Metrizably Fibered Generalized Ordered Spaces

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Abstract

In this paper we characterize generalized ordered spaces that are metrizably fibered in terms of certain quotient spaces and in terms of the existence of special open covers. We apply our results to give a new characterization of perfect generalized ordered spaces that have a σ-closed-discrete dense subset and to give examples of GO-spaces that are, or are not, metrizably fibered.

Key words and phrases: metrizably fibered, GO-space, generalized ordered space, LOTS, linearly ordered space, Souslin line, Big Bush, perfect space, σ-closed-discrete dense set.

Mathematical Reviews Classification numbers: Primary = 54F05; Secondary = 54E35

DRAFT OF August20, 2002

1 Introduction

Tkachuk introduced and studied metrizably fibered spaces in [T]. A topological space X is *metrizably fibered* provided there is a continuous function f from X to a metric space M with the property that the fiber $f^{-1}[m]$ of f is a metrizable subspace of X for each $m \in M$. The function $f : X \to M$ in that definition is said to be a *metric fibering* of X.

In this note we characterize generalized ordered spaces that are metrizably fibered, give several examples of generalized ordered spaces that are, or are not, metrizably fibered, and show that a perfect GO-space is metrizably fibered if and only if it has a σ-closed discrete dense subset.

Recall that a *generalized ordered space* (GO-space) is a Hausdorff space X equipped with a linear ordering such that X has a base of order-convex sets. If the topology of X coincides with the usual open interval topology of the given order, then we say that X is a *linearly ordered topological space* (LOTS). It is known that the class of generalized ordered spaces coincides with the class of topological subspaces of LOTS.

At several points in our paper we will use the term "relatively convex." Let $(X, <)$ be a linearly ordered set and let $Y \subseteq X$. A subset $S \subseteq Y$ is *relatively convex in* Y provided $b \in S$ whenever $a < b < c$ are points of Y and $a, c \in S$.

2 Metrizably Fibered GO-spaces

Lemma 2.1 *: Suppose $f : X \to M$ is a metric fibering of the GO-space X. Then X is first-countable and paracompact.*

Proof: Fix $x \in X$ and let $m = f(x)$. Then the set $f^{-1}[m]$ is a G_δ-subset of X, and $\{x\}$ is a G_δ-subset of the metric space $f^{-1}[m]$. Hence $\{x\}$ is a G_δ-set in X, so that X is first-countable.

If X is not paracompact, then there is a stationary set S in a regular uncountable cardinal κ that embeds in X as a closed subspace [EL]. But then S is also metrizably fibered by $f|_S$. For notational simplicity, we will assume that $f : S \to M$ is a metrizable fibering of S.

For each $n \geq 1$, let $\mathcal{G}(n)$ be an open cover of the metric space M by balls of radius $\frac{1}{n}$. Let $\mathcal{H}(n)$ be the family of convex components of the sets $f^{-1}[G]$ for $G \in \mathcal{G}(n)$. Let L be the set of limit points of S. Then L is also stationary in κ, and for each n and $\lambda \in L$ choose some $H(n,\lambda) \in \mathcal{H}(n)$ that contains λ. Then there is an ordinal $\alpha(\lambda, n) < \lambda$ such that $]\alpha(\lambda, n), \lambda] \cap S \subseteq H(n,\lambda)$. Apply the Pressing Down Lemma to the stationary set L to find an ordinal $\beta(n)$ and a cofinal set $L(n) \subseteq L$ such that $\alpha(\lambda, n) = \beta(n)$ for all $\lambda \in L(n)$. But then $[\beta, \rightarrow[\cap S \subseteq \mathrm{St}(\beta, \mathcal{H}(n))$ for each n, where $\beta \in S$ is any fixed ordinal greater than every $\beta(n)$. It follows that $[\beta, \rightarrow[\cap S$ is mapped to a single point $m \in M$ so that $[\beta, \rightarrow[\cap S \subseteq f^{-1}[m]$ and that is impossible because $f^{-1}[m]$ is metrizable. $\square$

Definition 2.2 *: Let X be a GO space and let $\mathcal{C}$ be a partition of X into closed, convex, metrizable subspaces. By $X/\mathcal{C}$ we mean the quotient space obtained from X by identifying each set in $\mathcal{C}$ to a point.*

Notice that the given ordering of X induces a natural linear ordering of $X/\mathcal{C}$, and van Wouwe (Proposition 1.2.3 of [vW]) proved

Proposition 2.3 *: Let X be a GO-space and let $\mathcal{C}$ be a partition of X into closed convex sets. Then $X/\mathcal{C}$ with the quotient topology is itself a GO-space with respect to the natural ordering, and the natural projection mapping is closed, continuous, and order-preserving.* $\square$

We are now able to characterize metrizably fibered GO-spaces using quotient spaces and special open covers.

Theorem 2.4 *: Let X be a GO-space. Then the following are equivalent:*

a) X is metrizably fibered;

b) X is paracompact and there is a sequence $\{\mathcal{H}(n) : n \geq 1\}$ of open covers of X such that for each $x \in X$, the set $\bigcap\{\mathrm{St}(x, \mathcal{H}(n)) : n \geq 1\}$ is a metrizable subspace of X;

c) there is a partition $\mathcal{C}$ of X into closed, convex, metrizable subspaces such that the quotient space $X/\mathcal{C}$ has a G_δ-diagonal;

d) there is continuous order-preserving mapping $g : X \to Y$ from X to a metrizable GO-space Y such that $g^{-1}[y]$ is a metrizable subset of X for each $y \in Y$.

Proof: Clearly $d) \Rightarrow a)$ so we prove $a) \Rightarrow b),\ b) \Rightarrow c)$ and $c) \Rightarrow d)$. However, see 2.5, below.

$\underline{a) \Rightarrow b)}$. Suppose $f : X \to M$ is a metric fibering of the GO-space X. Let $\mathcal{B}(n)$ be the collection of all open $\frac{1}{n}$ balls in M. Let $\mathcal{G}(n)$ be the collection of all convex components of all sets of the form $f^{-1}[B]$ for $B \in \mathcal{B}(n)$. Each $\mathcal{G}(n)$ is an open cover of X and for each $x \in X$ we have $\bigcap\{\mathrm{St}(x, \mathcal{G}(n)) : n \geq 1\} \subseteq f^{-1}[f(x)]$. Because $f^{-1}[f(x)]$ is metrizable, so is $\bigcap\{\mathrm{St}(x, \mathcal{G}(n)) : n \geq 1\}$, as required.

$\underline{b) \Rightarrow c)}$: Suppose that X is paracompact and the covers $\mathcal{G}(n)$ exist as in (b). Using paracompactness, we recursively define collections $\mathcal{H}(n)$ of X such that

i) each $\mathcal{H}(n)$ is a convex open cover of X that refines $\mathcal{G}(n)$;

ii) for each $n \geq 1$ and each $H \in \mathcal{H}(n+1)$, $\mathrm{St}(\mathrm{St}(\mathrm{St}(H, \mathcal{H}(n+1)), \mathcal{H}(n+1)), \mathcal{H}(n+1))$ is a subset of some member of $\mathcal{H}(n)$;

iii) for each $x \in X$ and each $n \geq 1$, $\mathrm{Cl}_X(\mathrm{St}(x, \mathcal{H}(n+1)) \subseteq St(x, \mathcal{H}(n))$.

For each $x \in X$ define $C_x = \bigcap\{\mathrm{St}(x, \mathcal{H}(n) : n \geq 1\}$. Because of (i), each C_x is a convex, metrizable subset of X. In the light of (iii), each C_x is closed in X. Finally, (ii) implies that if $C_x \cap C_y \neq \emptyset$, then $C_x = C_y$, i.e., the collection $\mathcal{C} = \{C_x : x \in X\}$ is a partition of X.

Let $Y = X/\mathcal{C}$ be the quotient space obtained by collapsing each member of $\mathcal{C}$ to a point. According to Proposition 2.3, Y is a GO-space under its natural ordering, and the projection mapping $\pi : X \to Y$ is a closed, continuous, order preserving mapping.

We show that Y has a G_δ-diagonal. For each open subset U of X, let $U^* = \bigcup\{C \in \mathcal{C} : C \subseteq U\}$. Because π is a closed mapping, each set U^* is open in X and its image $\pi[U^*]$ is open in the quotient space Y. Define

$$\mathcal{L}(n) = \{\pi[H^*] : H \in \mathcal{H}(n)\}.$$

To see that each $\mathcal{L}(n)$ is an open cover of Y, let $C_x \in \mathcal{C}$. Then for each $n \geq 1$, $C_x \subseteq \mathrm{St}(x, \mathcal{H}(n+1))$ and by (ii), some $H \in \mathcal{H}(n)$ has $\mathrm{St}(x, \mathcal{H}(n+1)) \subseteq H$. But then in the space Y we have $C_x \in \pi[H^*] \in \mathcal{L}(n)$. Thus $\mathcal{L}(n)$ is an open cover of Y. To complete the proof that Y has a G_δ-diagonal, suppose that $C_x \in \mathcal{C}$ and for each $n \geq 1$ we have $C_x \in \pi[H^*(n)]$ for some $H(n) \in \mathcal{H}(n)$. Then we have $x \in H(n)$ so that

$$C_x \subseteq \bigcap\{H(n) : n \geq 1\} \subseteq \bigcap\{\mathrm{St}(x, \mathcal{H}(n)) : n \geq 1\} = C_x.$$

Therefore, in the space Y, $\{C_x\} = \bigcap\{\pi[H^*(n)] : n \geq 1\}$, showing that the open covers $\mathcal{L}(n)$ are a G_δ-diagonal sequence for the space Y, as claimed.

c) $\Rightarrow$ d): According to c) we have a partition $\mathcal{C}$ of X such that the quotient space $Y = X/\mathcal{C}$ has a G_δ-diagonal. As noted in Proposition 2.3, the space Y with its quotient topology τ is a GO-space, so that according to a theorem of Przymusinski ([A]) there is a metrizable GO-topology σ on Y with $\sigma \subseteq \tau$. But then the mapping $\pi : X \to (Y, \sigma)$ is the mapping g described in d). $\square$

Remark 2.5 *:* Notice that, for $a) \Rightarrow b)$, all we need to know is that the range space Y has a G_δ-diagonal (and fibers of the mapping f are metrizable). In addition, the referee pointed out that *any* Tychonoff space X is metrizably fibered if and only if there is a sequence $\{\mathcal{U}(n) : n \geq 1\}$ of open covers of X such that $\mathcal{U}(n+1)$ star-refines $\mathcal{U}(n)$ for each n and such that $\bigcap\{\mathrm{St}(p, \mathcal{U}(n)) : n \geq 1\}$ is metrizable for each $p \in X$. Consequently the equivalence of a) and b) in the previous theorem is actually a consequence of the general theory of metrizably fibered spaces.

Theorem 2.6 *: Let X be a perfect GO-space. Then X is metrizably fibered if and only if X has a σ-closed-discrete dense set.*

Proof: First suppose that X has a σ-closed-discrete dense set. Then X is paracompact [F] and from [BHL] we know that there is a sequence $\mathcal{H}(n)$ of open covers of X such that for each $x \in X$ the set $\bigcap\{\mathrm{St}(x, \mathcal{H}(n) : n \geq 1\}$ is a countable set. But any countable GO-space is metrizable. Now apply part b) of Theorem 2.4.

Conversely, suppose X is perfect and metrizably fibered. We claim that every metrizable, convex, open subset U_0 of X is contained in a maximal open, convex metrizable subset of X. We can apply Zorn's lemma to the collection $\{V : V$ is a metrizable open convex subset of X and $U_0 \subseteq V\}$ partially ordered by inclusion, provided we show that if $\mathcal{C}$ is a chain of convex, open metrizable subspaces of X, then the subspace $S = \bigcup \mathcal{C}$ is metrizable. Fix $p \in S$. It will be enough to show that both $[p, \rightarrow[\cap S$ and $]\leftarrow, p] \cap S$ are metrizable.

We will consider $[p, \rightarrow[\cap S$, the other half being analogous. There are two cases. If some point $q \in X$ is both an upper bound for S and a limit point of S, then there is an increasing sequence x_n in S whose supremum is q. Then $[p, q] = \{q\} \cup (\bigcup\{[p, x_n] : n \geq 1\})$, so that the GO-space $[p, q]$ is a countable union of closed metrizable subspaces, whence $[p, q]$ is metrizable. Consequently, $S \cap [p, q[$ is also metrizable. The second case is where S has no supremum in X, i.e., the supremum of S is a gap of X. But then, because X is paracompact, a theorem of Faber [F] shows that there is a closed, discrete cofinal subset $T \subseteq [p, \rightarrow[\cap S$. Using the set T, we can write $[p, \rightarrow[\cap S$ as a discrete union of closed metrizable convex subspaces and thereby conclude that $[p, \rightarrow[\cap S$ is metrizable.

Now let $\mathcal{E}$ be the collection of all maximal, open, convex, metrizable subspaces of X. Then members of $\mathcal{E}$ are pairwise disjoint, so that $E_0 = \bigcup \mathcal{E}$ is a metrizable open subspace of X. Because X is perfect, E_0 is the union of countably many closed subspaces X_n of X, each metrizable, and hence X_n has a dense subset D_n that is σ-closed-discrete in X. Therefore E_0 also has a dense subset that is σ-closed-discrete in X.

Let $Y = X - E_0$. Then Y is closed in X. We claim that if $y_1 < y_2$ are points of Y and if $T = [y_1, y_2] \cap Y$ is metrizable, then $[y_1, y_2]$ is a metrizable subspace of X. Clearly $[y_1, y_2] = T \cup ([y_1, y_2] \cap E_0)$. From above, the latter set is metrizable and is an F_σ-subset of X. Hence $[y_1, y_2]$ is the union of countably many closed metrizable subspaces. Because $[y_1, y_2]$ is a GO-space, it must be metrizable, as claimed. Consequently, $]y_1, y_2[$ must be contained in some member of $\mathcal{E}$. But then $[y_1, y_2] \cap Y = \{y_1, y_2\}$. Thus, any relatively convex, metrizable subset of Y has at most two points.

To complete the proof, observe that the subspace Y of X is also metrizably fibered so that, according to Theorem 2.4, there is a sequence $\mathcal{H}(n)$ of relatively open, relatively convex covers of Y with the property that for each $y \in Y$, the set $C_y = \bigcap\{\mathrm{St}(y, \mathcal{H}(n) : n \geq 1\}$ is metrizable. But then, by the previous paragraph, C_y has at most two points. Now invoke the main theorem in [BHL] to conclude that the subspace Y has a dense subset that is relatively σ-closed-discrete in Y, and hence also σ-closed discrete in X. As noted above, E_0 also contains a dense subset that is σ-closed-discrete in X. Hence X has such a subset.$\square$

Question 2.7 *:* Is there a ZFC example of a perfect GO-space that is not metrizably fibered? In the light of Theorem 2.6 that question is equivalent to an old question of R.W. Heath, namely "Is there a ZFC example of a perfect GO-space that does not have a σ-closed-discrete dense subset?"(Note that a Souslin space would be a perfect GO-space that does not have such a dense subset, but consistently, Souslin spaces do not exist.)

As noted in the Introduction, the class of GO-spaces is precisely the class of subspaces of linearly ordered topological spaces. The key step in the proof of that assertion is to construct, for any GO-space $(X, \tau, <)$, a canonical LOTS X^* that contains X as a closed subspace. To define X^*, let λ be the usual open-interval topology of the given order $<$. Define

$$R = \{x \in X : [x, \rightarrow[\in \tau - \lambda\}$$

and

$$L = \{x \in X :]\leftarrow, x] \in \tau - \lambda\}$$

and then let X^* be the lexicographically ordered set

$$\{(x, 0) : x \in X\} \cup \{(x, n) : x \in R,\ n \leq 0\} \cup \{(x, n) : x \in L,\ n \geq 0\}.$$

It is known that many topological properties at X are "inherited" by X^*. One such example, proved in [L2] and needed below, is:

Lemma 2.8 *: The GO-space X is metrizable if and only if the LOTS X^* is metrizable*

However, there are some topological properties that X^* does not "inherit" from X, the best known being the property "each closed subet of X is a G_δ-subset of X." Therefore, it is natural to ask whether a GO-space X is metrizably fibered if and only if the LOTS X^* is also metrizably fibered, and the answer is given by:

Proposition 2.9 *: A GO-space X is metrizably fibered if and only if the LOTS X^* is metrizably fibered.*

Proof: The only part of the theorem that requires proof is that if the GO-space X is metrizably fibered, then so is X^*. Apply Theorem 2.4 to show that if X is metrizably fibered, then there is a metrizable GO-space Y and a continuous, increasing map $f : X \to Y$ such that for each $y \in Y,\ f^{-1}[y]$ is metrizable. Clearly the fibers of f are convex subsets of X.

Let $\psi : X^* \to X$ be given by $\psi(x, n) = x$ for each $(x, n) \in X^*$. Then ψ is continuous and order-preserving. Let $g = f \circ \psi$. Then g is a continuous, order-preserving mapping from X^* to M. Fix any $m \in M$. Then $g^{-1}[m] = \{(x, n) \in X^* : x \in f^{-1}[m]\}$. Let $Z = f^{-1}[m]$ topologized as a subspace of X. Then Z is a metrizable GO-space, so that we can apply the canonical construction to find a LOTS Z^* that contains Z as a closed subspace. Because Z is metrizable, Lemma 2.8 shows that Z^* is also metrizable. Because Z is a convex subset of X, the space $g^{-1}[m]$ is a subspace of Z^* and is therefore metrizable, as required. $\square$

3 Examples

As the next two examples show, there are GO-spaces that can be seen to be metrizably fibered without invoking the characterizations given in the previous section, because the necessary continuous mapping onto a metric space is obvious.

Example 3.1 *: The lexicographic square and the double arrow space are metrizably fibered compact LOTS. (Because the lexicographic square is metrizably fibered, we see that a compact LOTS can be metrizably fibered without being perfect.) The Sorgenfrey line and the Michael line are GO-spaces that are metrizably fibered.* $\square$

Example 3.2 *: The space $E(Y, X)$ of [BL2] and [BL1] is a Čech complete, non-metrizable space that has weight ω_1 and has a σ-closed-discrete dense subset. Hence $E(Y, X)$ is metrizably fibered.*

Proof: The space $E(Y, X)$ is constructed by starting with a special metric space $X \subset D^\omega$ described by A. Stone in [S], where D is a discrete space with cardinality ω_1. Let Y be the closure of X in D^ω and note that, according to a theorem of Herrlich [H] Y is a LOTS under some linear order. The LOTS $E(Y, X)$ is constructed by "splitting" each point of Y into two consecutive points and using the lexicographic ordering. We see that $E(Y, X)$ is metrically fibered because the projectiuon mapping that re-collapses the points of $E(Y, X)$ that were split is the required metric fibering. Alternatively, because $E(X, Y)$ has a σ-closed-discrete dense subset, it is metrizably fibered, by Theorem 2.6. $\square$

We next give examples showing how our results can be used to show that certain spaces are *not* metrizably fibered.

The next example is due to Tkachuk [T] who used it to show that there are first-countable compact Hausdorff spaces that are not metrizably fibered. Our results make the proof shorter.

Example 3.3 *: The lexicographic cube* $[0,1]^3$ *is not metrizably fibered.*

Proof: If $X = [0,1]^3$ were metrizably fibered, then by Theorem 2.4 there would be a continuous order-preserving mapping f from X onto a metrizable GO-space Y. Note that Y is compact and hence separable. For each $x \in [0,1]$ the interval $[(x,0,0),(x,1,1)]$, being a copy of the lexicographic square, is non-metrizable and therefore cannot be a subset of any fiber of f, so that $f(x,0,0) < f(x,1,1)$ in Y. But then $\{]f(x,0,0), f(x,1,1)[\ : x \in [0,1]\}$ is an uncountable collection of pairwise disjoint, non-degenerate, connected convex subsets of the separable GO-space Y, and that is impossible. □

Example 3.4 *The "big bush" in [B] is a non-metrizable paracompact LOTS with a point-countable base that is not metrizably fibered.*

Proof: Let X be the "big bush." The space X is the set

$$\{f : [0,\alpha] \to \mathbb{R} : \alpha < \omega_1,\ \forall \beta < \alpha,\ f(\beta) \in \mathbb{P},\ \text{and}\ f(\alpha) \in \mathbb{Q}\}$$

where $\mathbb{Q}$, $\mathbb{P}$, and $\mathbb{R}$ denote, respectively, the usual sets of rational, irrational, and real numbers. Equip X with the open interval topology of the lexicographic order.

The key feature of X is that it has no metrizable convex sets larger than a single point. For contradiction, suppose X is metrizably fibered, and consider the open covers $\mathcal{H}(n)$ in Theorem 2.4. The convex sets C_x must be singletons, showing that the space X must have a G_δ-diagonal, and that is impossible because X is a LOTS and is not metrizable. [L1] □

Our final example concerns Souslin spaces, i.e., linearly ordered spaces that are not separable and yet have countable cellularity. No connectedness or completeness is assumed. The existence of such spaces is undecidable in ZFC. Our result slightly improves an example in [T] which begins with a compact, connected Souslin space having no separable open intervals.

Example 3.5 *: A Souslin space cannot be metrizably fibered.*

Proof: Any Souslin space is perfect. Hence, Theorem 2.6 shows that if it were metrizably fibered, then it would have a σ-closed-discrete dense set D. But then D would be countable, and that is impossible. □

References

[A] Alster, K., Subparacompactness in Cartesian products of generalized ordered spaces, *Fundamenta Math.*, 87 (1975), 7-28.

[B] Bennett, H., Point-countability in ordered spaces, *Proc. Amer. Math. Soc.*, 28 (1971), 598-606.

[BHL] Bennett, H., Heath, R., and Lutzer, D., GO-spaces with σ-closed-discrete dense subsets, *Proc. Amer. Math. Soc.*, 129 (2000) 931-939.

[BL1] Bennett, H. and Lutzer, D., Diagonal conditions in ordered spaces, *Fundamenta Math.* 153 (1997), 99-123.

[BL2] Bennett, H. and Lutzer, D., A metric space of A.H. Stone and an example concerning σ-minimal bases, *Proc. Amer. Math. Soc.*, 126 (1998), 2191-2196.

[EL] Engelking, R. and Lutzer, D., Paracompactness in ordered spaces, *Fundamenta Math.*, 94 (1976), 49-58.

[F] Faber, M., Metrizability in generalized ordered spaces, *Math. Centre Tracts* no. 53, 1974, Amsterdam.

[H] Herrlich, H., Ordnungsfahigkeit total-discontinuierlicher Raume, *Math. Ann.*, 159 (1965), 77-80.

[L1] Lutzer, D., A metrization theorem for linearly ordered topological spaces, *Proc. Amer. Math. Soc.* 22 (1969) 557–558.

[L2] Lutzer, D., On generalized ordered spaces, *Dissertationes Math* 1971, vol. 89.

[S] Stone. A., On σ-discreteness and Borel isomorphism, *Amer. J. Math.* 85 (1963), 655-666.

[T] Tkachuk, V., A glance at compact spaces which map "nicely" onto the metrizable ones, *Topology Proc.* 19 (1994), 321-334.

[vW] van Wouwe, J., *GO-spaces and generalizations of metrizability*, Mathematical Centre Tract 104, Mathematical Centre, Amsterdam, 1979.