \in T \subset S(n)$. Therefore, X is hereditarily paracompact. $\square$

(3.6) Lemma: *If X is a generalized ordered space with a quasi-G_δ-diagonal, then X has Property III.*

Proof: Let $\{\mathcal{G}'(n) : n \geq 1\}$ be a sequence of collections of open convex subsets of X such that if x and y are distinct points of X, then for some n, $x \in St(x, \mathcal{G}'(n)) \subset X - \{y\}$. For each n, let $U(n) = \cup\mathcal{G}'(n)$ and let $\mathcal{G}(n)$ be a collection of convex open sets in X that covers $U(n)$ and has the property that for each $G \in \mathcal{G}(n)$, the closure of G in X is a subset of $U(n)$. Because X is hereditarily paracompact by (3.2), the open cover $\mathcal{G}(n)$ of the subspace $U(n)$ has a σ-discrete refinement by closed subsets of X. Specifically, for each $n \geq 1$ there is a collection $\mathcal{H}(n) = \cup\{\mathcal{H}(n, k) : k \geq 1\}$ where:

a) $\mathcal{H}(n, k)$ is a collection of closed subsets of X (and not just closed subsets of $U(n)$);

b) $\mathcal{H}(n, k)$ is a discrete collection in the subspace $U(n)$ of X;

c) $\cup\{\mathcal{H}(n, k) : k \geq 1\}$ refines $\mathcal{G}(n)$ and covers $U(n)$.

According to (2.2), for each $H \in \mathcal{H}(n, k)$ we may choose a set $E(H)$ that is closed in H (and hence also in X), discrete-in-itself, and both cofinal and coinitial in H.

Define $D(0) = \{x \in X : x \text{ is isolated in X}\} = U(0)$. Clearly $D(0)$ is discrete-in-itself and is relatively closed in $U(0)$. For each $n, k \geq 1$, let $U(n, k) = U(n)$, and $D(n, k) = \cup\{E(H) : H \in \mathcal{H}(n, k)\}$. Because $\mathcal{H}(n, k)$ is a discrete collection in the subspace $U(n, k)$, the set $D(n, k)$ is closed in $U(n, k)$ and is discrete-in-itself.

To complete the proof that X has Property III, suppose V is open in X and $x \in V$. We may assume that V is convex. If x is an isolated point of X, then $x \in U(0)$ and $x \in D(0) \cap V$, so assume that x is not isolated. The there must be a point $y \in V - \{x\}$. Without loss of generality, we may assume $x < y$. Then $[x, y] \subset V$ and there is an n such that $x \in St(x, \mathcal{G}(n)) \subset X - \{y\}$. Because members of the collection $\mathcal{G}(n)$ are convex, it follows that $St(x, \mathcal{G}(n))$ is a subset of $]\leftarrow, y[$. Choose k so that $x \in \cup\mathcal{H}(n, k)$, and then let H be the unique member of $\mathcal{H}(n, k)$ that contains x. Choose $G \in \mathcal{G}(n)$ with $H \subset G$.

Because the set $E(H)$ is cofinal in H, there must be a point $z \in E(H)$ with $x \leq z$. Then $\{x, z\} \subset H \subset G$ so that $z \in St(x, \mathcal{G}(n)) \subset]\leftarrow, y[$. But then $z \in [x, y] \subset V$, showing that $z \in E(H) \cap V \subset D(n, k) \cap V$. Therefore we have shown that $x \in U(n, k)$ and $V \cap D(n, k) \neq \phi$, as required. $\square$

(3.7) Lemma: *Suppose* $X = \cup\{X(k) : k \geq 1\}$ *where each* $X(k)$ *is a discrete-in-itself subspace of* X. *Then* X *has Property III.*

Proof: Let $D(k) = X(k)$ and let $U(k) = \cup\{V : V \text{ is open in } X \text{ and } card(V \cap X(k)) \leq 1\}$. Then $U(k)$ is open in X for each $k \geq 1$, and $D(k)$ is relatively closed in $U(k)$. Suppose that G is open in X and $p \in G$. Choose k so that $p \in X(k)$. Then $p \in U(k)$ and $p \in G \cap D(k) \neq \phi$, as required to show that X has Property III. $\square$

Combining Lemmas 3.4 through 3.7 completes the proof of (3.3) In addition, Lemma 3.7 will be useful in showing that a certain space in Section 5 has Property III.

4. Property III in ordered spaces.

Among general topological spaces, Property III behaves strangely. Proposition (6.6), below, shows that any topological space Y can be embedded as a closed subset of a topological space X, where X has Property III. Clearly, then, Property III is not hereditary in general spaces. However, even in general spaces we have:

(4.1) Lemma: *If* X *is a topological space having Property III and if* Y *is an open subspace of* X, *then* Y *has Property III.*

Proof: Let $U(n)$ and $D(n)$ be the subsets of X guaranteed by Property III. Let $V(n) = Y \cap U(n)$ and let $E(n) = Y \cap D(n)$. It is easy to check that Y has Property III with respect to $V(n)$ and $E(n)$. $\square$

Property III has more interesting ramifications in the class of generalized ordered spaces, as our next two results show.

(4.2) Proposition: *If* X *is a generalized ordered space having Property III, then* X *is hereditarily paracompact.*

Proof: We will first show that X is paracompact. If X is not paracompact, by (2.1) there must be a regular cardinal κ having a stationary set S that embeds in X as a closed subspace by a monotonic homeomorphism. We will assume that the embedding is increasing, and will identify S with its image in X. There are two cases to consider.

Case 1: If S is cofinal in X, then let $D(n)$ and $U(n)$ be the sets given in the definition of Property III for X. Let $J = \{i \geq 1 : D(i) \text{ is bounded above in X}\}$. Because S is cofinal in X, for each $j \in J$, we may choose $\beta(j)$ in S with $D(j) \subset]\leftarrow, \beta(j)[$. Because κ has uncountable cofinality, some β in S has $\beta(j) \leq \beta$ for each $j \in J$.

For any $\alpha \in S$, let α' denote the first element of S that is greater than α. Let $I = \{i : i \geq 1 \text{ and } i \notin J\}$. For each $i \in I$, let $L(i) = \{\sigma \in S : \sigma > \beta \text{ and }]\beta, \sigma'[\cap D(i) \neq \phi \text{ and } \sigma \in U(i)\}$. We claim that $S \cap]\beta, \rightarrow[= \cup\{L(i) : i \in I\}$. To prove that claim, fix

p in $S\cap]\beta,\rightarrow[$. Then the set $V=]\beta,p'[$ is open in X and contains p, so there is a $k\geq 1$ with $p\in U(k)$ and $V\cap D(k)\neq\phi$. If $k\in J$, then $D(k)\subset]\leftarrow,\beta(k)[\subset]\leftarrow,\beta[$ and hence $D(k)\cap V=D(k)\cap]\beta,p'[\subset]\leftarrow,\beta[\cap]\beta,p'[=\phi$, contrary to $D(k)\cap V\neq\phi$. Therefore, k is not a member of J, so that $k\in I$. Note that $p>\beta$ and $]\beta,p'[\cap D(k)=V\cap D(k)\neq\phi$, and $p\in U(k)$. Hence $p\in L(k)$ as required to show that $S\cap]\beta,\rightarrow[\subset\cup\{L(i):i\in I\}$. Next consider any $q\in\cup\{L(i):i\in I\}$. For such a point q, choose $i\in I$ with $q\in L(i)$. Then $q\in S$ and $q>\beta$, and so $q\in S\cap]\beta,\rightarrow[$. Therefore $S\cap]\beta,\rightarrow[=\cup\{L(i):i\in I\}$.

Because $S\cap]\beta,\rightarrow[$ is stationary, there must be some index $k\in I$ for which the set $L(k)$ is also stationary. Let M be the set of all points of $L(k)$ that are not isolated in $L(k)$. Then M is also stationary.

Fix $s\in M$. Because $s\in U(k)$ and $D(k)$ is discrete in $U(k)$, the point s cannot be a limit point of $[0,s[\cap D(k)$. However, s is a limit point of $L(k)$, so there must be a point $p(s)\in L(k)\subset S$ with $p(s)<s$ and with $D(k)\cap[0,s[\subset[0,p(s)[$. Apply the Pressing Down Lemma to the regressive function $p:M\rightarrow S$ to find a point $z\in S$ and a cofinal set $T\subset M$ such that $p(t)=z$ for every $t\in T$.

Because T is cofinal in S and hence also in X, we know that $D(k)=\cup\{D(k)\cap[0,t[:t\in T\}$. But for each $t\in T$ we have $D(k)\cap[0,t[\subset[0,p(t)[=[0,z[$ and, therefore, $D(k)$ is bounded by z, contrary to $k\in I$. This completes case 1.

Case 2: If S is not cofinal in X, then let u be the supremum of S in the Dedekind completion of X. Then u is either a gap of X or a pseudo- gap, i.e., a point u of X for which the set $[u,\rightarrow[$ is open in X even though u has no immediate predecessor in X. Let Y be the interval $]\leftarrow,u[$ of the space X. Then Y is open in X, so that Y inherits Property III from X, according to (4.1), and Y is a generalized ordered space, and S is cofinal in Y. Now apply Case 1 to the subspace Y to obtain the same contradiction.

Cases 1 and 2 force us to conclude that any generalized ordered space X with Property III must be paracompact. To show hereditary paracompactness of X, it is enough to show that every open subspace Z of X will be paracompact. As in case 2 above, any such subspace Z inherits Property III and is itself a generalized ordered space, so that cases 1 and 2 show that Z is paracompact, as required. $\square$

(4.3) Theorem: *If X is a generalized ordered space having Property III, then every subspace of X also has Property III.*

Proof: Let $U(n)$ and $D(n)$ be the sets guaranteed by the definition of Property III for the space X. Let Y be any subspace of X. Let $U'(n)=Y\cap U(n)$ and $D'(n)=Y\cap D(n)$ for each n.

Let $V(n) = U(n) - D(n)$ and let $\{R(n, \alpha) : \alpha \in A(n)\}$ be the family of all convex components of $V(n)$ in X. Let $V'(n) = Y \cap V(n)$. Then $V'(n) = \cup\{R(n, \alpha) \cap Y : \alpha \in A(n)\}$ and the sets $R(n, \alpha) \cap Y$ are pairwise disjoint, relatively open and relatively closed subsets of the set $V'(n)$. Also note that $V'(n)$ is relatively open in Y.

Because X is hereditarily paracompact, it follows from (2.2) that for each $\alpha \in A(n)$ there is a set $E'(n, \alpha) \subset R(n, \alpha) \cap Y$ that is discrete-in-itself, relatively closed in $R(n, \alpha) \cap Y$, and both cofinal and coinitial in $R(n, \alpha) \cap Y$. Then $E'(n, \alpha)$ is also relatively closed in $V'(n)$, and the set $E'(n) = \cup\{E'(n, \alpha) : \alpha \in A(n)\}$ is also relatively closed in $V'(n)$. In addition, $E'(n)$ is discrete-in -itself.

To complete the proof we will show that the sequences $D'(n)$ and $U'(n)$, together with $E'(n)$ and $V'(n)$, satisfy the definition of Property III for the subspace Y. Clearly $U'(n)$ and $V'(n)$ are open in Y, while $D'(n)$ and $E'(n)$ are discrete in themselves, and relatively closed in $U'(n)$ and $V'(n)$ respectively. Now suppose that p is a point of a relatively open set $G \cap Y$ of Y, where G is a convex open subset of X. Choose n so that $p \in U(n)$ and $G \cap D(n) \neq \phi$. Choose $d \in D(n) \cap G$. There are several cases to consider.

Case 1: If $d \in Y$, then $d \in D'(n)$ so we have $p \in U'(n)$ and $D'(n) \cap (G \cap Y) \neq \phi$, as required.

Case 2: If $p \in D(n)$, then $p \in D'(n)$ and hence $p \in D'(n) \cap (G \cap Y) \neq \phi$, as required.

Case 3: If $d \notin Y$ *and* $p \notin D(n)$, then observe that $p \in Y$ forces $p \neq d$. Without loss of generality, assume $p < d$. In this case we have $p \in V(n)$ so there is a unique $\alpha \in A(n)$ with $p \in R(n, \alpha)$. Because $p \in Y$ and because $E'(n, \alpha)$ is a cofinal subset of $Y \cap R(n, \alpha)$, there is some point $e \in E'(n, \alpha)$ with $p \leq e$. Because $d \in D(n)$, we know that $d \notin V(n)$. Because $R(n, \alpha)$ is a convex component of $V(n)$, either $R(n, \alpha) \subset] \leftarrow, d[$ or else $R(n, \alpha) \subset]d, \rightarrow [$. Because $p \in R(n, \alpha) \cap] \leftarrow, d[$, it must be the case that $R(n, \alpha)$ is a subset of $] \leftarrow, d[$. But then from $e \in E'(n, \alpha) \subset R(n, \alpha)$ we conclude $e < d$ so that we have $p \leq e < d$. Now both p and d belong to the convex subset G of X, and so $e \in G$. But then from $e \in E'(n, \alpha) \subset R(n, \alpha) \cap Y \subset Y$, it follows that $e \in E'(n) \cap (G \cap Y)$. Since also $p \in V(n) \cap Y = V'(n)$ we have proved that, in case 3, $p \in V'(n)$ *and* $E'(n) \cap (G \cap Y) \neq \phi$, as required. $\square$

5. Examples concerning Property III for ordered spaces.

(5.1) Example: *The Lexicographic Square* $L = [0,1] \times [0,1]$ *is a compact linearly ordered space having Property III. Hence Property III does not imply quasi-developability or the existence of a point- countable base, even in the presence of compactness.*

Proof: Let Q be the set of rational numbers in the interval $[0,1]$ of the real line. For each $q \in Q$ define six sets as follows:

$U(q) = L$ and $D(q) = \{(q, 1/2)\}$;

$V(q) = [0,1] \times]0,1[$ and $E(q) = \{(x,q) : 0 \leq x \leq < 1\}$;

$W(q) = [(0,0),(0,1)[\times](1,0),(1,1)]$ and $F(q) = \{(0,q),(1,q)\}$.

The sets $D(q)$, $E(q)$, *and* $F(q)$ are discrete in themselves and are relatively closed, respectively, in $U(q)$, $V(q)$, and $W(q)$ which are open in L. (Note that $W(q)$ is the union of two open convex sets in L). One uses the sets $U(q)$ and $D(q)$ to verify Property III for points of the form $(x,0)$ and $(y,1)$ for $0 < x \leq 1$ and for $0 \leq y < 1$. One uses the sets $V(q)$ and $E(q)$ to verify the property for points of the form (x,t) with both x and t lying in $]0,1[$. One uses the sets $W(q)$ and $F(q)$ to verify the property for the points $(0,0)$ and $(1,1)$. □

(5.2) <u>Example</u>: *Familiar ordered spaces and Property III.*

(5.2-a) It is easy to give a direct proof that the usual Sorgenfrey line S has Property III. The space S is separable, hereditarily Lindelöf, and perfect, but does not have a point-countable base. It is also instructive to think of S as the subspace $\{(x,1) : 0 < x < 1\}$ of the Lexicographic Square L in (5.1), in order to understand the relationship between the sets $U(n)$ and $D(n)$ of the space X and the subspace Y of X in the proof of (4.3).□

(5.2-b) The Michael line has a σ-disjoint base, has Property III, and is paracompact but is not perfect and is not Lindelöf. □

(5.2-c) The usual space of all countable ordinals is not paracompact and therefore cannot have Property III. However, this space does have a dense subset that is the union of countably many subspaces, each of which is discrete-in-itself. Thus Property III is much stronger than the property "X has a σ-discrete-in-itself dense subset." □

(5.3) <u>Example</u>: *There is a linearly ordered topological space having a point-countable base but not having Property III.*

Proof: It is known that it is consistent with and independent of the usual ZFC axioms that a Souslin space exists, i.e., a non- separable linearly ordered topological space that has countable cellularity. It is also known that if there is a Souslin space, then there is a Souslin space S with a point-countable base $[Be1, P]$. Being perfect but not metrizable, S cannot be quasi-developable. Therefore, in the light of Theorem 1.5, S cannot have Property III. A second example, not requiring any set theory beyond ZFC, is described in Example 2.11 of $[Be2]$. This space has a point- countable base but is not quasi-developable so that, by Theorem 1.5, it cannot have Property III. □

(5.4) <u>Example</u>: *There is a generalized ordered space having Property III but which is not first countable at any point.*

Proof: Let ω_1 be the first uncountable ordinal. Let X be the set of all sequences $\langle x(i)\rangle$ in $[0,\omega_1]$ with the property that for some n, $x(i) < \omega_1$ for $i \leq n$ and $x(i) = \omega_1$ for each $i > n$. Let $\prec$ denote the lexicographic ordering of X, so that $x \prec y$ provided that for some integer d, $x(i) = y(i)$ for $1 \leq i < d$, and $x(d) < y(d)$ in the ordinal space $[0,\omega_1]$. Topologize X by the usual order topology of $\prec$. Using a sequence of lemmas, we will show that X has Property III and is not first countable at any point.

For any $x \in X$, define $length(x)$ to be the least integer n such that $\omega_1 = x(n+1) = x(n+2) = ...$ It follows that the constant sequence $(\omega_1, \omega_1, \omega_1, ...)$ is the unique point of X with length 0. We will use the notation $x = (x(1), ...x(m), \overline{\omega_1})$ to denote the point of X with $x(i) = \omega_1$ for each $i > m$. In using this notation, we do not necessarily mean to suggest that $length(x)$ is m, because it might happen that one of the point's earlier coordinates is also equal to ω_1. Note that for any point $y \in X$, if $y(k) = \omega_1$, then every subsequent coordinate of y is also ω_1 so that $length(y)$ is less than k.

(5.4.a) <u>Lemma</u>: *Each set $D(n) = \{x \in X : \ length(x) \ is \ n\}$ is discrete-in-itself, and X has Property III.*

Proof: The set $D(0)$ is a singleton, so we need only consider $D(n)$ for $n > 0$. Fix $x = (x(1), ..., x(n), \overline{\omega_1})$ in $D(n)$. Define a point p of X by $p(i) = x(i)$ for $1 \leq i \leq n$, and $p(n+1) = 0$, and $p(k) = \omega_1$ for $k > n+1$. Define $q \in X$ by $q(i) = x(i)$ for $1 \leq i \leq (n-1)$, and $q(n) = x(n)+1$, and $q(k) = \omega_1$ for $k > n$. Then $p \prec x \prec q$. We claim that $]p, q[\cap D(n) = \{x\}$. For suppose $y \in]p, q[\cap D(n)$. Let d be the first coordinate in which y differs from p. Then for $1 \leq i < d$, we have $p(i) = y(i)$ and $p(d) < y(d)$. There are three cases to consider.

Case 1: if $d < n$. Then for $1 \leq i < d$, we have $y(i) = p(i) = x(i) = q(i)$, and $y(d) > p(d) = x(d) = q(d)$. But that is enough to force the point y to be larger than q, contrary to $y \prec q$.

Case 2: if $d = n$. Then for $1 \leq i < n$ we have $y(i) = p(i) = x(i) = q(i)$ and $y(n) > p(n) = x(n)$. Because $y \prec q$, there are only two possibilities. One is that $y(n) < q(n)$. But in that case we have $y(n) < q(n) = x(n)+1$, so we conclude that $x(n) = p(n) < y(n) < q(n) = x(n)+1$, i.e., that some ordinal lies strictly between $x(n)$ and $x(n)+1$, which is impossible. The other possibility is that $y(n) = q(n) = x(n)+1$ and that there is an index $m > n$ for which $y(m) < q(m)$. But $m > n = length(y)$, and so we have $\omega_1 = y(m) < q(m) < \omega_1$, another contradiction.

Case 3: if $d > (n+1)$. But then $x(i) = p(i) = y(i)$ for $1 \leq i < d$. Because $d \geq (n+1) > n$ we have $x(i) = p(i) = y(i)$ for $1 \leq i \leq n$. But then $y = x$ because both y and x belong to $D(n)$. $\square$

The fact that X has Property III now follows from Proposition (3.7), above.

(5.4.b) Lemma: *Let $x \in X$. Then X is not first countable at x.*

Proof: First observe that if u and v are points of X with $u \prec v$, then the integer d, defined as the first coordinate where u and v differ, satisfies $d \leq (n+1)$ where n is $length(v)$. For suppose $d > (n+1)$. Write $v = (v(1), v(2), ..., v(n), \overline{\omega_1})$ and $u = (u(1), u(2), ..., u(m), \overline{\omega_1})$. Then $v(i) = u(i)$ for $1 \leq i < d$, and in particular $u(n+1) = v(n+1) = \omega_1$. But then, because $d > (n+1)$, we also have $u(d) = \omega_1$. Because $u \prec v$ and d is the first coordinate where the two points differ, we must have $\omega_1 = u(d) < v(d) < \omega_1$ and that is impossible. Therefore $d \leq (n+1)$ as claimed.

Now fix $x \in X$. The set $[x, \rightarrow [$ is not open in X (because it is easy to see that X has no first point and that x has no immediate predecessor in X). Therefore, it will be enough to show that there is no strictly increasing sequence $y(1) \prec y(2) \prec ...$ that converges to x. Let n be $length(x)$ and let $x = (x(1), ..., x(n), \overline{\omega_1})$. For each k, let $d(k)$ be the first coordinate in which the points $y(k)$ and x differ. According to the observation above, $d(k) \leq n+1$ for each k. There are two cases to consider.

Case 1: Suppose there are infinitely many k with $d(k) \leq n$. Define a new point z of X by $z = (x(1), x(2), ...x(n), 0, \overline{\omega_1})$. Then $z \prec x$. Since there are infinitely many values of k with $d(k) \leq n$, we may choose one point $y(k)$ with $z \prec y(k)$ and $d(k) \leq n$. Writing $y(k,i)$ for the $i-th$ coordinate of the point $y(k)$, we then have $y(k,i) = x(i) = z(i)$ for each $i < d(k)$, and $y(k, d(k)) < x(d(k)) = z(d(k))$. Hence $d(k)$ is also the number of the first coordinate in which $y(k)$ and z differ, and in the $d(k)$ coordinate y and z differ in a way that is incompatible with $z \prec y(k)$. Thus Case 1 is impossible.

Case 2: Suppose there are only finitely many k with $d(k) \leq n$. In this case, infinitely many values of k have $d(k) = (n+1)$. Passing to a subsequence if necessary, we may assume that there are no k with $d(k) \leq n$. But from above, every $d(k)$ has $d(k) \leq (n+1)$ and therefore $d(k) = (n+1)$ for each k. Then $y(k,i) = x(i)$ for all $i < (n+1)$ and $y(k, n+1) < x(n+1) = \omega_1$. Choose an ordinal α with $y(k, n+1) < \alpha < \omega_1$ for each k. Let $z = (x(1), x(2), ..., x(n), \alpha, \overline{\omega_1})$. Then $z \prec x$ and for each k, we have $y(k) \prec z$. Therefore, even in Case 2, the sequence $y(k)$ cannot converge to x. $\square$

(5.4.c) Remark: A much easier example of a generalized ordered space with Property III that is not first countable at one point can be obtained by starting with the usual ordinal space $[0, \omega_1]$ and isolating each countable ordinal. In a way, the more complicated example given in (5.4) can be thought of as the result of attaching a copy of this space to each isolated point of itself, and repeating the process inductively. $\square$

6. Property III in more general spaces.

The referee of our paper posed a series of questions about the role of Property III in more general spaces. In this section we outline a few positive results and describe a general construction that provides negative answers to many other questions about Property III.

(6.1) Proposition: *A topological space X has Property III if either of the following holds:*

a) *X has a dense set $\cup\{E(n)n < \omega\}$ such that each $E(n)$ is a closed discrete subspace of X;*

b) *$X = \cup\{X(n) : n < \omega\}$ where each X(n) is a discrete, but not necessarily closed, subspace of X.*

Proof: To show that a space satisfying (a) must have Property III, let $U(n) = X$ and $D(n) = E(n)$. As for (b), it was proved in (3.7). $\square$

It follows from (6.1) that every separable space has Property III, e.g., the compact space 2^c. It also follows that many generalized metric spaces have Property III in the light of our next result.

(6.2) Corollary: *If X is stratifiable, developable, a σ-space, or semistratifiable, then X has Property III.*

Proof: It will be enough to prove the corollary for semistratifiable spaces, because all of the other spaces mentioned in the corollary belong to that class. Recall that a semistratifiable space admits a family $g(n, x)$ of open sets such that $x \in g(n, x)$ for each $x \in X$ and each $n < \omega$ such that if $y \in g(n, x_n)$ for each n, then the sequence $\{x_n\}$ clusters at the point y. Also recall from $[BW]$ that any semistratifiable space has the covering Property D, i.e., if $G(x)$ is an open neighborhood of x for each $x \in X$, then there is a closed discrete subset D of X such that $\cup\{G(x) : x \in D\} = X$. For each n, let $D(n)$ be a closed discrete subspace of X such that $\{g(n, x) : x \in D(n)\}$ covers X. It follows that $\cup\{D(n) : n < \omega\}$ is dense in X. Now apply (6.1-a). $\square$

(6.3) Question: In the light of (3.6), it is natural to ask whether every regular topological space with a G_δ-diagonal has Property III. We conjecture that the answer is negative.

(6.4) Remark: It is easy to see that if X has Property III and is hereditarily Lindelöf, then X is separable. In that behavior, spaces with Property III resemble many generalized metric spaces. However, unlike the situation in generalized metric spaces, one cannot say that a Lindelöf space with Property III must be separable. E. Michael $[M]$ has used CH to construct a Lindelöf, non-separable GO-space with a G_δ-diagonal. According to (3.6), that space has Property III. Burke and Davis constructed the same kind of space under weaker hypotheses in $[BD]$.

(6.5) Construction: For any topological space $(X, \mathcal{T})$, define

$$B(X) = (X\times\{0\}) \cup (X\times\{1/k : k \geq 1\}).$$

For any open subset $U \subset X$ define

$$S(n, U) = (U\times\{0\}) \cup (U\times\{1/k : k \geq n\}).$$

Topologize B(X) in such a way that each point $(x, 1/k)$ is isolated, and so that basic neighborhoods of each point $(x, 0)$ have the form $S(n, U)$ where $n < \omega$ and U is an open subset of X that contains x.

(6.6). Proposition: *For any topological space $(X, \mathcal{T})$, the space $B(X)$ has Property III and its closed subspace $X\times\{0\}$ is homeomorphic to $(X, \mathcal{T})$.* $\square$

Because of its simple structure, $B(X)$ mimics many of the topological properties of X.

(6.7) Proposition: *Let $P \in$ {regularity, normality, paracompactness, hereditary paracompactness, monotone normality, a point- countable base, a σ-point-finite base, a σ-disjoint base}. Then $B(X)$ has P if and only if X has P.*

Proof: The proofs all follow the same patterns, so we will give only two of them. We begin by noting that each of the listed properties is closed-hereditary, so that if $B(X)$ has the property, then so does X. Thus, it is enough to prove that if X has P, then so does $B(X)$.

Case 1: Suppose P is normality. Suppose that A is closed and U is open in $B(X)$ with $A \subset U$. We will find an open set W in $B(X)$ having $A \subset W \subset cl(W) \subset U$. Identify X with the subspace $X\times\{0\}$ of $B(X)$. Then $A \cap X$ and $U \cap X$ are closed and open, respectively, so that there is an open subset V of X with $A \cap X \subset V \subset cl(V) \subset U \cap X$. Define $W = (S(1, V) \cap U) \cup (A - X)$. Then W is open in $B(X)$ and $A \subset W \subset U$. To show that $cl(W) \subset U$, suppose that p is a limit point of W in $B(X)$. Then either $p \in cl(A-X)$ or else $p \in cl(S(1, V) \cap W)$. In the first case, $p \in cl(A) = A \subset U$. In the second case, if p has the form $(x, 1/k)$, then p is isolated in $B(X)$ so that $p \in (S(1, V) \cap U) \subset U$. If, in the second case, $p = (x, 0)$, then consider any open neighborhood G of x in X. Then the open subset $S(1, G)$ of $B(X)$ must meet $S(1, V) \cap U$. But any point of $S(1, G) \cap (S(1, V) \cap U$ has its first coordinate in $G \cap V$ so that x is a limit point of V and hence $p = (x, 0) \in cl(V) \subset (U \cap X)$. Therefore $p \in U$ as claimed, so that $B(X)$ is normal.

Case 2: Suppose P is hereditary paracompactness. Let $\mathcal{U}$ be any collection of open sets in $B(X)$ and let $Y = \cup\mathcal{U}$. Refining $\mathcal{U}$ by basic open sets if necessary, we may assume that $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_1 = \{S(n_\alpha, V_\alpha) : \alpha \in A\}$ and where $\mathcal{U}_2$ is a collection of isolated points of $B(X)$. The family $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ is an open cover of some subspace Z of X. Because X is hereditarily paracompact, there is an open refinement $\{W_\alpha : \alpha \in A\}$ of $\mathcal{V}$ that is locally finite in Z and has $W_\alpha \subset V_\alpha$ for each $\alpha \in A$. (Of course, many of the sets V_α can be empty.) Define $\mathcal{W}_1 = \{S(n_\alpha, V_\alpha) : \alpha \in A\}$ and $\mathcal{W}_2 = \{\{(x, 1/k)\} : (x, 1/k) \in Y - \cup\mathcal{W}_1\}$. Then $\mathcal{W}_1 \cup \mathcal{W}_2$ is an open collection that refines $\mathcal{U}$, covers Y, and is locally finite in Y. It now follows that $B(X)$ is hereditarily paracompact. $\square$

(6.8) <u>Corollary</u>: *The following examples exist:*

a) *a hereditarily paracompact, first countable space with Property III that has a closed subspace without Property III;*

b) *a normal space with Property III that is not countably paracompact;*

c) *a monotonically normal space with Property III that is not paracompact.*

Proof: Example 2.11 of [*Be*2] is a hereditarily paracompact LOTS X with a point-countable base that does not have a σ-disjoint base and therefore does not have Property III in the light of (3.2) above. Then $B(X)$ is the space described in (a).

Next let Y be any Dowker space [*R*], i.e. a normal space that is not countably paracompact. Then $B(Y)$ is the space described in (b).

Finally let Z be the usual space of countable ordinals. Then Z is monotonically normal but not paracompact, and $B(Z)$ is the space described in (c). We note that in the light of this example, Property III is not closed- hereditary in monotonically normal spaces. $\square$

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