Reduction of Order

Suppose that y_1 is a solution to the problem

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

Our goal is to find a second linearly independent solution y_2.

The motivation for this approach is the method of variation of parameters seen earlier in the class. We seek a solution in the form

$$y_2(x) = v(x)y_1(x).$$

This implies that

$$y_2' = v'y_1 + vy_1',$$

and

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substituting these expressions into (1) gives

$$\left[v''y_1 + 2v'y_1' + vy_1''\right] + p(x)\left[v'y_1 + vy_1'\right] + q(x)\left[vy_1\right] = 0.$$

Collecting the terms multiplying v we get

$$v(y_1'' + p(x)y_1' + q(x)y_1) = 0$$

so the equation for v simplifies to

$$y_1v'' + (2y_1' + p(x)y_1)v' = 0.$$

Thus we obtain

$$v'' + \frac{(2y_1' + p(x)y_1)}{y_1}v' = 0. \tag{2}$$

Setting $w = v'$ and $P = \dfrac{(2y_1' + p(x)y_1)}{y_1}$ this equation reduces to the first order linear equation

$$w' + Pw = 0.$$

A solution is $w = \exp\left(-\int^x P(s)\,ds\right)$ so that

$$v' = \exp\left(-\int^x P(s)\,ds\right) = \exp\left(-\int^x \left[\frac{2y_1'}{y_1} + p(x)\right]\,ds\right) = \frac{1}{y_1^2}\exp\left(-\int^x p(x)\,ds\right)$$

and hence

$$v = \int \frac{1}{y_1^2(x)}\exp\left(-\int^x p(x)\,ds\right)\,dx.$$

So we have

$$y_2 = y_1\int \frac{1}{y_1^2(x)}\exp\left(-\int^x p(x)\,ds\right)\,dx.$$

As an example consider $y'' - 2r_0 y' + r_0^2 y = 0$. The characteristic equation has a double root $r = r_0$ so one solution is $y_1 = e^{r_0 x}$. Here $p(x) = -2r_0$ and $q(x) = r_0^2$. We seek $y_2 = v y_1$ and obtain :

$$y_2 = e^{r_0 x} \int \frac{1}{e^{2r_0 x}} \exp\left(- \int^x -2r_0 \, ds \right) dx = e^{r_0 x} \int dx = x e^{r_0 x}.$$