Method of Undetermined Coefficients

Our goal is to find the general solution of $\boxed{ay'' + by' + cy = f(x)\ (*)}$

The general solution of $(*)$ is obtained as $\boxed{y = y_h + y_p}$ where

1. y_h is the general solution of the homogeneous problem, i.e. $y_h = c_1 y_1 + c_2 y_2$ where y_1, y_2 are two linearly independent solutions of $ay'' + by' + cy = 0$.
2. y_p is (any) particular solution of the nonhomogeneous problem $(*)$.

The main problem then is to find y_p.

Remarks on the *Method of Undetermined Coefficients*

Remark:

1. The general solution of the homogeneous problem is given as a sum of numbers times terms of the form
$$p(x), \quad p(x)e^{ax}, \quad p(x)e^{\alpha x}\cos(\beta x), \quad p(x)e^{\alpha x}\sin(\beta x) \tag{1}$$
where $p(x)$ is a polynomial in x. No other types of solutions are possible!
2. This remains true for the nonhomogeneous problem $(*)$ provided the right hand side $f(x)$ is also given as a sum of terms of the form (1).
3. The main thing is to find y_p and here we consider the case of $f(x)$ in the form (1).

We consider $p(x) = Cx^m + \cdots$ is a polynomial of degree m.

$$ay'' + by' + cy = p(x)e^{r_0 x} \quad \Rightarrow \quad y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{r_0 x}$$

1. $s = 0$ if r_0 is not a root of the characteristic polynomial $\boxed{ar^2 + br + c = 0 \quad (\dagger)}$.
2. $s = 1$ if r_0 is a simple root of $(\dagger)$.
3. $s = 2$ if r_0 is a double root of $(\dagger)$.

N.B. The above case includes the case $r_0 = 0$ in which case the right side is $p(x)$.

$$ay'' + by' + cy = \begin{cases} p(x)e^{\alpha x}\cos(\beta x) \\ \text{or} \\ p(x)e^{\alpha x}\sin(\beta x) \end{cases} \quad \Rightarrow \quad y_p = \begin{array}{l} x^s(A_m x^m + \cdots + A_1 x + A_0)e^{\alpha x}\cos(\beta x) \\ +x^s(B_m x^m + \cdots + B_1 x + B_0)e^{\alpha x}\sin(\beta x) \end{array}$$

1. $s = 0$ if $r_0 = \alpha + i\beta$ is not a root of $(\dagger)$.
2. $s = 1$ if $r_0 = \alpha + i\beta$ is a simple root of $(\dagger)$.