Off-Diagonal Metrization Theorems

by

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Abstract:

Experience shows that there is a strong parallel between metrization theory for compact spaces and for linearly ordered spaces in terms of diagonal conditions. Recent theorems of Gruenhage, Pelant, Kombarov, and Stepanova have described metrizability of compact (and related) spaces in terms of the off- diagonal behavior of those spaces, i.e., in terms of properties of $X^2 - \Delta$. In this paper, we show that these off-diagonal results have no analogs for linearly ordered topological spaces by constructing a non- metrizable, first countable LOTS X that is paracompact off of the diagonal, has a locally finite rectangular open cover of $X^2 - \Delta$, and admits a collection $\mathcal{U}$ of subsets of $X^2 - \Delta$ that is σ-locally finite in $X^2 - \Delta$, covers $X^2 - \Delta$, and consists of co-zero subsets of X^2. Provided $b = \omega_1$, our example contains a Lindelöf subspace Y that has a countable rectangular open cover of $Y^2 - \Delta$ and yet does not have a G_δ-diagonal, thereby answering a question of Kombarov. In addition, we consider the role of much stronger off-diagonal covering conditions such as the Lindelöf property and hereditary paracompactness.

MR numbers (1991): 54F05, 54E35, 54G20

Key words and phrases: Off diagonal metrization theorems, paracompact off of the diagonal, LOTS, GO space, G_δ-diagonal, rectangular open cover, metrization.

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1. Introduction.

Experience has shown that there is a parallelism between certain metrization theorems for compact Hausdorff spaces and for linearly ordered topological spaces. The best known example is: if X is a compact Hausdorff space or a linearly ordered space then X is metrizable if it has a G_δ-diagonal ([S], [L1]). Another example is: if X is paracompact and can be p-embedded in a compact space or in a LOTS, then X is metrizable if it has a G_δ-diagonal ([Bo],[O],[L1]). A more recent example is: if X has a small diagonal, then X is metrizable if X is a Lindelöf linearly ordered space [BL], or if the Continuum Hypothesis holds and X is a compact Hausdorff space [JS]. Theorems of this type might well be called "diagonal metrization theorems" and an attempted explanation of the parallelism appears in [L2].

Recently, there have been metrization results that involve the bevavior of a compact space X off of the diagonal, i.e., that involve properties of the subspace $X^2 - \Delta = \{(x,y) \in X^2 : x \neq y\}$ of X^2. For example, Gruenhage ([G]) proved that a compact Hausdorff space is metrizable if it is <u>paracompact off of the diagonal</u>, i.e., if the subspace $X^2 - \Delta$ of X^2 is paracompact. This result was generalized to the class of paracompact Σ-spaces in [GP]. A related result, due to Kombarov [K], shows that a paracompact Σ-space X has a G_δ-diagonal if and only if there is an open cover of $X^2 - \Delta$ that is locally finite in $X^2 - \Delta$ and consists of rectangular open sets, i.e., sets of the form $G \times H$ where G and H are open in X and disjoint. In what follows, we will refer to such an open cover of $X^2 - \Delta$ as a <u>Kombarov cover</u>. Finally, E. Stepanova [St] proved that a paracompact p-space X is metrizable if and only if some familty of subsets of $X^2 - \Delta$ is a σ-locally finite (in $X^2 - \Delta$) cover of $X^2 - \Delta$ by functionally open (i.e., co- zero) subsets of X^2.

Based upon the parallelism between compact and linearly ordered spaces for diagonal metrization theorems, it is reasonable to ask whether the off-diagonal results of Gruenhage, Pelant, Kombarov, and Stepanova have analogs for linearly ordered spaces with the usual open interval topology. The bottom line is that they do not, as we show in Sections 2 and 3, below. There are two primary examples in our paper. Each is a linearly ordered topological space, is paracompact off of the diagonal, has a Kombarov cover and a functionally open cover of the type studied by Stepanova, and is non-metrizable. The first is extremely simple – it is a reordered version of the usual space of ordinals less than or equal to ω_1, with all countable ordinals made discrete. Unfortunately that space is not first countable, and to obtain a first countable example one must work harder. Our second example is a linearly ordered space M^* constructed from the familiar Michael line (see 2.2, below). In addition to the properties mentioned above, under CH or $b = \omega_1$, this space contains a subspace L^* that gives a consistent answer to a question posed by Kombarov in [K]: L^* is regular, Lindelöf, and admits a countable recangular open cover of $(L^*)^2 - \Delta$, and yet does not have a G_δ-diagonal. It is interesting to note that, even though they do not guarantee metrizability in a LOTS, the special covers studied by Kombarov and Stepanova are of interest in ordered space theory because they do yield paracompactness. (Indeed, they yield paracompactness in the larger class of monotonically normal spaces – see 3.4.) Finally, Section 4 begins the study of stronger off-diagonal conditions for an ordered space X, e.g., that X is Lindelöf off of the diagonal or hereditarily paracompact off of the diagonal.

By a linearly ordered topological space (LOTS) we mean a linearly ordered set with the usual open interval topology of the given ordering. Subspaces of a LOTS might fail to be linearly ordered spaces because their subspace and order topologies might not coincide. Such spaces are called generalized ordered spaces and can be characterized internally as Hausdorff spaces with a linear order that have a base for their topology consisting of convex sets. There is a significant difference between metrization theory for LOTS and for GO-spaces. For example, the G_δ-diagonal metrization theorems for compact Hausdorff spaces and for linearly ordered spaces have no analogs for generalized ordered spaces.

A note on notation: Because the underlying linearly ordered sets in this paper are often lexicographic products whose points are ordered pairs, familiar interval notation will be a special problem. For example, there seems to be no right way to denote the interval stretching from one ordered pair in a lexicographic product to another. In this paper, we will adopt a suggestion of K.P. Hart and use the symbol $< a, b >$ to denote an ordered pair, reserving symbols such as (a, b) and $[a, b)$ to denote intervals in linearly ordered sets. Thus, $[< a, b >, < c, d >)$ might denote a half open interval in the lexicographic square.

2. A non-metrizable LOTS that is paracompact off of the diagonal

We begin with a very easy example showing that the results of Gruenhage and Pelant have no analog for linearly ordered spaces in general.

2.1 **Example**: *There is a Lindelöf LOTS that is non-metrizable and hereditarily paracompact off of the diagonal.*

Proof: Let Z denote the usual set of all integers. Consider the lexicographically ordered set $X = ([0, \omega_1) \times Z) \cup \{< \omega_1, 0 >\}$ with its open interval topology. This space is a Lindelöf, non-metrizable LOTS and one easily checks that it is paracompact (and even hereditarily paracompact) off of the diagonal. $\square$

Unfortunately the space in (2.1) is not first countable. To get a first countable example, we let P, Q, and R denote, respectively, the usual sets of irrational, rational, and real numbers, and we consider an extension of the familiar Michael line M.

2.2 **Example** *There is a first countable, non-metrizable LOTS that is paracompact off of the diagonal.*

Proof: Consider the lexicographically ordered set $M^* = (R \times \{0\}) \cup (P \times Z)$. This linearly ordered space contains the usual Michael line M as the closed subspace $R \times \{0\}$, and M^* is first countable, hereditarily paracompact, and even quasi-developable. However, it is not perfect and not metrizable.

To see that the space M^* is paracompact off of the diagonal, we invoke the following easily proved result. In the next proposition, we will use X^d to denote the derived set of a space X, i.e., the set of all non- isolated points of X.

2.3 **Proposition**: *Suppose X is a hereditarily paracompact space in which points are G_δ-sets. If X^d is the union of countably many closed, discrete subsets of X, then X is paracompact off of the diagonal.*

Proof: Let $\mathcal{U}$ be any open cover of $X^2 - \Delta$. For each $p \in X^d$, $\mathcal{U}$ is an open cover of $Y_p = (\{p\} \times X) - \{< p, p >\}$. Because $X - \{p\}$ is paracompact and is an F_σ in X, there is a locally finite (in X) collection $\mathcal{V}_p(n)$ of open subsets of X such that if $\mathcal{V}_p = \bigcup\{\mathcal{V}_p(n) : n \geq 1\}$, then $\{\{p\} \times V : V \in \mathcal{V}_p\}$ refines $\mathcal{U}$ and covers Y_p. For each $V \in \mathcal{V}_p$ choose $U(p, V) \in \mathcal{U}$ with $\{p\} \times V \subset U(p, V)$.

Write $X^d = \bigcup\{D(n) : n \geq 1\}$ where each $D(n)$ is a closed, discrete subset of X. Find open sets $G(p, n)$ of X such that $p \in G(p, n)$ and such that $\{G(p, n) : p \in D(n)\}$ is a discrete collection in X. Let $\mathcal{W}(n) = \{(G(p, n) \times V) \cap U(d, V) : p \in D(n),\ V \in \mathcal{V}_p(n)\}$. Then $\bigcup\{\mathcal{W}(n) : n \geq 1\}$ is the required σ-locally finite open cover of $X^2 - \Delta$ that refines $\mathcal{U}$. $\square$

2.4 **Corollary**: *If X is a GO-space that is first countable and paracompact, and if X^d is the union of countably many closed, discrete subsets of X, then X is paracompact off of the diagonal. In particular, the space M^* in (2.2) is paracompact off of the diagonal.*

Proof: If a GO-space is first countable and paracompact, then it is hereditarily paracompact, so that (2.3) yields the desired conclusion. $\square$

2.5 **Remark**: Given suitable set theory, we can sharpen Example (2.2). If CH holds, or if $b = \omega_1$, then there is a set $P^{'} \subset P$ that is concentrated on the rationals ([vD, Theorem 10.2]). Then topologize $L = Q \cup P^{'}$ as a subspace of M and construct L^* as in (2.2). The resulting space L^* is a non-metrizable, first countable, Lindelöf LOTS that is paracompact off of the diagonal. (The GO-space L is due to E. Michael [M]. See (3.6) below for further details.)

The property of paracompactness off of the diagonal combines in interesting ways with other properties of linearly ordered spaces. A result due to Gruenhage and Pelant [GP] shows:

2.6 **Proposition**: *Suppose X is a Lindelöf LOTS that is paracompact off of the diagonal. Then X has a point-countable base.*

Proof: Gruenhage and Pelant [GP] have shown that if X is any space that is Lindelöf and paracompact off of the diagonal, then X admits a point- countable open cover $\mathcal{U}$ with the property that if $a \neq b$ are points of X, then some $U \in \mathcal{U}$ has $a \in U \subset X - \{b\}$. If X is a LOTS, then we can assume that members of $\mathcal{U}$ are order-convex. Let $\mathcal{B} = \{U_1 \cap U_2 : U_i \in \mathcal{U}\}$. Then $\mathcal{B}$ is the required point-countable base for the LOTS X. $\square$

The space in (2.5), constructed under CH or $b = \omega_1$, is Lindelöf and paracompact off of the diagonal, and it has a base that is point countable in a very strong way – the base is actually σ- disjoint. That makes one wonder whether, with suitable set theoretic hypotheses, one could strengthen the conclusion of (2.6) to "X has a σ-point finite base." One approach would be to determine whether such a space has Property III (see [BL2]), a necessary and sufficient condition for an ordered space with a point countable base to have a σ-point-finite base. A related question would be to determine whether there can be a Souslin space (i.e., a non-separable LOTS with the countable chain condition) that is paracompact off of the diagonal. Such a space would be a Souslin space with a point-countable base, but could not have a σ-point-finite base. It is already known that the existence of a Souslin space with a point-countable base is consistent with ZFC ([B],[P]).

3. A non-metrizable LOTS with special covers of $X^2 - \Delta$

Neither the existence of a Kombarov cover for a LOTS X (i.e., a locally finite cover of $X^2 - \Delta$ by open sets of the form $U \times V$ where U and V are disjoint open sets in X) nor the existence of a co- zero cover of $X^2 - \Delta$ of the type studied by Stepanova (see Section 1) forces X to be metrizable or (equivalently) to have a G_δ-diagonal. The easiest example is:

3.1 **Example**: *The non-metrizable space X of (2.1) is a LOTS that admits a Kombarov cover and a σ-locally finite open cover of $X^2 - \Delta$ by subsets of $X^2 - \Delta$ that are co-zero sets in X^2.*

Proof: For each $\alpha < \omega_1$, let $H(\alpha) = (\alpha, \omega_1] \times \{\alpha\}$ and $V(\alpha) = \{\alpha\} \times (\alpha, \omega_1]$. Let $\mathcal{W} = \{H(\alpha), V(\alpha) : \alpha < \omega_1\}$. Then $\mathcal{W}$ is a locally finite open cover of $X^2 - \Delta$ by sets that are both rectangular and are co-zero subsets of X^2. $\square$

As noted before, the space in (3.1) is not first countable. M^* gives a first-countable example.

3.2 **Example**: *The non-metrizable LOTS M^* of (2.2) admits a Kombarov cover but does not have a G_δ-diagonal. In addition, there is a σ-locally finite cover of $(M^*)^2 - \Delta$ by open subsets of $(M^*)^2 - \Delta$ that are co-zero subsets of $(M^*)^2$.*

Proof: Throughout this proof we will write $X = M^*$. Because X is a non-metrizable LOTS, it cannot have a G_δ-diagonal. It remains to show that X has a locally finite, rectangular open cover of $X^2-\Delta$ and a σ-locally finite (in $X^2-\Delta$) cover by open co-zero subsets of X^2.

Husek and Pelant [HP] have proved that every metrizable space Y has a Kombarov cover of $Y^2-\Delta$. In particular, if $Y=R$ is the usual space of real numbers, there is a family $\mathcal{C}=\{C(\alpha):\alpha\in A\}$ that is locally finite in $Y^2-\Delta$, has $\bigcup\mathcal{C}=Y^2-\Delta$, and consists of sets of the form $C(\alpha)=G(\alpha)\times H(\alpha)$ where $G(\alpha)$ and $H(\alpha)$ are disjoint open sets in the usual topology of the real numbers. Note that the index set A must be countable.

Write $G(\alpha)$ as the union of its convex components, say $G(\alpha)=\bigcup\{G(\alpha,i):i\geq 1\}$ where the sets $G(\alpha,i)=(a(\alpha,i),b(\alpha,i))$ are pairwise disjoint open intervals. Similarly, $H(\alpha)=\bigcup\{H(\alpha,j):j\geq 1\}$ where the sets $H(\alpha,j)=(c(\alpha,j),d(\alpha,j))$ are pairwise disjoint open intervals. Notice that $G(\alpha,i)\cap H(\alpha,j)=\emptyset$ for every choice of i and j.

For each real number x, define $u(x)=0=v(x)$ if x is rational, and for any irrational x let $u(x)=+\infty$ and $v(x)=-\infty$. For irrational x, the ordered pairs $<x,u(x)>$ and $<x,v(x)>$ are gaps of the lexicographically ordered space $X=M^*$ of (2.2), while for a rational number x, the pair $<x,u(x)>=<x,0>=<x,v(x)>$ is a point of X. We will use these gaps or points as ends of certain convex subsets of $X=M^*$. For example, $(<2,u(2)>,<\pi,v(\pi)>)=\{<x,k>\in X:2<x<\pi\}$. We begin with the following easily verified fact:

<u>Claim 1</u>: For any real numbers $a<b$, $(<a,u(a)>,<b,v(b)>)$ is a covex open set in X and a point $<x,k>\in X$ belongs to $(<a,u(a)>,<b,v(b)>)$ if and only if $a<x<b$.

Next, for $\alpha\in A$ and $i,\ j\geq 1$, let

$$G^*(\alpha,i)=(<a(\alpha,i),u(a(\alpha,i))>,<b(\alpha,i),v(b(\alpha,i)>);$$

$$G^*(\alpha)=\bigcup\{G^*(\alpha,i):i\geq 1\};$$

$$H^*(\alpha,j)=(<c(\alpha,j),u(c(\alpha,j))>,<d(\alpha,j),v(d(\alpha,j))>);$$

and

$$H^*(\alpha)=\bigcup\{H^*(\alpha,j):j\geq 1\}.$$

Let $\mathcal{C}^*=\{G^*(\alpha)\times H^*(\alpha):\alpha\in A\}$. Then one easily verifies:

<u>Claim 2</u>: $G^*(\alpha)$ and $H^*(\alpha)$ are disjoint open subsets of X so that $\bigcup\mathcal{C}^*\subset X^2-\Delta$.

<u>Claim 3</u>: Let $<<x,m>,<y,n>>\in X^2-\Delta$. Then $<<x,m>,<y,n>>\notin\bigcup\mathcal{C}^*$ if and only if $x=y\in P$. To verify "$\Rightarrow$," suppose that $<<x,m>,<y,n>>\notin\bigcup\mathcal{C}^*$ and yet $x\neq y$. Then for some $\alpha\in A$ and $i,j\geq 1$ we have $<x,y>\in G(\alpha,i)\times H(\alpha,j)$. It follows from Claim 1 that $<<x,m>,<y,n>>\in G^*(\alpha,i)\times H^*(\alpha,j)\subset\bigcup\mathcal{C}^*$, contrary

to $<< x, m >, < y, n >> \notin \bigcup \mathcal{C}^*$. Thus, $x = y$. But then we have $<< x, m >, < x, n >> = << x, m >, < y, n >> \in X^2 - \Delta$ which forces $m \neq n$ so that $x \in P$. Conversely, if $x = y \in P$ then in the light of Claim 1, $<< x, m >, < x, n >> \notin G^*(\alpha, i) \times H^*(\alpha, j)$ for each $\alpha \in A$ because $G(\alpha, i) \cap H(\alpha, j) = \emptyset$.

<u>Claim 4</u>: The collection $\mathcal{C}^*$ is locally finite in $X^2 - \Delta$. To verify that claim, we first consider a point $<< x, m >, < y, n >> \in X^2 - \Delta$ with $x \neq y$. Because $\mathcal{C}$ is locally finite in $Y^2 - \Delta$, there are rational numbers p, q, r, s with $p < x < q$ and $r < y < s$ such that the set $N = (p, q) \times (r, s) \subset M^2 - \Delta$ and meets only a finite number of members of $\mathcal{C}$. But then the set

$$N^* = (< p, 0) >, < q, 0 >) \times (< r, 0 >, < s, 0 >)$$

meets only a finite number of members of $\mathcal{C}^*$. Next consider a point $<< x, m >, < x, n >> \in X^2 - \Delta$. Then $x \in P$ and, because of Claim 3, $<< x, m >, < x, n >>$ belongs to no member of $\mathcal{C}^*$. Then $N^* = \{<< x, m >, < x, n >>\}$ is a neighborhood of $<< x, m >, < x, n >>$ meeting no member of $\mathcal{C}^*$. This establishes Claim 4.

It now follows that $\mathcal{S} = \mathcal{C}^* \cup \{\{<< x, m >, < x, n >>\} : x \in P \text{ and } m \neq n\}$ is a locally finite rectangular open cover of $X^2 - \Delta$, as required.

To prove the final assertion in (3.2), we will construct a collection $\mathcal{D}$ of subsets of $X^2 - \Delta$ such that:

a) $\mathcal{D}$ is σ-locally finite in $X^2 - \Delta$;
b) each member of $\mathcal{D}$ is a co-zero set in X^2;
c) $\mathcal{D}$ covers $X^2 - \Delta$.

For each $\alpha \in A$ and $i, j \geq 1$ let $\mathcal{D}(\alpha, i, j) = \{G^*(\alpha, i) \times H^*(\alpha, j)\}$. Being a product of two co-zero sets in X, each $G^*(\alpha, i) \times H^*(\alpha, j)$ is a co-zero set in X^2. Because A is countable, the collection $\bigcup\{\mathcal{D}(\alpha, i, j) : \alpha \in A,\ i, j \geq 1\}$ is a σ-locally finite collection in $X^2 - \Delta$. Furthermore, $\bigcup\{G^*(\alpha, i) \times H^*(\alpha, j) : \alpha \in A, i, j \geq 1\} = \bigcup \mathcal{C}^*$. We now let $\mathcal{D}(0) = \{\{<< x, m >, < x, n >>\} \subset X^2 - \bigcup \mathcal{C}^* : x \in P \text{ and } m,\ n \text{ are distinct integers }\}$. Then $\mathcal{D}(0)$ is a locally finite collection in $X^2 - \Delta$ whose members are open co-zero subsets of X^2, so that $\bigcup\{\mathcal{D}(\alpha, i, j) : \alpha \in A, i, j \geq 1\} \cup \mathcal{D}(0)$ is the required open cover of $X^2 - \Delta$. $\square$

Even though the special covers of $X^2 - \Delta$ as in (3.2) do not yield a G_δ-diagonal for a LOTS X, they do have interesting consequences. We begin with a technical proposition about certain stationary sets with their usual topologies.

3.3 **Proposition**: *Let S be a stationary subset of an uncountable regular cardinal κ. Then there is no Kombarov cover of $S^2 - \Delta$ and no σ-locally finite open cover of of $S^2 - \Delta$ by* *subsets of $S^2 - \Delta$ that are co-zero sets (or even F_σ-sets) in S^2.*

Proof: In this proof, if $L \subset S$ then the term "convex component of L" will mean "convex component of L in the set S." As a first step in the proof the reader can apply the Pressing Down Lemma to the stationary set S^d of all non-isolated points of S to prove:

(*) If $\mathcal{L}$ is a σ-locally finite open cover of S, then the family of all $L \in \mathcal{L}$ that have a convex component that is cofinal in S is non-empty and finite.

Now, for contradiction, suppose that $\mathcal{C} = \{U(\alpha) \times V(\alpha) : \alpha \in A\}$ is a locally finite open cover of $S^2 - \Delta$ where $U(\alpha)$ and $V(\alpha)$ are disjoint open subsets of S. For each $s \in S^d$, apply (*) to the subspace $(S - \{s\}) \times \{s\}$ of $S^2 - \Delta$ to find some $\alpha(s) \in A$ such that $s \in V(\alpha(s))$ and some convex component of $U(\alpha(s))$ is cofinal in S. For each $s \in S^d$, there is a point $f(s) \in S$ with $f(s) < s$ and $S \cap (f(s), s] \subset V(\alpha(s))$. Apply the Pressing Down Lemma to the function f to find a point $s_0 \in S$ and a stationary subset $T \subset S^d$ such that $f(t) = s_0$ for each $t \in T$.

Let t_0 be the first element of T. Then $t_0 > s_0$, and local finiteness of $\mathcal{C}$ forces the set $F = \{\alpha \in A : t_0 \in V(\alpha),$ and $U(\alpha)$ has a convex component that is cofinal in $S\}$ to be nonempty and finite. Furthermore, if $t \in T - \{t_0\}$ then we have $t_0 \in (s_0, t] = (f(t), t] \subset V(\alpha(t))$ so that $\alpha(t) \in F$ for every $t \in T$. Because F is finite, for some $\beta \in F$ we conclude that the set $T^* = \{t \in T : \alpha(t) = \beta\}$ is stationary in S.

Because $U(\beta)$ contains a convex component that is cofinal in S, we may choose $x_1 \in U(\beta) \cap (s_0, \rightarrow)$. Because T^* is stationary in S we may choose $t_1 \in T^*$ with $t_1 > x_1$. But then we have $x_1 \in (s_0, t_1] = (f(t_1), t_1] \subset V(\alpha(t_1)) = V(\beta)$ so that $< x_1, x_1 > \in U(\beta) \times V(\beta)$ and that is impossible because $U(\beta) \times V(\beta) \subset S^2 - \Delta$. That contradiction completes the proof that no Kombarov cover of $S^2 - \Delta$ can exist.

For the second half of the proof, suppose that $\mathcal{U}$ is a σ- locally finite collection of subsets of $S^2 - \Delta$ that covers $S^2 - \Delta$ and whose members are each F_σ-subsets of S^2. Write $\mathcal{U} = \bigcup\{\mathcal{U}(n) : n \geq 1\}$ where each $\mathcal{U}(n)$ is locally finite in $S^2 - \Delta$. The Pressing Down Lemma shows:

(**) if K is an F_σ-subset of S^2 with $K \subset S^2 - \Delta$, then for some $\beta < \kappa$, $K \cap ((\beta, \rightarrow))^2 = \emptyset$.

For each $\alpha \in S$, let $H(\alpha) = (S \cap (\alpha, \rightarrow)) \times \{\alpha\}$. In the light of (*) above, for each $\alpha \in S$ there is an $n = n(\alpha)$ such that some member $U(\alpha) \in \mathcal{U}(n)$ contains a tail of $H(\alpha)$, i.e., contains $([\gamma, \rightarrow) \cap S) \times \{\alpha\}$ for some $\gamma \in S$. Define $S(k) = \{\alpha \in S : n(\alpha) = k\}$. Then for some $k \geq 1$, $S(k)$ is stationary in $[0, \kappa)$. Fix such a k. For each $\alpha \in S(k)$, choose $U(\alpha) \in \mathcal{U}(k)$ such that $U(\alpha)$ contains a tail of $H(\alpha)$ and then, using (**), choose $\beta(\alpha) \in (\alpha, \kappa)$ such that $U(\alpha) \cap ((\beta(\alpha), \kappa))^2 = \emptyset$.

By a β interlaced net we will mean a strictly increasing, well ordered net $\{\alpha_\mu : \mu < M\}$ such that $\alpha_\mu \in S(k)$ and if $\mu < \nu < M$ then $\alpha_\mu < \beta(\alpha_\mu) < \alpha_\nu$. Let $C = \{\gamma < \kappa : \gamma$ is the supremum of some β-interlaced net $\}$. It is easy to see that C is a closed, unbounded subset of $[0, \kappa)$ so that, $S(k)$ being stationary, we may choose $\delta \in C \cap S(k)$. Let M be the cofinality of δ and find a β- interlaced net $\{\alpha_\mu : \mu < M\}$ having δ as supremum. For each $\mu < M$, as noted above, $U(\alpha_\mu)$ contains a tail of $H(\alpha_\mu)$, say $([\eta_\mu, \kappa) \cap S) \times \{\alpha_\mu\} \subset U(\alpha_\mu)$. Because $\delta < \kappa$ and κ is regular, $\eta^* = \sup\{\eta_\mu : \mu < M\}$ is less than κ. Thus $([\eta^*, \kappa) \cap S) \times \{\alpha_\mu\} \subset U(\alpha_\mu)$ so that each point of the set $H^* = ([\eta^*, \kappa) \cap S) \times \{\delta\}$ is a limit point of $\bigcup\{([\eta^*, \kappa) \cap S) \times \{\alpha_\mu\} : \mu < M\}$ and hence also a limit point of $\bigcup\{U(\alpha_\mu) : \mu < M\}$. But that is impossible because the sets $U(\alpha_\mu)$ are all chosen from $\mathcal{U}(k)$ which is a locally finite collection and no point of H^* is a limit point of any set $U(\alpha_\mu)$. $\square$

3.4 **Corollary**: *Suppose X is monotonically normal and admits a Kombarov cover of $X^2 - \Delta$ or a σ-locally finite collection of subsets of $X^2 - \Delta$ that covers $X^2 - \Delta$ and consists of open F_σ-subsets of X^2 . Then X is hereditarily paracompact. In particular, any GO space X that admits such a cover of $X^2 - \Delta$ is hereditarily paracompact.*

Proof: If X is not hereditarily paracompact, then (by a result of Balogh and Rudin [BR]), X contains a subspace S that is homeomorphic to a stationary set in some regular uncountable cardinal. Restricting the given covering of $X^2 - \Delta$ to $S^2 - \Delta$ yields a contradiction of (3.3). The second assertion of the corollary now follows, because every GO-space is monotonically normal [HLZ]. $\square$

In his paper [K], Kombarov pointed out that a regular hereditarily Lindelöf space X will have a G_δ-diagonal provided there is a countable open cover of $X^2 - \Delta$ by rectangular open sets. (It is enough to know that X is perfect for Kombarov's assertion to hold.) He then asked whether the same result holds for spaces that are Lindelöf, but not hereditarily so. Using the Continuum Hypothesis (CH) or the weaker hypothesis $b = \omega_1$, we will construct a subspace of the space M^* in Example (2.2) that answers Kombarov's question in the negative.

We begin by constructing a countable open cover of $(M^*)^2 - \Delta$ by rectangular open sets. This contrasts with the rectangular open cover constructed in (3.2) which was not countable.

3.5 **Lemma**: *Let M^* be the space of (2.2). Then there is a countable cover of $(M^*)^2 - \Delta$ by rectangular open sets.*

Proof: Let R, P, Q, and Z denote, respectively, the usual space of real numbers and the sets of irrational numbers, rational numbers, and integers. Because R^2 is hereditarily Lindelöf, there is a countable rectangular open cover $\{U_n \times V_n : n \geq 1\}$ of $R^2 - \Delta$ where each U_n and V_n is an open interval in R with rational endpoints. Write $U_n = (a_n, b_n)$ and $V_n = (c_n, d_n)$, and in the space M^* define $U_n^* = (< a_n, 0 >, < b_n, 0 >)$ and $V_n^* = (< c_n, 0 >, < d_n, 0 >)$. For each $n \in Z$ define $G_n = \{< x, n >: x \in P\}$. Then G_n is an open subset of X and it is easy to verify that the collection

$$\{U_n^* \times V_n^* : n \geq 1\} \cup \{G_m \times G_n : m, n \in Z \text{ and } m \neq n\}$$

is the required countable, rectangular open cover of $(M^*)^2 - \Delta$. $\square$

3.6 **Example**: *Assume CH or $b = \omega_1$. Then there is a Lindelöf linearly ordered space Y that admits a countable rectangular open cover of $Y^2 - \Delta$ and yet does not have a G_δ-diagonal.*

Proof: Recall that, assuming the Continuum Hypothesis (or even the weaker hypothesis that $b = \omega_1$), there is an uncountable dense-in-itself subset L of R that contains and is concentrated on the set Q. As Michael noted [M], when topologized as a subspace of M, L is a Lindelöf non- metrizable space. Starting with L, one creates the lexicographically ordered set $L^* = (L \times \{0\}) \cup ((P \cap L) \times Z)$. With its usual open interval topology, L^* is a Lindelöf LOTS and is a subspace of the space M^* of (2.2). It is easy to see that L^* inherits a countable rectangular open cover from M^*. Because the LOTS L^* contains the non-metrizable subspace L, we know that L^* cannot have a G_δ- diagonal. $\square$

4. Stronger off-diagonal properties.

There are two natural ways to strengthen the property "paracompact off of the diagonal." One is to strengthen the covering condition from paracompact to a stronger property such as Lindelöf, and the other is to consider hereditary paracompactness off of the diagonal.

For a regular space to be Lindelöf off of the diagonal is a very strong hypothesis and immediately gives a G_δ-diagonal. It is easy to prove:

4.1 **Lemma**: *A regular space X is Lindelöf off of the diagonal if and only if X^2 is Lindelöf and X has a G_δ- diagonal.*

Proof: First, suppose X is Lindelöf off of the diagonal. Fix $p \in X$. Because $X - \{p\}$ embeds in $X^2 - \Delta$ as a closed subset, we see that X is a Lindelöf space. Because Δ is homeomorphic to X, we see that $X^2 = (X^2 - \Delta) \cup \Delta$ is the union of two Lindelöf subspaces, so that X^2 is Lindelöf. Next observe that because X^2 is regular and $X^2 - \Delta$ Lindelöf, we may cover $X^2 - \Delta$ with countably many open subsets of X^2 whose closures in X^2 miss Δ. Hence Δ is a G_δ-subset of X^2.

Conversely, if X has a G_δ-diagonal and X^2 is Lindelöf, then $X^2 - \Delta$ is an F_σ-subset of a Lindelöf space, so that X is Lindelöf off of the diagonal. $\square$

4.2 **Proposition**: *Suppose X is a LOTS or is a generalized ordered space that can be p-embedded in some LOTS. If X is Lindelöf off of the diagonal, then X is metrizable and there is a monotonic homeomorphism from X onto a subspace of the real line.*

Proof: According to (4.1), X has a G_δ-diagonal and it is known that a generalized ordered space that can be p-embedded in a LOTS and has a G_δ-diagonal is metrizable [L1]. Lemma 4.1 also establishes that X is a Lindelöf space, so X is separable and metrizable. To complete the proof, recall that any separable metrizable LOTS embeds in the real line by a monotonic homeomorphism. $\square$

Certain local conditions on a LOTS X combine with the property "X is paracompact off of the diagonal" to give metrization theorems. For example:

4.3 **Corollary**: *A linearly ordered space X is metrizable if it is paracompact off of the diagonal and is one of:*

a) locally compact
b) locally connected
c) locally separable.

Proof: Because b) implies a), it will be enough to verify that a) and c) each yield metrizability. If X is paracompact off of the diagonal then it is paracompact, so that it is enough to prove local metrizability in each case. If a) holds, then Gruenhage's theorem gives local metrizability. So suppose c) holds. Because a space that is separable and paracompact off of the diagonal is actually Lindelöf off of the diagonal, we know that locally the space X is a LOTS that is Lindelöf off of the diagonal. Then (4.2) yields local metrizability, as required. $\square$

From 4.2 we obtain a structure theorem for arbitrary generalized ordered spaces that are Lindelöf off of the diagonal, namely:

4.4 **Lemma**: *Let X be any generalized ordered space that is Lindelöf off of the diagonal. Then X is homeomorphic to a space obtained by modifying a subspace of the real line by isolating certain points and by making Sorgenfrey modifications at certain other points.*

Proof: Let $\mathcal{T}$ be the given topology of X and let $\mathcal{I}$ be the open interval topology of the given ordering of X. Then the LOTS $(X,\mathcal{I})$ is Lindelöf off of the diagonal, so that (4.2) gives a monotonic homeomorphism from $(X,\mathcal{I})$ onto a subspace S of the real line. Examining the way that $\mathcal{T}$ is obtained from $\mathcal{I}$ completes the proof. $\square$

Proposition 4.2 and Lemma 4.4 raise as many questions as they answer. For example, one might wonder whether the hypothesis that the GO-space X can be p-embedded in some LOTS is actually necessary in (4.2). Might it be true that *any* generalized ordered space that is Lindelöf off of the diagonal must be metrizable? Lemma (4.3) reduces the problem to asking about certain generalized ordered modifications of certain subspaces of the real line. For such special GO-spaces, could one show that if X is a GO-space constructed on the real line and X is Lindelöf off of the diagonal, then X must be separable? As our next examples show, both of those questions have axiom-sensitive answers.

4.5 **Examples**: *Assuming CH, there is a non-metrizable GO-space that is Lindelöf off of the diagonal.*

Proof: E. Michael [M] showed that, assuming CH, there is an uncountable subset X of what is now called the Michael line such that $X \times X$ is a Lindelöf space. This space X is a GO-space and, because X^2 has a G_δ- diagonal, (4.1) shows that X is Lindelöf off of the diagonal. This example shows that the hypothesis about p-embedding is essential in (4.2) and that one cannot prove separability for a GO-space that is Lindelöf off of the diagonal. $\square$

Isolated points are at the heart of Example 4.5. But even without any isolated points, there is plenty of pathology.

4.6 **Example**: *The existence of separable, non-metrizable GO- spaces that are Lindelöf off of the diagonal is axiom-sensitive.*

Proof: Under CH, Michael [M] showed that there is an uncountable dense-in-itself subset X of the Sorgenfrey line such that X^2 is Lindelöf but X^3 is not normal. This X is a GO-space that is Lindelöf off of the diagonal and is not metrizable, showing once again the need for the p-embedding hypothesis in (4.2). On the other hand, Baumgartner [Ba] and Todorčević [T] proved (respectively) that, under PFA and OCA, X^2 cannot be Lindelöf for any uncountable subset X of the Sorgenfrey line. A more extensive discussion of this issue appears in [BMo], near their Theorem 3.5. $\square$

A second very strong covering hypothesis for $X^2 - \Delta$ is hereditary paracompactness. At the present time, the ramifications of that hypothesis are not well understood. We begin with an easy example showing that one will need additional assumptions such as first countability if interesting results are to be obtained.

4.7 **Example**: *The non-first countable space X of (2.1) is a non-metrizable LOTS and is hereditarily paracompact off of the diagonal.* $\square$

In contrast to (4.7), ordered spaces that are *first countable* and hereditarily paracompact off of the diagonal do have certain strong properties. For example, it is well known that if X is any topolgical space such that the product of X with the convergent sequence $\{\frac{1}{n} : n \geq 1\} \cup \{0\}$ is hereditarily normal, then X is perfectly normal. Essentially the same proof shows:

4.8 **Proposition**: *Suppose X is a first-countable GO space such that $X^2 - \Delta$ is hereditarily normal. Then X is perfectly normal.* $\square$

Proposition 4.8 shows that the space M^*, used several times above when a first countable example was needed, will not be useful in the study of hereditary paracompactness off of the diagonal, because M^* is certainly not perfectly normal. It is possible to give a more direct proof that M^* is not hereditarily paracompact off of the diagonal, as follows.

4.9 **Example**: *The space M^* of (2.2) is not hereditarily paracompact off of the diagonal.*

Proof: Let Q, P be the usual sets of rational and irrational numbers, respectively. To verify that M^* is not hereditarily paracompact off of the diagonal, it will be enough to prove that if M denotes the usual Michael line, then $Y = M^2 - \Delta$ is not hereditarilty paracompact, because Y can be embedded in $(M^*)^2 - \Delta$.

Fix a rational number q and let $Z = \{< x, y >\in Y : if\ x = q,\ then\ y \in P\}$. Let $W_0 = Z - (\{q\} \times M)$ and for each $x \in P$ let $W_x = (M \times \{x\}) \cap Z$. Then $\mathcal{W} = \{W_x : x = 0\ or\ x \in P\}$ is an open cover of Z. For contradiction, suppose there is a locally finite open cover $\mathcal{U}$ of Z that refines $\mathcal{W}$. For each $x \in P$ choose $U(x) \in \mathcal{U}$ with $(q, x) \in U(x)$. Because W_x is the only member of $\mathcal{W}$ that contains (q, x), we know that $U(x) \subset W_x$ so there is a positive $e(x)$ such that $(q - e(x), q + e(x)) \times \{x\} \subset U(x)$. Let $P_n = \{x \in P : e(x) \geq \frac{1}{n}\}$. Because $P = \bigcup\{P(n) : n \geq 1\}$, some set $P(n)$ has a rational limit point s in the usual topology of the real numbers. Choose a rational number $t \in (q - \frac{1}{n}, q)$ with $t \neq s$. Then (s, t) is a limit point of the set $\bigcup\{U(x) : x \in P(n)\}$ in the space Z even though $(s, t) \notin \mathrm{cl}(U(x))$ for every $x \in P(n)$, and that is impossible because $\mathcal{U}$ is locally finite in Z. $\square$

One of the first questions that one encounters when studying hereditary paracompactness off of the diagonal in linearly ordered spaces is suggested by the result of Gruenhage and Pelant (see 2.6, above):

4.10 **Question**: Suppose X is a LOTS that is first countable and hereditarily paracompact off of the diagonal. Must X have a point-countable base?

Perhaps Souslin spaces – linearly ordered topological spaces that are not separable and yet have countable cellularity – might be a source of examples in the study of hereditary paracompactness off of the diagonal. (Note that no connectedness or completeness is assumed for Souslin spaces.) Such spaces are always first countable, Lindelöf, and perfect, and it is known that such spaces can have point countable bases ([B] and [P]). It would be interesting to know:

4.11 **Question**: Is it possible that a Souslin space can be hereditarily paracompact off of the diagonal?

The techniques that Mary Ellen Rudin used in [Ru] can be slightly modified to show that a Souslin space whose order is complete cannot be hereditarily paracompact, or even hereditarily normal, off of the diagonal, and this limits the kinds of examples that one might find.

The distinction between LOTS and GO-spaces is likely to be important in studying the role of hereditary paracompactness off of the diagonal. The consistent example of Michael described in (4.6) makes it clear that for GO- spaces, the hypothesis of hereditary paracompactness off of the diagonal (or even the property of having a hereditarily Lindelöf square) gives almost nothing in terms of special base properties such as point-countable bases or metrizability.

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