6 Non-homogeneous Heat Problems

Up to this point all the problems we have considered for the heat or wave equation we what we call homogeneous problems. This means that for an interval $0 < x < \ell$ the problems were of the form

$$\boxed{\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \\ \mathcal{B}_0(u) &= 0, \quad \mathcal{B}_1(u) = 0 \\ u(x,0) &= \varphi(x) \end{aligned}} \qquad \boxed{\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \\ \mathcal{B}_0(u) &= 0, \quad \mathcal{B}_1(u) = 0 \\ u(x,0) &= f(x), \ u_t(x,0) = g(x) \end{aligned}}$$

In contrast, in this chapter we are concerned with the more general non-homogeneous cases:

$$\boxed{\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + F(x,t), \\ \mathcal{B}_0(u) &= \gamma_0(t), \quad \mathcal{B}_1(u) = \gamma_1(t) \\ u(x,0) &= \varphi(x) \end{aligned}} \qquad \boxed{\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) + F(x,t), \\ \mathcal{B}_0(u) &= \gamma_0(t), \quad \mathcal{B}_1(u) = \gamma_1(t) \\ u(x,0) &= f(x), \ u_t(x,0) = g(x) \end{aligned}}$$

where $\gamma_j(t)$ and $F(x,t)$ are known source terms.

Here we have used the notation $\mathcal{B}_j(u)$ to indicate a boundary condition. So for example we might have

$$\mathcal{B}_0(u) = \alpha_0 u_x(0,t) + \beta_0 u(0,t), \quad \mathcal{B}_1(u) = \alpha_1 u_x(1,t) + \beta_1 u(1,t).$$

Specifically then for Dirichlet boundary conditions we have $\mathcal{B}_0(u) = u(0,t)$, $\mathcal{B}_1(u) = u(1,t)$ and for Neumann conditions we have $\mathcal{B}_0(u) = u_x(0,t)$, $\mathcal{B}_1(u) = u_x(1,t)$.

6.1 Non-Homogeneous Equation, Homogeneous Dirichlet BCs

We first show how to solve a non-homogeneous heat problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + F(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.1}$$

Let us recall from all our examples involving Fourier series and Sturm-Liouville problems we have

$$\lambda_n = -\mu_n^2, \quad \mu_n = \frac{n\pi}{\ell}, \quad \varphi_n(x) = \sin(\mu_n x)$$

and for the non-homogeneous problem, instead of looking for a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{k\lambda_n t} \varphi_n(x)$$

as we would if $F(x,t) \equiv 0$, we look for

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \varphi_n(x) \tag{6.2}$$

We can notice from the initial data that

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} c_n(0) \varphi_n(x)$$

where $b_n \equiv c_n(0)$ are the Fourier Sine coefficients of φ, i.e.,

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x). \tag{6.3}$$

where

$$\boxed{b_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \varphi_n(x)\, dx.} \tag{6.4}$$

Next we find $\{F_n(t)\}$ so that

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \varphi_n(x), \tag{6.5}$$

by setting

$$\boxed{F_n(t) = \frac{2}{\ell} \int_0^{\ell} F(x,t) \varphi_n(x)\, dx.} \tag{6.6}$$

Putting these things together we can differentiate (6.2) with respect to t to obtain

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \frac{dc_n}{dt}(t) \varphi_n(x),$$

and twice respect to x to obtain

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} c_n(t) \frac{d^2 \varphi_n}{dx^2}(x) = \sum_{n=1}^{\infty} \lambda_n c_n(t) \varphi_n(x).$$

Then plugging these expressions and (6.5) into the PDE (6.1) we arrive at

$$\sum_{n=1}^{\infty} \left(\frac{dc_n}{dt}(t) - k\lambda_n c_n(t) - F_n(t) \right) \varphi_n(x) = 0.$$

Then using the orthogonality of the functions $\{\varphi_n\}$ we obtain an infinite sequence of ODES

$$\frac{dc_n}{dt}(t) - k\lambda_n c_n(t) = F_n(t), \quad n = 1, 2, \cdots .$$

These equations are first order linear ODEs which we can easily solve by multiplying both sides by the integrating factor

$$e^{-k\lambda_n t}$$

which give

$$\frac{d}{dt}\left(e^{-k\lambda_n t} c_n(t)\right) = e^{-k\lambda_n t} F_n(t).$$

We integrate both sides from $t = 0$ to t to obtain

$$e^{-k\lambda_n t} c_n(t) - c_n(0) = \int_0^t e^{-k\lambda_n \tau} F_n(\tau)\, d\tau.$$

Thus we get

$$c_n(t) = e^{k\lambda_n t} b_n + e^{k\lambda_n t} \int_0^t e^{-k\lambda_n \tau} F_n(\tau)\, d\tau$$

where $b_n = c_n(0)$ and from (6.2) we obtain

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \varphi_n(x) + \sum_{n=1}^{\infty} \left(\int_0^t e^{k\lambda_n (t-\tau)} F_n(\tau)\, d\tau \right) \varphi_n(x),} \tag{6.7}$$

Let us consider an example with Dirichlet boundary conditions.

Example 6.1. In our first example we consider the case in which $F(x,t) = f(x)$ does not depend on t.

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) + f(x), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.8}$$

We know that the solution is given by (6.7)

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \varphi_n(x) + \sum_{n=1}^{\infty} \left(\int_0^t e^{k\lambda_n (t-\tau)} F_n(\tau)\, d\tau \right) \varphi_n(x)$$

where

$$c_n(t) = e^{k\lambda_n t} c_n(0) + e^{k\lambda_n t} \int_0^t e^{-k\lambda_n \tau} F_n(\tau)\, d\tau$$

and $c_n(0) = b_n$ are the Fourier Sine coefficients of φ, i.e.,

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \varphi_n(x)$$

so that $c_n = c_n(0)$ are given by

$$b_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\varphi_n(x)\,dx. \tag{6.9}$$

In this case

$$F_n(t) \equiv f_n = \frac{2}{\ell}\int_0^\ell f(x)\varphi_n(x)\,dx.$$

Thus we have

$$c_n(t) = e^{k\lambda_n t}b_n + e^{k\lambda_n t}\int_0^t e^{-k\lambda_n \tau} f_n\,d\tau$$

where f_n is independent of τ so we can compute

$$\int_0^t e^{-k\lambda_n\tau} f_n\,d\tau = f_n \frac{e^{-k\lambda_n\tau}}{-k\lambda_n}\bigg|_{\tau=0}^{\tau=t} = f_n\left(\frac{1-e^{-k\lambda_n t}}{k\lambda_n}\right).$$

Thus we have

$$c_n(t) = e^{k\lambda_n t}b_n + e^{k\lambda_n t} f_n\left(\frac{1-e^{-k\lambda_n t}}{k\lambda_n}\right)$$

or

$$c_n(t) = e^{k\lambda_n t}b_n + f_n\left(\frac{e^{k\lambda_n t}-1}{k\lambda_n}\right)$$

Therefore the solution can be written

$$u(x,t) = \sum_{n=1}^\infty b_n e^{k\lambda_n t}\varphi_n(x) + \sum_{n=1}^\infty f_n\left(\frac{e^{k\lambda_n t}-1}{k\lambda_n}\right)\varphi_n(x).$$

Notice that as $t\to\infty$ the solution converges to a time independent steady state solution

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^\infty \left(\frac{f_n}{-k\lambda_n}\right)\varphi_n(x) \equiv G(x).$$

Notice that this "steady state" function $G(x)$ is a solution to the problem with initial condition $u(x,0) = G(x)$, i.e., $u(x,t) = G(x)$ satisfies $u_t - ku_{xx} = f(x)$ since (note u is independent of t so $u_t = 0$

$$\begin{aligned} u_t - ku_{xx} &= -k\sum_{n=1}^\infty \left(\frac{f_n}{-k\lambda_n}\right)\varphi_n''(x) \\ &= \sum_{n=1}^\infty \left(\frac{f_n}{\lambda_n}\right)\lambda_n\varphi_n(x) = \sum_{n=1}^\infty f_n\varphi_n(x) = f. \end{aligned}$$

And clearly $u(x,t)$ satisfies the boundary conditions since every φ_n does.

Remark 6.1. We learn two very important things from this example:

1. The solution approaches a (generally) non-constant *steady state*, i.e., the solution consists of two parts which are often referred to as the *transient* and the *steady state*.

$$u(x,t) = u_{\text{trans}}(x,t) + u_{\text{ss}}(x,t) \approx u_{\text{ss}}(x,t) \quad \text{as} \quad t \to \infty.$$

For this problem we have

$$u_{\text{ss}}(x,t) = G(x)$$

which is independent of t.

2. While we did not take advantage of this fact, it is clear from looking at the derivation of the solution that the *principle of Superposition* holds which would have allowed us to analyze two simpler problems rather than one harder one. The solution to the problem

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + F(x,t), \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned}$$

can be written as a sum $u(x,t) = w(x,t) + v(x,t)$ where w and v are the solutions of the simpler problems

$$\boxed{\begin{aligned} w_t(x,t) &= kw_{xx}(x,t) + F(x,t), \\ w(x,0) &= 0, \quad w(\ell,t) = 0 \\ w(x,0) &= 0 \end{aligned}} \qquad \boxed{\begin{aligned} v_t(x,t) &= kv_{xx}(x,t), \\ v(x,0) &= 0, \quad v(\ell,t) = 0 \\ v(x,0) &= \varphi(x) \end{aligned}}$$

3. For a time independent forcing term, i.e., for $F(x,t) = f(x)$, and homogeneous Dirichlet boundary conditions the solution $u(x,t)$ converges to a steady state function $G(x)$ as t goes to infinity. To find $G(x)$ we only need to solve the associated steady state problem for (6.8). Recall that (6.8) is

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + f(x), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned}$$

and the steady state problem is obtained by setting $u_t = 0$

$$\begin{aligned} \psi''(x) &= -\frac{1}{k} f(x), \quad 0 < x < \ell, \\ \psi(0) &= 0, \quad \psi(\ell) = 0. \end{aligned}$$

Notice this is a non-homogeneous second order constant coefficient boundary value problem.

Example 6.2. Find the steady state solution for the heat problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) - 6x, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= 0, \quad u(1,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned}$$

As described in the remark the steady state problem is obtained by setting $u_t = 0$ and solving the non-homogeneous BVP

$$\begin{aligned} \psi''(x) &= 6x, \quad 0 < x < 1, \\ \psi(0) &= 0, \quad \psi(1) = 0 \end{aligned}$$

For this problem we apply the techniques from an elementary ODE class. Namely, we know that the general solution is the sum of the general solution of the homogenous problem ψ_h and any particular solution ψ_p. The general solution of the homogeneous problem $\psi''(x) = 0$ is $\psi_h(x) = c_1 x + c_2$ and it is clear that $\psi_p(x) = x^3$ is a particular solution. *N.B. Remember we learned two methods to find a particular solution: Undetermined Coefficients, Variation of Parameters.*

So we have $\psi(x) = \psi_h(x) + \psi_p(x) = c_1 x + c_2 + x^3$. Now we try to find c_1 and c_2 so that the boundary conditions are satisfied. We need

$$0 = \psi(0) = c_2, \quad \text{and} \quad 0 = \psi(1) = c_1 + 1^3$$

which implies $c_1 = -1$ and

$$\psi(x) = x^3 - x.$$

Thus for every initial condition $\varphi(x)$ the solution $u(x,t)$ to this forced heat problem satisfies

$$\lim_{t\to\infty} u(x,t) = \psi(x).$$

In this next example we show that the steady state solution may be time dependent.

Time Dependent steady State

Example 6.3. Consider the problem

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) + f(x)\sin(t), \ 0 < x < 1, \\ u(0,t) &= 0, \quad u(1,t) = 0 \\ u(x,0) &= 0. \end{aligned}$$

In this example we have set $\ell = 1$ and for the initial condition and forcing terms we have set $\varphi(x) = 0$ and $F(x,t) = f(x)\sin(t)$. Notice that, by the superposition principle, there is no lose in generality by taking the initial condition $\varphi = 0$ since the problem breaks up into two independent problems, one depending on φ and the other depending of the forcing term $F(x,t)$. Furthermore the part corresponding to a non-zero initial condition will decay

exponentially to zero as t tends to infinity and so it will not contribute to the steady state solution.

For $F(x,t) = f(x)\sin(t)$ (6.7) becomes

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t e^{k\lambda_n(t-\tau)} F_n(\tau)\, d\tau \right) \sin(n\pi x)$$

where we compute $F_n(t)$ using (6.6) which gives

$$F_n(\tau) = 2\sin(\tau) \int_0^1 f(x)\varphi_n(x)\, dx = \sin(\tau) f_n, \quad f_n = 2\int_0^1 f(x)\varphi_n(x)\, dx.$$

Thus we have

$$\begin{aligned}
\int_0^t e^{k\lambda_n(t-\tau)} F_n(\tau)\, d\tau &= \int_0^t e^{k\lambda_n(t-\tau)} \sin(\tau) f_n\, d\tau \\
&= f_n e^{k\lambda_n t} \int_0^t e^{-k\lambda_n \tau} \sin(\tau)\, d\tau && (6.10) \\
&\text{(see calculation below)} && (6.11) \\
&= f_n \frac{\left[e^{k\lambda_n t} - \cos(t) - k\lambda_n \sin(t)\right]}{(1+(k\lambda_n)^2)}. && (6.12)
\end{aligned}$$

In the above we need to use a special form of integration by parts which we carry out here

$$\begin{aligned}
\int_0^t e^{-k\lambda_n \tau} \sin(\tau)\, d\tau &= \int_0^t e^{-k\lambda_n \tau} (-\cos(\tau))'\, d\tau \\
&= e^{-k\lambda_n \tau}(-\cos(\tau))\Big|_0^t - \int_0^t (-k\lambda_n) e^{-k\lambda_n \tau}(-\cos(\tau))\, d\tau \\
&= -e^{-k\lambda_n t}\cos(t) + 1 - k\lambda_n \int_0^t e^{-k\lambda_n \tau}(\sin(\tau))'\, d\tau \\
&= -e^{-k\lambda_n t}\cos(t) + 1 - k\lambda_n \left[e^{-k\lambda_n \tau}\sin(\tau)\Big|_0^t + k\lambda_n \int_0^t e^{-k\lambda_n \tau}\sin(\tau)\, d\tau \right] \\
&= -e^{-k\lambda_n t}\cos(t) + 1 - k\lambda_n e^{-k\lambda_n t)}\sin(t) - (k\lambda_n)^2 \int_0^t e^{-k\lambda_n \tau}\sin(\tau)\, d\tau.
\end{aligned}$$

Thus we have

$$(1+(k\lambda_n)^2) \int_0^t e^{-k\lambda_n \tau}\sin(\tau)\, d\tau = -e^{-k\lambda_n t}\cos(t) + 1 - k\lambda_n e^{-k\lambda_n t)}\sin(t)$$

so that

$$\int_0^t e^{-k\lambda_n \tau} \sin(\tau)\, d\tau = \frac{-e^{-k\lambda_n t}\cos(t) + 1 - k\lambda_n e^{-k\lambda_n t}\sin(t)}{(1+(k\lambda_n)^2)}.$$

Finally, then

$$e^{k\lambda_n t}\int_0^t e^{-k\lambda_n \tau} \sin(\tau)\, d\tau = \frac{-\cos(t) - k\lambda_n \sin(t) + e^{k\lambda_n t}}{(1+(k\lambda_n)^2)}.$$

With this we can write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{-\cos(t) - k\lambda_n \sin(t) + e^{k\lambda_n t}}{(1+(k\lambda_n)^2)} \right) f_n \sin(n\pi x) \tag{6.13}$$

Notice that in this case the steady state (which we denote by $v(x,t)$) is not independent of time. Namely we have

$$v(x,t) = \sum_{n=1}^{\infty} \left(\frac{-\cos(t) - k\lambda_n \sin(t)}{(1+(k\lambda_n)^2)} \right) f_n \sin(n\pi x)$$

which is a 2π periodic funtion of t. Notice that

$$\left(\frac{1}{\sqrt{(1+(k\lambda_n)^2)}} \right)^2 + \left(\frac{-k\lambda_n}{\sqrt{(1+(k\lambda_n)^2)}} \right)^2 = 1,$$

which implies there is an angle α_n so that

$$\sin(\alpha_n) = \frac{1}{\sqrt{1+(k\lambda_n)^2}}, \quad \cos(\alpha_n) = \frac{-k\lambda_n}{\sqrt{1+(k\lambda_n)^2}}$$

So that

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} \left(-\sin(\alpha_n)\cos(t) - \cos(\alpha_n)\sin(t) \right) f_n \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \sin(t-\alpha_n)\, f_n \sin(n\pi x) \end{aligned}$$

6.2 Non-Homogeneous Equation, Homogeneous Neumann BCs

As it turns out very little changes if we change the boundary conditions. We show this by considering the case of Neumann boundary conditions. Let us consider the problem

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) + F(x,t), \\ u_x(0,t) &= 0, \quad u_x(\ell,t) = 0 \\ u(x,0) &= \varphi(x) \end{aligned}$$

which can be written as a sum $u(x,t) = w(x,t) + z(x,t)$ where w and v are the solutions of the simpler problems

$$\boxed{\begin{aligned} w_t(x,t) &= kw_{xx}(x,t) + F(x,t), \\ w_x(x,0) &= 0, \quad w_x(\ell,t) = 0 \\ w(x,0) &= 0 \end{aligned}} \qquad \boxed{\begin{aligned} z_t(x,t) &= kz_{xx}(x,t), \\ z_x(x,0) &= 0, \quad z_x(\ell,t) = 0 \\ z(x,0) &= \varphi(x) \end{aligned}}$$

The main difference in this case is that the eigenvalues and eigenfunctions change. Recall that for Neumann conditions zero is a eigenvalue. We have $\lambda_0 = 0$, $\varphi_0(x) = 1$, $\mu_n = (n\pi/\ell)$, $\lambda_n = -\mu_n^2$, $\varphi_n(x) = \cos(\mu_n x)$.

For the problem for $z(x,t)$ we have We compute

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \varphi_n(x). \tag{6.14}$$

where

$$a_n = \frac{2}{\ell} \int_0^\ell \varphi(x)\varphi_n(x)\, dx. \tag{6.15}$$

Then the solution to the $v(x,t)$ equation is

$$z(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{k\lambda_n t} \varphi_n(x).$$

Next for the $w(x,t)$ problem we find $\{F_n(t)\}$ so that

$$F(x,t) = \frac{F_0(t)}{2} + \sum_{n=1}^{\infty} F_n(t)\varphi_n(x), \tag{6.16}$$

where

$$F_n(t) = \frac{2}{\ell} \int_0^\ell F(x,t)\varphi_n(x)\, dx. \tag{6.17}$$

We seek a solution to the w problem in the form

$$w(x,t) = \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t)\varphi_n(x)$$

and repeat verbatim the calculation carried out in the Dirichlet case except that now the initial condition for w is zero so that $c_n(0) =$ for all $n = 0, 1, 2, \cdots$.

In particular, using the orthogonality of the functions $\{\varphi_n\}$ we obtain an infinite sequence of ODES

$$\frac{dc_n}{dt}(t) - k\lambda_n c_n(t) = F_n(t), \quad n = 0, 1, 2, \cdots,$$

which gives

$$c_n(t) = e^{k\lambda_n t} \int_0^t e^{-k\lambda_n \tau} F_n(\tau)\, d\tau.$$

For $n = 0$ we have

$$c_0(t) = \int_0^t F_n(\tau)\, d\tau.$$

$$w(x,t) = \frac{\int_0^t F_n(\tau)\, d\tau}{2} + \sum_{n=1}^{\infty} \left(\int_0^t e^{k\lambda_n (t-\tau)} F_n(\tau)\, d\tau \right) \varphi_n(x) \tag{6.18}$$

Finally we obtain the desired solution $u(x,t)$ as

$$u(x,t) = z(x,t) + w(x,t)$$

6.3 Non-homogeneous Dirichlet Boundary Conditions

In this section we consider forcing through Dirichlet boundary conditions

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= \gamma_0(t), \quad u(\ell,t) = \gamma_1(t) \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.19}$$

In order to obtain a continuous solution we also need to impose the compatibility conditions

$$\varphi(0) = \gamma_0(0), \quad \varphi(0) = \gamma_1(0).$$

Our method to solve this problem is to transform it into a problem like the ones found in the previous section. In order to do this we introduce the function

$$h(x,t) = \gamma_0(t) + \frac{x}{\ell}(\gamma_1(t) - \gamma_0(t)).$$

Then we introduce a new function $v(x,t)$ by

$$v(x,t) = u(x,t) - h(x,t).$$

Our goal is to see what problem $v(x,t)$ satisfies. To this end we note that

$$h_{xx}(x,t) = 0 \quad h_t(x,t) = \frac{d\gamma_0}{dt}(t) + \frac{x}{\ell}\left(\frac{d\gamma_1}{dt}(t) - \frac{d\gamma_0}{dt}(t) \right)$$

We see that

$$v_t - k v_{xx} = (u(x,t) - h(x,t))_t - k\,(u(x,t) - h(x,t))_{xx} = -h_t(x,t)$$

and

$$v(0,t) = u(0,t) - h(0,t) = 0, \qquad v(\ell,t) = u(\ell,t) - h(\ell,t) = 0,$$

$$v(x,0) = u(x,0) - h(x,0) = \varphi(x) - \left(\gamma_0(0) + \frac{x}{\ell}(\gamma_1(0) - \gamma_0(0))\right) \equiv v_0(x). \tag{6.20}$$

Collecting this information we find that $v(x,t)$ satisfies

$$\begin{aligned} v_t(x,t) &= kv_{xx}(x,t) - h_t(x,t), \quad 0 < x < \ell, \quad t > 0 \\ v(0,t) &= 0, \quad v(\ell,t) = 0 \\ v(x,0) &= v_0(x) \end{aligned} \tag{6.21}$$

So we can apply the results of the last section to obtain a formula for $v(x,t)$.

Once we do this (see below) we can obtain the desired solution from

$$u(x,t) = v(x,t) + h(x,t).$$

Following the procedure outlined in (6.1)-(6.7) we proceed as follows. First as in (6.6) we compute $\{h_n(t)\}$ so that

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\varphi_n(x), \tag{6.22}$$

where

$$h_n(t) = \frac{2}{\ell}\int_0^{\ell} h(x,t)\varphi_n(x)\,dx. \tag{6.23}$$

Then for the initial condition we compute

$$\varphi(x) = \sum_{n=1}^{\infty} b_n\varphi_n(x). \tag{6.24}$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} \varphi_n(x)v_0(x)\,dx. \tag{6.25}$$

Combining these results we obtain

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x) - \sum_{n=1}^{\infty}\left(\int_0^t e^{k\lambda_n(t-\tau)}\frac{dh_n}{d\tau}(\tau)\,d\tau\right)\varphi_n(x) \tag{6.26}$$

and finally

$$u(x,t) = v(x,t) + h(x,t).$$

Example 6.4. Let us consider a very special case of the previous example. Suppose that $\gamma_0(t) = \alpha$ and $\gamma_1(t) = \beta$ (are constants) so that (6.19) becomes

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= \alpha, \quad u(\ell,t) = \beta \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.27}$$

In this case

$$h(x,t) = \alpha + (\beta - \alpha)\frac{x}{\ell} \equiv U(x)$$

and we have $h_t(x,t) = 0$ so that the integral terms in (6.26) are all zero. Also we have $v_0(x) = \varphi(x) - h(x,0) = \varphi(x) - U(x)$ and the equation for $v(x,t) = u(x,t) - h(x,t)$ becomes

$$\begin{aligned} v_t(x,t) &= kv_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ v(0,t) &= 0, \quad v(\ell,t) = 0 \\ v(x,0) &= v_0(x) \end{aligned}$$

and with we find

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x), \quad b_n = \frac{2}{\ell}\int_0^\ell v_0(x)\varphi_n(x)\,dx$$

and findally

$$u(x,t) = U(x) + \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\varphi_n(x).$$

Notice in this case that as $t \to \infty$ all the exponential terms in the sum tend to zero and we have

$$\lim_{t\to\infty} u(x,t) = U(x).$$

This represents a nonzero and non constant steady state temperature profile.

Example 6.5. We now consider one final example where the boundary forcing function is not a constant.

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0, \quad u(\ell,t) = \sin(t) \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.28}$$

In this case $\gamma_0(t) = 0$, $\gamma_1(t) = \sin(t)$ and $h(x,t) = (x/\ell)\sin(t)$. We need to compute

$$h_n(t) = \frac{2}{\ell}\int_0^\ell h(x,t)\varphi_n(x)\,dx = \sin(t)\,\frac{2}{\ell}\int_0^\ell x\varphi_n(x)\,dx \equiv \sin(t)c_n$$

and

$$b_n = \frac{2}{\ell}\int_0^\ell v_0(x)\varphi_n(x)\,dx.$$

Notice in this example, since $h(x,0) = 0$ we have $v_0(x) = \varphi(x) - h(x,0) = \varphi(x)$ (see (6.20)).

We can compute c_n explicitly and we have

$$
\begin{aligned}
\frac{2}{\ell}\int_0^\ell x\varphi_n(x)\,dx &= \frac{2}{\ell}\int_0^\ell x\sin(\mu_n x)\,dx \\
&= \frac{2}{\ell}\int_0^\ell x\left(-\frac{\cos(\mu_n x)}{\mu_n}\right)'\,dx \\
&= \frac{2}{\ell}\left[x\left(-\frac{\cos(\mu_n x)}{\mu_n}\right)\Big|_0^\ell - \int_0^\ell\left(-\frac{\cos(\mu_n x)}{\mu_n}\right)\,dx\right] \\
&= \frac{2}{\ell}\left[-\frac{\ell\cos(\mu_n\ell)}{\mu_n} + \frac{\sin(\mu_n x)}{\mu_n^2}\Big|_0^\ell\right] \\
&= \frac{2\ell(-1)^{n+1}}{n\pi}
\end{aligned}
$$

Then using (6.26) we have

$$
\begin{aligned}
v(x,t) &= \sum_{n=1}^\infty b_n e^{k\lambda_n t}\varphi_n(x) - \sum_{n=1}^\infty\left(\int_0^t e^{k\lambda_n(t-\tau)}h_n'(\tau)\,d\tau\right)\varphi_n(x) \\
&= \sum_{n=1}^\infty b_n e^{k\lambda_n t}\varphi_n(x) - \sum_{n=1}^\infty\left(\int_0^t e^{k\lambda_n(t-\tau)}\cos(\tau)\,d\tau\right)c_n\,\varphi_n(x) \\
&= \sum_{n=1}^\infty b_n e^{k\lambda_n t}\varphi_n(x) - \sum_{n=1}^\infty\left(\frac{k\lambda_n e^{k\lambda_n t} - k\lambda_n\cos(t) + \sin(t)}{1+(k\lambda_n)^2}\right)c_n\,\varphi_n(x) \\
&= \sum_{n=1}^\infty\left(b_n e^{k\lambda_n t} - \frac{k\lambda_n e^{k\lambda_n t}}{1+(k\lambda_n)^2}\,c_n\right)\varphi_n(x) + \sum_{n=1}^\infty\left(\frac{k\lambda_n\cos(t)-\sin(t)}{1+(k\lambda_n)^2}\right)c_n\,\varphi_n(x) \\
&= v_{\text{trans}}(x,t) + v_{\text{ss}}(x,t).
\end{aligned}
$$

Note that $v_{\text{trans}}(x,t)\to 0$ as $t\to\infty$ and $v_{\text{ss}}(x,t)$ is a steady state periodic function of t.

Finally then we return to our original solution $u(x,t)$ to obtain $u(x,t) = v(x,t) + h(x,t)$ which gives

$$u(x,t) = u_{\text{trans}}(x,t) + u_{\text{ss}}(x,t)$$

where

$$u_{\text{trans}}(x,t) = v_{\text{trans}}(x,t) = \sum_{n=1}^\infty\left(b_n e^{k\lambda_n t} - \frac{k\lambda_n e^{k\lambda_n t}}{1+(k\lambda_n)^2}\,c_n\right)\varphi_n(x)$$

and

$$u_{\text{ss}}(x,t) = \frac{x}{\ell}\sin(t) + v_{\text{ss}}(x,t) = \frac{x}{\ell}\sin(t) + \sum_{n=1}^\infty\left(\frac{k\lambda_n\cos(t)-\sin(t)}{1+(k\lambda_n)^2}\right)c_n\,\varphi_n(x).$$

So we find that forcing the boundary with a periodic function produces a solution that consists of a transient part that goes to zero plus a steady state part that is a periodic solution with the same period as the forcing term.

6.4 Non-homogeneous Neumann Boundary Conditions

In this section we consider forcing through Neumann boundary conditions.

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \\ u_x(0,t) &= \gamma_0(t), \quad u_x(\ell,t) = \gamma_1(t) \\ u(x,0) &= \varphi(x) \end{aligned} \tag{6.29}$$

The primary difference between this problem and that considered in the previous section (i.e., (6.19)) is that we need a different function $h(x,t)$.

In order to find an appropriate function $h(x,t)$ lets us examine the properties we desire. We want a function that satisfies two conditions:

$$h_x(0,t) = \gamma_0(t), \quad h_x(\ell,t) = \gamma_1(t)$$

First we find two functions $\alpha_0(x)$ and $\alpha_1(x)$ satisfying

$$\alpha_0'(0) = 1, \quad \alpha_0'(\ell) = 0, \quad \alpha_1'(0) = 0, \quad \alpha_1'(\ell) = 1.$$

We find, for example,

$$\alpha_0(x) = -\frac{(\ell - x)^2}{2\ell}, \quad \alpha_1(x) = \frac{x^2}{2\ell}.$$

Then we take

$$h(x,t) = \alpha_0(x)\gamma_0(t) + \alpha_1(x)\gamma_1(t)$$

and set

$$u(x,t) = v(x,t) - h(x,t),$$

which implies

$$v(x,t) = u(x,t) - h(x,t).$$

With this we can compute (just as above)

$$v_t - kv_{xx} = (u(x,t) - h(x,t))_t - k\,(u(x,t) - h(x,t))_{xx} = -h_t(x,t) + kh_{xx}(x,t)$$

where

$$h_{xx} = -\frac{1}{\ell}\gamma_0(t) + \frac{1}{\ell}\gamma_1(t), \quad h_t(x,t) = \alpha_0(x)\frac{d\gamma_0(t)}{dt} + \alpha_1(x)\frac{d\gamma_1(t)}{dt}.$$

So we set

$$F(x,t) = -h_t(x,t) + kh_{xx}(x,t).$$

So to find the function $v(x,t)$ we need only solve

$$\begin{aligned} v_t(x,t) &= kv_{xx}(x,t) + F(x,t), \quad 0 < x < \ell, \quad t > 0 \\ v(0,t) &= 0, \quad v(\ell,t) = 0 \\ v(x,0) &= v_0(x) = \varphi(x) - h(x,0). \end{aligned} \tag{6.30}$$

Notice once again that the initial condition v_0 is not just the original initial condition. Finally we obtain

$$u(x,t) = v(x,t) + h(x,t).$$

7 Assignment

1. Solve the non-homogeneous heat problem

$$u_t(x,t) = u_{xx}(x,t) + F(x,t), \quad 0 < x < \pi, \quad t > 0$$

   (a) $F(x,t) = 2\sin(3x)$, BC: $u(0,t) = 0$, $u(\pi,t) = 0$, IC: $u(x,0) = \sin(2x)$.
   Also find the steady state.

   (b) $F(x,t) = \sin(x) - 2\sin(2x)$, BC: $u(0,t) = 0$, $u(\pi,t) = 0$, IC: $u(x,0) = 0$.
   Also find the steady state.

   (c) $F(x,t) = e^{-t}\sin(x)$, BC: $u(0,t) = 0$, $u(\pi,t) = 0$, IC: $u(x,0) = 0$.
   Also find the steady state.

   (d) $F(x,t) = -\cos(x)$, BC: $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, IC: $u(x,0) = 1$.
   Also find the steady state solution.

2. Find the steady state solution $\psi(x)$ for the problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \cos(x), \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) &= 0, \quad u(\pi,t) = 0 \\ u(x,0) &= x(\pi - x) \end{aligned}$$

3. Find the steady state solution $\psi(x)$ for the problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + 2, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= 0, \quad u(1,t) = 0 \\ u(x,0) &= \sin(x)e^x \end{aligned}$$

4. Solve the heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$

   with non-homogeneous boundary conditions

   (a) BC: $u(0,t) = 1$, $u(1,t) = 1$ IC: $u(x,0) = 0$

   (b) BC: $u_x(0,t) = 1$, $u_x(1,t) = 1$, IC: $u(x,0) = 0$

   (c) BC: $u(0,t) = 0$, $u(1,t) = 2$, IC: $u(x,0) = 0$

   (d) BC: $u(0,t) = 0$, $u(1,t) = e^{-t}$, IC: $u(x,0) = x$