3 Initial Value Problem for the Heat Equation

3.1 Derivation of the equations

Suppose that a function u represents the temperature at a point x on a rod. The value of this function will change with time t as the heat spreads over the length of the rod. Thus $u = u(x,t)$ is a function of the spatial point x and the time t. Our first objective is to derive a partial differential equation satisfied by the temperature under some standard assumptions. We assume that the rod has a constant cross section surrounded by insulation so that the heat can only flow along the rod and not into the surrounding media. We also assume the rod is very long so we can neglect what happens at the ends – indeed for practical purposes we can assume the rod is infinite in extent. We assume that the temperature is uniform across each cross section. Let c be the *specific heat* of the material, i.e., *the amount of heat energy (e.g., in calories) needed to raise the temperature of a unit mass of the material one degree centigrade.* We will assume that c does not depend on t but it may depend on x. We also let ρ be the linear density of the material which may also depend on x. Then the amount of heat energy in an interval of the rod between points $x = a$ and $x = b$ is given by

$$\int_a^b u(x,t)c(x)\rho(x)\,dx.$$

The rate of change of heat energy in the portion of the rod from $x = a$ to $x = b$ is given by

$$\frac{d}{dt}\int_a^b u(x,t)c(x)\rho(x)\,dx.$$

Now the rate of change of energy in $[a,b]$ is the rate at which heat enters and leaves this portion of the rod through its ends at $x = a$ and $x = b$. Thus we can write

$$\frac{d}{dt}\int_a^b u(x,t)c(x)\rho(x)\,dx = F(a,t) - F(b,t),$$

where $F(x,t)$ is the *flux*, i.e., *the rate at which heat energy passes the point x.* We make the convention that $F(x,t) \geq 0$ if the heat energy is flowing from left to right. Now if u is continuously differentiable then we can also write

$$\frac{d}{dt}\int_a^b u(x,t)c(x)\rho(x)\,dx = \int_a^b \frac{\partial u}{\partial t}(x,t)c(x)\rho(x)\,dx.$$

And, if F is continuously differentiable then we can also write

$$F(a,t) - F(b,t) = -F(x,t)\Big|_{x=a}^{x=b} = -\int_a^b \frac{\partial F}{\partial x}(x,t)\,dx.$$

Thus if u and F are sufficiently smooth (continuously differentiable), then

$$\int_a^b \left[\frac{\partial u}{\partial t}(x,t)c(x)\rho(x) + \frac{\partial F}{\partial x}(x,t)\right] dx = 0 \qquad \forall \quad a,\ b. \tag{3.1}$$

Now we appeal to an important fundamental principal: If for a smooth function f we have

$$\int_a^b f(x)\, dx = 0 \quad \text{for every pair} \quad a,\ b, \quad \text{Then} \quad f(x) = 0\ \ \forall\ \ x. \tag{3.2}$$

To see why this is true suppose there is a number x_0 so that $f(x_0) \neq 0$ (without loss of generality we can assume $f(x_0) > 0$ since otherwise we can consider the function $(-f)$). Now since f is assumed to be smooth there must exist numbers $a < x_0 < b$ so that $f(x) > 0$ on $[a, b]$. But then we would have

$$\int_a^b f(x)\, dx > 0,$$

which is a contradiction to our assumption in (3.2).

Therefore we can conclude from (3.1) and (3.2) that

$$\frac{\partial u}{\partial t}(x,t)c(x)\rho(x) + \frac{\partial F}{\partial x}(x,t) \qquad \forall\ \ x, \quad t. \tag{3.3}$$

Next we can obtain a relation for the heat flux F in terms of the temperature u by appealing to *Fourier's law* which states that

$$F(x,t) = -k\frac{\partial u}{\partial x}(x,t),$$

where k (the *heat conductivity*) is a property of the material in the rod and may depend on x. In particular, Fourier's law states that the rate at which heat energy crosses a surface (i.e., the heat flux $F(x,t)$) is proportional to the temperature gradient (in one dimension this means $\partial u/\partial x$) at the surface.

Using this in (3.3) we have

$$\frac{\partial u}{\partial t}(x,t)c(x)\rho(x) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right) = 0.$$

Finally then we can write

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right). \tag{3.4}$$

In this course we will consider the simplest case in which $c = \rho = 1$ and k is also a constant. Thus we can write our heat equation as

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t). \tag{3.5}$$

3.2 Solution of the Initial Value Problem

The objective of this section is to derive a formula for the solution to Initial Value Problem (IVP) for the one dimensional heat equation on $\mathbb{R} = \{x \ : \ -\infty < x < \infty\}$. This problem can be stated in mathematical terms as

$$u_t(x,t) = k u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \tag{3.6}$$

$$u(x,0) = f(x), \quad -\infty < x < \infty, \tag{3.7}$$

$$|u(x,t)| < M < \infty \quad \text{for some} \quad M \quad \forall\, x \in \mathbb{R}, \quad t \geq 0. \tag{3.8}$$

Here $f(x)$ in (3.7) is the initial temperature distribution (or initial condition) and the condition in (3.8) says that we we seek a bounded solution.

A careful derivation of the solution to the heat problem (3.6)-(3.8) is much more involved and technical than our relatively simple calculations using characteristics for the wave equation. In this course we will present an important formula for the solution and discuss some of its properties. A more detailed discussion of the derivation is contained in two Appendices to this set of notes.

The solution that we give is written in terms of the so-called *the Gaussian or heat kernel* by

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy. \tag{3.9}$$

The heat kernel is

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-x^2/(4kt)} \tag{3.10}$$

and the solution (3.9) is the convolution in the x variable over the whole real number line of $S(x,t)$ with the initial temperature distribution $f(x)$.

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, f(y)\, dy.$$

One of the exercises for this section is to show that $S(x,t)$ satisfies the heat equation. The most difficult thing to show is that the solution given in (3.9) converges to the initial condition $f(x)$ as t goes to zero. This point is the main reason for the discussion on delta sequences and the delta function provided in the Appendix A. The results of this Appendix show that

$$f(x) = u(x,0) = \lim_{t \downarrow 0} \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy.$$

The heat kernel is derived in the Appendix (section B). At this point you might want to go to the Appendix to read the derivation of the solution of the IVP for the heat equation (3.6). In the Appendix we show how the heat kernel allows us to obtain the solution (3.9) by way of delta-sequences.

It would seem fairly obvious that if no additional heat is being applied to the rod then the rod cannot become hotter than it starts out at time $t = 0$. More precisely, the first statement of the 2nd law of thermodynamics, which says that heat flows from a hot to a cold body, tells us the obvious fact that an ice cube must melt on a hot day, rather than becoming colder. This means that a point on the rod cannot become hotter but might possibly become cooler since heat flows from hot to cold. Mathematically this means that we should have something like

$$\sup_{(x,t)} |u(x,t)| \leq \sup_{x} |u(x,0)|.$$

This can be made more precise in the form of a result called the *Maximum Principle.*

There are a couple of important points we need to address at this point. The first is that we will only concern ourselves in this class with bounded and piecewise continuous initial conditions $f(x)$. So, in particular, we assume that there is a constant $M_0 > 0$ so that

$$|f(x)| \leq M_0 \quad \text{for all } x \in \mathbb{R}.$$

Next, in this class, we will only consider so-called *strict solutions* by which we mean solutions which are sufficiently differentiable that the equation is satisfied in the sense of regular calculus. Namely we will study problems for which, for $t > 0$, u, u_t, u_x and u_{xx} are all continuous.

With these assumptions we can give a mathematical proof of the maximum pricnciple for the heat equation. The proof is given in Appendix D.

Theorem 3.1. *Assume the u is a bounded strict solution to* (3.6)-(3.8) *with $|f(x)| \leq M_0$ for all $x \in \mathbb{R}$ and let $T > 0$ be arbitrary. Define*

$$B_T = \{(x,t) \,:\, x \in \mathbb{R}, \quad 0 < t < T\}, \quad \overline{B_T} = \{(x,t) \,:\, x \in \mathbb{R}, \quad 0 \leq t \leq T\}.$$

Since u is bounded there is an M so that $u(x,t) \leq M$ for all $(x,t) \in \overline{B_T}$. Then we have

$$u(x,t) \leq M_0 \quad \textit{for all} \quad (x,t) \in \overline{B_T}.$$

Under the same assumptions as above we can now show that the solution to our heat problem is unique. This means that every solution (satisfying the conditions of the Theorem) must be given by (3.9).

Theorem 3.2. *Assume that $|f(x)| \leq M_0$ for all $x \in \mathbb{R}$. Then there is a unique, bounded, strict solution to* (3.6)-(3.8) *satisfying*

$$u(x,t) \leq M_0 \quad \textit{for all} \quad x \in \mathbb{R} \textit{ and } t > 0.$$

In addition u is actually C^∞ in x and t.

Proof. **Part 1. Solutions Exist.** We show that (3.9) is a solution to (3.6)-(3.8) proving that a solution exists. To this end we first notice that the heat kernel at $(x - y)$ for any fixed x and t, given (3.10) by

$$S(x-y,t) = \frac{1}{\sqrt{4k\pi t}} e^{-(x-y)^2/(4kt)}$$

goes to zero exponentially fast as $|y| \to \infty$ so that the improper integral defining the convolution has no problem converging. Furthermore we can differential the convolution with respect to x or t as many times as we like and pull the derivatives inside the integral without problem due to the rapid convergence to zero of the kernel as $|y| \to \infty$. Namely as an example we can compute u_t and we have (for any fixed $x \in \mathbb{R}$ and any fixed $t > 0$)

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) =& \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4k\pi t}}\right)\int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy \\ &+ \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} e^{-(x-y)^2/(4kt)}\, f(y)\, dy \\ =& -\frac{1}{2\sqrt{4k\pi}\, t^{3/2}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy \\ &+ \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \frac{(x-y)^2}{4kt^2} e^{-(x-y)^2/(4kt)}\, f(y)\, dy\end{aligned}$$

While the formulas become unwieldy the calculations are justified, the integrals exist and we have

$$\frac{\partial^k u}{\partial t^k} = \int_{\mathbb{R}} \frac{\partial^k}{\partial t^k} S(x-y,t)\, f(y)\, dy,$$

and

$$\frac{\partial^k u}{\partial x^k} = \int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} S(x-y,t)\, f(y)\, dy.$$

Thus, for example, we have (for any $x \in \mathbb{R}$ and any $t > 0$)

$$u_t - ku_{xx} = \int \left[S_t(x-y,t) - kS_{xx}(x-y,t)\right]\, f(y)\, dy = 0.$$

This follows from your first homework problem in this chapter. Therefore u is a strict solution of the heat equation. We also calim that it satisfies the initial conditions as a consequence of the discussion on delta sequences given in the Appendix A. Namely we claim that

$$\lim_{t\to 0} u(x,t) = f(x) \quad \text{ for each } \; x.$$

In particular recalling the definition of the delta sequence γ_r in (A.3) we can notice that

$$S(x-y,t) = \gamma_r(x-y) = \sqrt{\frac{r}{\pi}}\, e^{-r(x-y)^2} \quad \text{with} \quad r = \frac{1}{4kt}.$$

Therefore we have

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\, f(y)\, dy = \int_{\mathbb{R}} \gamma_r(y-x)\, f(y)\, dy$$

since $\gamma_r(x-y) = \gamma_r(y-x)$. Notice that as $t \to 0$ we have $r \to \infty$ so we can apply Theorem A.1 (together with part 3 of Remark A.1 concerning shifting the value from zero in a delta sequence) to conclude that $u(x,0) = f(x)$.

Part 2. **Show Solution is Bounded**. To show this we use the facts that for any x, y and $t > 0$

$$S(x-y,t) > 0 \quad \text{and} \quad \int_{\mathbb{R}} S(x-y,t)\,dy = 1.$$

So we have

$$|u(x,t)| = \left| \int_{-\infty}^{\infty} S(x-y,t)\, f(y)\,dy \right| \le M_0 \int_{\mathbb{R}} S(x-y,t)\,dy \le M_0.$$

Part 3. **The Solution is Unique.** To prove this we use the maximum principle. Suppose that u and v are two bounded solutions of (3.6)-(3.8) with the same initial function $f(x)$. Then $w = u - v$ is a solution of (3.6)-(3.8) with initial function $w(x,0) = 0$. Now by the maximum principle we have

$$w(x,t) = u(x,t) - v(x,t) \le \max_{x \in \mathbb{R}} \left(u(x,0) - v(x,0) \right) = 0.$$

Reversing the roles of u and v we also obtain that

$$-w(x,t) = v(x,t) - u(x,t) \le \max_{x \in \mathbb{R}} \left(v(x,0) - u(x,0) \right) = 0.$$

So we

$$w(x,t) \le 0, \quad -w(x,t) \le 0 \quad \Rightarrow \quad w(x,t) = 0$$

and we conclude that $u(x,t) = v(x,t)$ so that (3.6)-(3.8) has a single solution.

□

It should be obvious that it is extremely difficult to compute explicitly the values of u for general initial data f. From a computational point of view it would be nice to be able to write the solution in terms of known functions. For functions that are piecewise constant we can express the solution in terms of the so called "error function" denoted by $\operatorname{erf}(x)$ and given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\,dx.$$

Notice that we can use the properties of integrals to deduce that

$$\operatorname{erf}(-x) = -\operatorname{erf}(x).$$

So for example if we have the initial function

$$f(x) = \begin{cases} \alpha, & x < 0 \\ 0, & x \ge 0 \end{cases}$$

then

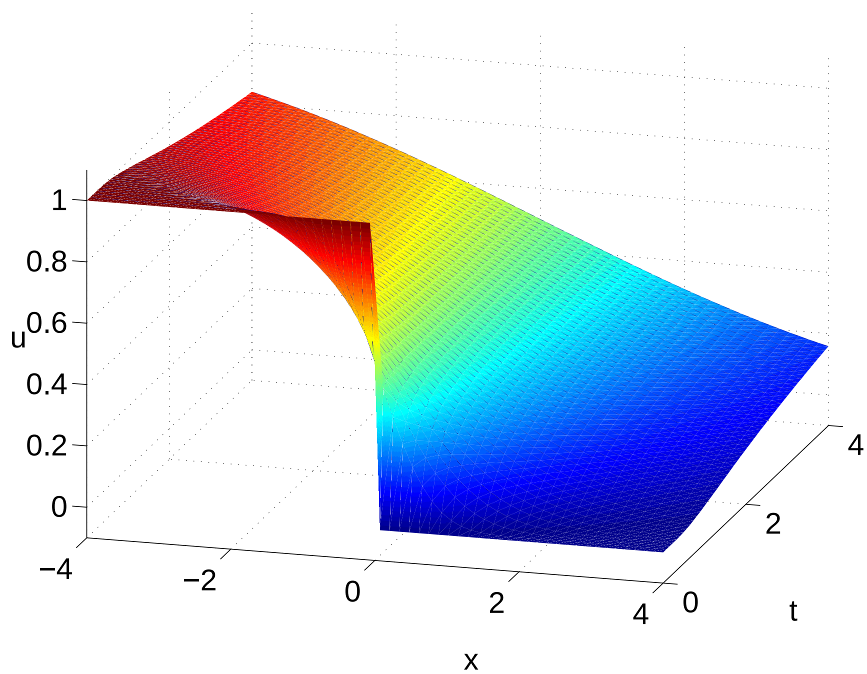
$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\,dy = \frac{\alpha}{\sqrt{4k\pi t}} \int_{-\infty}^{0} e^{-(x-y)^2/(4kt)}\,dy.$$

Make the change of variables

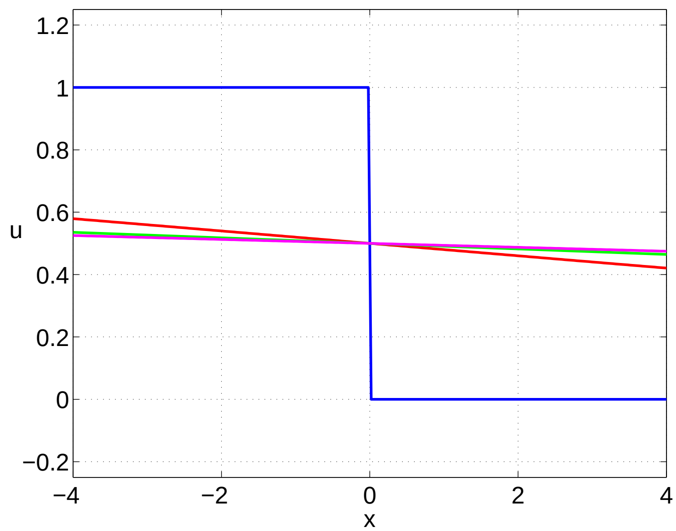
$$z = \frac{(y-x)}{\sqrt{4kt}}, \quad \Rightarrow \quad dz = \frac{1}{\sqrt{4kt}}\, dy.$$

This gives

$$\begin{aligned} u(x,t) &= \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{-x/\sqrt{4kt}} e^{-z^2}\, dz \\ &= \frac{\alpha}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-z^2}\, dz \\ &= \frac{\alpha}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-z^2}\, dz - \int_0^{x/\sqrt{4kt}} e^{-z^2}\, dz \right) \\ &= \frac{\alpha}{2} \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}} \right) \right). \end{aligned}$$



Solution Surface



Solution Curves $t = 100$, $t = 500$, $t = 1000$

3.3 Dirichlet Condition on a Half Line

$$u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0 \tag{3.11}$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \tag{3.12}$$

$$u(0,t) = 0 \tag{3.13}$$

$$|u(x,t)| < M < \infty \quad \text{for some} \quad M$$

To solve this problem we extend the initial data f as an odd function to all of $\mathbb{R}$ as

$$F_0(x) = \begin{cases} f(x) & 0 < x < \infty \\ -f(-x) & -\infty < x < 0 \end{cases}.$$

Then we consider the problem

$$v_t(x,t) = k v_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \tag{3.14}$$

$$v(x,0) = F_0(x), \quad -\infty < x < \infty,$$

$$|v(x,t)| < M < \infty \quad \text{for some} \quad M.$$

We can solve the problem (3.14) using (3.9) to obtain

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, F_0(y)\, dy \\
&= \frac{1}{\sqrt{4k\pi t}} \left[\int_{-\infty}^{0} e^{-(x-y)^2/(4kt)}\, F_0(y)\, dy + \int_{0}^{\infty} e^{-(x-y)^2/(4kt)}\, F_0(y)\, dy \right] \\
&= \frac{1}{\sqrt{4k\pi t}} \left[-\int_{0}^{\infty} e^{-(x+y)^2/(4kt)}\, f(y)\, dy + \int_{0}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy \right] \\
&= \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[e^{-(x-y)^2/(4kt)} - e^{-(x+y)^2/(4kt)} \right] f(y)\, dy.
\end{aligned}$$

Now this function solves the heat equation on $(-\infty, \infty)$ and at $t = 0$ it is $F_0(x)$ so it satisfies (3.6) also on $(0, \infty)$ and

$$v(0,t) = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[e^{-y^2/(4kt)} - e^{-y^2/(4kt)} \right] f(y)\, dy = 0.$$

So the solution to (3.11) is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[e^{-(x-y)^2/(4kt)} - e^{-(x+y)^2/(4kt)} \right] f(y)\, dy. \tag{3.15}$$

3.4 Neumann Condition on a Half Line

$$\begin{aligned}
u_t(x,t) &= k u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0 \qquad (3.16)\\
u(x,0) &= f(x), \quad 0 < x < \infty, \\
u_x(0,t) &= 0 \\
|u(x,t)| &< M < \infty \quad \text{for some } M
\end{aligned}$$

To solve this problem we extend the initial data f as an even function to all of $\mathbb{R}$ as

$$F_e(x) = \begin{cases} f(x) & 0 < x < \infty \\ f(-x) & -\infty < x < 0 \end{cases}.$$

Then we solve the problem

$$\begin{aligned}
v_t(x,t) &= k v_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0 \qquad (3.17)\\
v(x,0) &= F_e(x), \quad -\infty < x < \infty, \\
|v(x,t)| &< M < \infty \quad \text{for some } M
\end{aligned}$$

to obtain

$$
\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, F_e(y)\, dy \\
&= \frac{1}{\sqrt{4k\pi t}} \left[\int_{-\infty}^{0} e^{-(x-y)^2/(4kt)}\, F_e(y)\, dy + \int_{0}^{\infty} e^{-(x-y)^2/(4kt)}\, F_e(y)\, dy \right] \\
&= \frac{1}{\sqrt{4k\pi t}} \left[\int_{0}^{\infty} e^{-(x+y)^2/(4kt)}\, f(y)\, dy + \int_{0}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\, dy \right] \\
&= \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[e^{-(x-y)^2/(4kt)} + e^{-(x+y)^2/(4kt)} \right] f(y)\, dy.
\end{aligned}
$$

Now this function solves the heat equation on $(-\infty, \infty)$ and at $t = 0$ it is $F_e(x)$ so it satisfies (3.6) also on $(0, \infty)$ and

$$
v_x(0,t) = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[\frac{-2(x+y)}{4kt} e^{-(x+y)^2/(4kt)} + \frac{-2(x-y)}{4kt} e^{-(x-y)^2/(4kt)} \right] f(y)\, dy = 0.
$$

So we have

$$
v_x(0,t) = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \frac{2y}{4kt} \left[-e^{-y^2/(4kt)} + e^{-y^2/(4kt)} \right] f(y)\, dy = 0.
$$

So the solution to (3.16) is

$$
u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{0}^{\infty} \left[e^{-(x-y)^2/(4kt)} + e^{-(x+y)^2/(4kt)} \right] f(y)\, dy. \tag{3.18}
$$

3.5 Assignment Heat Equation

1. Show that for all $t > 0$ and all $x \in \mathbb{R}$ $S(x,t) = \dfrac{1}{\sqrt{4k\pi t}} e^{-x^2/(4kt)}$ satisfies the heat equation $\dfrac{\partial S}{\partial t}(x,t) = k \dfrac{\partial^2 S}{\partial x^2}(x,t)$.

2. Consider the IVP (3.6)-(3.8) with $f(x) = 1$ for all $x \in \mathbb{R}$. How would you argue that the problem has a unique solution. Clearly the function $u(x,t) = 1$ for all x and t solves the problem (3.6)-(3.8). So, assuming solutions are unique, the remarkable formula

$$
1 = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, dy
$$

must hold for all x, k and t.

Show that this is indeed the case as follows: introduce the change of variables

$$
s = \frac{(x-y)}{\sqrt{4kt}} \quad \text{which implies} \quad ds = -\frac{dy}{\sqrt{4kt}}
$$

and, for any fixed but arbitrary x and t we obtain

$$1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2}\, ds.$$

Use our results from Section 3.2 of these notes to conclude this is indeed the case.

3. Consider the following heat problem

$$u_t(x,t) = u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x) = \begin{cases} T_0, & x < 0 \\ T_1, & x > 0 \end{cases}.$$

Find the solution in terms of the complementary error function erfc(x) defined by

$$\text{erfc}\,(x) = \frac{2}{\pi} \int_x^{\infty} e^{-s^2}\, ds.$$

$$\begin{aligned} u(x,t) &= \frac{T_0}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-s^2}\, ds + \frac{T_1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4t}} e^{-s^2}\, ds \\ &= \frac{T_0}{2} \text{erfc}\left(\frac{x}{\sqrt{4t}}\right) - \frac{T_1}{2} \text{erfc}\left(\frac{x}{\sqrt{4t}}\right) \end{aligned} \tag{3.19}$$

4. Assume that $k = 1$ and show that

$$u(x,t) = \frac{1}{(1+4t)^{1/2}} \exp\left(-\frac{x^2}{(1+4t)}\right)$$

is a solution to the heat equation (3.6)-(3.8) with

$$u(x,0) = f(x) = e^{-x^2}$$

and satisfying

$$\lim_{|a|\to\infty} u_x(a,t) = 0, \quad \int_{\mathbb{R}} f(x)^2\, dx < \infty.$$

Since the solution is unique we must have

$$\frac{1}{(1+4t)^{1/2}} \exp\left(-\frac{x^2}{(1+4t)}\right) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, e^{-y^2}\, dy.$$

5. Show that the following functions all satisfy the heat equation on $\mathbb{R}$ with the given initial condition. In these problems A and B are arbitrary constants.

   (a) $f(x) = Ax + B$ and $u(x,t) = Ax + B$

   (b) $f(x) = Ax^3 + B$ and $u(x,t) = A(x^3 + 6ktx) + B$

   (c) $f(x) = Ae^x$ and $u(x,t) = Ae^{kt+x}$

6. Suppose that $u(x,t)$ is a bounded strict solution to the heat equation on the whole line with $k=1$ and with initial condition $u(x,0)=\arctan(x)$. Let $Q=\{(x,t) : -\infty < x < \infty,\ t \ge 0\}$. Find M_0 so that

$$\max_{(x,t)\in Q} |u(x,t)| \le M_0.$$

7. Let $I_k = \displaystyle\int_{-\infty}^{\infty} \frac{\sin^2(k(x-1))}{\pi k (x-1)^2} \cos\left(\frac{\pi x}{4}\right)\, dx$. Find $\displaystyle\lim_{k\to\infty} I_k$.

8. The solution to the Initial, Boundary Value Problem for the heat problem

$$\begin{aligned} &u_t(x,t) = k u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0 \\ &u_x(0,t) = 0 \\ &u(x,0) = f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}. \end{aligned}$$

   Using the formula (3.18), i.e.,

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} \left[e^{-(x-y)^2/(4kt)} + e^{-(x+y)^2/(4kt)} \right] f(y)\, dy$$

   the solution with our initial condition can be written as

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{f_1(x,t)}^{f_2(x,t)} e^{-s^2}\, ds.$$

   Find $f_1(x,t)$ and $f_2(x,t)$. (Hint: Write the integral as a sum of two integrals and then use the substitution $s = (x-y)/\sqrt{4kt}$ in the first integral and $s = (x+y)/\sqrt{4kt}$ in the second.)

A Appendix: Delta Sequences and the Delta Function

In order to understand how the solution presented in (3.9) converges to the initial condition $f(x)$ as t goes to zero involves the need for a very brief introduction to a more advanced topic - the *delta sequence*. This involves showing that

$$f(x) = u(x,0) = \lim_{t\downarrow 0} \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} f(y)\, dy. \tag{A.1}$$

To help understand why this formula holds we will describe both delta sequences and the delta function.

Let us define a pair of sequences of functions

$$f_r(x) = \begin{cases} r, & |x| \le 1/(2r) \\ 0, & |x| > 1/(2r) \end{cases}, \tag{A.2}$$

$$\gamma_r(x) = \sqrt{\frac{r}{\pi}}\, e^{-rx^2}. \tag{A.3}$$

The functions γ_r are called Gaussians.

For every r these functions satisfy:

1. $f_r(x) \ge 0$ for all x and $\displaystyle\int_{-\infty}^{\infty} f_r(x)\, dx = 1$;
2. $\displaystyle\lim_{r\to\infty} f_r(x) = 0$ for all $x \ne 0$ and $\displaystyle\lim_{r\to\infty} f_r(0) = \infty$;
3. $\gamma_r(x) \ge 0$ for all x and $\displaystyle\int_{-\infty}^{\infty} \gamma_r(x)\, dx = 1$;
4. $\displaystyle\lim_{r\to\infty} \gamma_r(x) = 0$ for all $x \ne 0$ and $\displaystyle\lim_{r\to\infty} \gamma_r(0) = \infty$;

The only one of these properties requiring any serious work is the second conclusion of part 3. To see this make the change of variables $y = \sqrt{r}x$ to obtain

$$\int_{-\infty}^{\infty} \gamma_r(x)\, dx = \sqrt{\frac{r}{\pi}} \int_{-\infty}^{\infty} e^{-rx^2}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

This last integral can be evaluated using a trick and polar coordinates as follows. Let

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

Then using the dummy variable integration y we can also write

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

so we have

$$\begin{aligned}
I^2 &= \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \left(\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-x^2}\,dx\right)\left(\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-y^2}\,dy\right) \\
&= \frac{4}{\pi}\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy \\
&= \frac{4}{\pi}\int_0^\infty\int_0^{\pi/2} e^{-r^2}\,r\,d\theta\,dr \\
&= \left(\frac{4}{\pi}\right)\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\int_0^\infty e^{-u}\,du \\
&= 1.
\end{aligned}$$

Now we state the main property of the sequences

Theorem A.1. *Let $\varphi(x)$ be a bounded function which is continuous at $x = 0$. Then*

(a) $\displaystyle\lim_{r\to\infty}\int_{-\infty}^{\infty} f_r(x)g(x)\,dx = g(0)$

(b) $\displaystyle\lim_{r\to\infty}\int_{-\infty}^{\infty} \gamma_r(x)g(x)\,dx = g(0)$

We give an indication without complete details for the proof of Theorem A.1 for a general delta sequence as defined below.

Definition A.1. Any family of functions $\varphi_r(x)$ is called a *delta sequence* if it has the following properties

1. $\varphi_r(x) \geq 0 \quad \forall \quad x$,
2. $\displaystyle\int_{-\infty}^{\infty} \varphi_r(x)\,dx = 1$,
3. for every $\epsilon > 0$ and $c > 0$ (no matter how small), there is a r_0 so that

$$\int_{|x|\geq c} \varphi_r(x)\,dx < \epsilon \quad \forall \quad r > r_0.$$

Notice that a consequence of the above definition is that

$$\lim_{r\to\infty}\varphi_r(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}.$$

Theorem A.1 follows immediately from the next result.

Theorem A.2. *Let $g(x)$ be a bounded function which is continuous at $x = 0$ and $\varphi_r(x)$ be a delta sequence. Then we have*

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \varphi_r(x)g(x)\,dx = g(0)$$

Proof. Take $\epsilon > 0$ an arbitrary small number. We will show that there is a number r_0 so that for all $r > r_0$ we have

$$\left| \int_{-\infty}^{\infty} \varphi_r(x)g(x)\,dx - g(0) \right| < \epsilon.$$

Since g is bounded let us assume that

$$\sup_{x\in\mathbb{R}} |g(x)| \le M < \infty.$$

With the ϵ above we now take $c > 0$ so that

$$\max_{|x|\le c} |g(x) - g(0)| \le \frac{\epsilon}{2}$$

which is possible since we have assumed that g is continuous near $x = 0$. Now with ϵ and c fixed we choose r_0 in Definition A.1 so that

$$\int_{|x|\ge c} \varphi_r(x)\,dx < \frac{\epsilon}{4M}.$$

Then for all $r > r_0$ we have

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} \varphi_r(x)g(x)\,dx - g(0) \right| &= \left| \int_{-\infty}^{\infty} \varphi_r(x)g(x)\,dx - g(0)\int_{-\infty}^{\infty} \varphi_r(x)\,dx \right| \\
&\le \int_{-\infty}^{\infty} \varphi_r(x)\,|g(x) - g(0)|\,dx \\
&= \int_{|x|\le c} \varphi_r(x)\,|g(x) - g(0)|\,dx \\
&\qquad + \int_{|x|\ge c} \varphi_r(x)\,|g(x) - g(0)|\,dx \\
&\le \sup_{|x|\le c} |g(x) - g(0)| \int_{|x|\le c} \varphi_r(x)\,dx \\
&\qquad + 2M \int_{|x|\ge c} \varphi_r(x)\,dx \\
&\le \frac{\epsilon}{2} \int_{\mathbb{R}} \varphi_r(x)\,dx + 2M\left(\frac{\epsilon}{4M}\right) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Remark A.1. 1. For any delta sequence φ_r we define the Dirac delta function by the defining property

$$\delta(x) = \lim_{r\to\infty} \varphi_r(x). \tag{A.4}$$

where formula (A.4) needs to be interpreted in the following sense: For any continuous g we have

$$g(0) = \lim_{r\to\infty} \int_{-\infty}^{\infty} \varphi_r(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, g(x)\, dx.$$

2. If $\delta(x)$ were really a function then $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$.

3. But as a function this makes no sense. Thus the delta function must be interpreted as a *generalized function* or *distribution*.

4. The defining property of the delta function can be shifted to any real number $x = a$ by simply shifting the delta sequence.

$$\int_{-\infty}^{\infty} \delta(x-a)\, g(x)\, dx = \lim_{r\to\infty} \int_{-\infty}^{\infty} \varphi_r(x-a)\, g(x)\, dx = g(a).$$

5. Other examples of delta-sequences

   (a) $f_k(x) = \dfrac{k}{\pi(1+k^2x^2)}$ as $k \to \infty$.

   (b) $f_k(x) = \dfrac{\sin^2(kx)}{\pi k x^2}$ as $k \to \infty$.

6. There are also examples of delta-sequences that do not satisfy all the conditions stated in our definition.

   (a) $f_k(x) = \begin{cases} \dfrac{\sin((k+1/2)x)}{2\pi \sin(x/2)}, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ as $k \to \infty$. This called the Dirichlet Kernel used for Fourier series.

   (b) $f_k(x) = \dfrac{\sin(kx)}{\pi x}$ as $k \to \infty$. This is the so-called "Sinc function" which is used in many applications in engineering.

   (c) Let $I_k = \int_{-1}^{1} (1-x^2)^k\, dx$ then $f_k(x) = \begin{cases} \dfrac{(1-x^2)^k}{I_k}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ as $k \to \infty$. This kernel is used in a proof of the Weirstrass Approximation Theorem.

B Appendix: Derivation of the Heat Kernel

Now we return to the problem (3.6)-(3.8). Our objective is to derive the heat kernel $S(x,t)$ and learn some of its properties. No matter how you decide to approach this derivation some considerable work will be involved. In this section we approach the problem through motivation from the need for the solution to provide the delta function at $t = 0$ and we want to do the work using only very elementary methods. For this approach we do make one important assumption - for each fixed time value t the heat flux approaches zero as $|x| \to \infty$.

Under the assumption that the heat flux $F(x,t) = ku_x(x,t)$ has the property that

$$\lim_{-a \to -\infty} F(-a,t) = \lim_{a \to \infty} F(a,t) = 0 \quad \text{for all} \quad t$$

we can integrate the heat equation with respect to x to obtain

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\,dx &= \int_{-\infty}^{\infty} k u_{xx}(x,t)\,dx \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial x} F(x,t)\,dx \\ &= -\lim_{a\to\infty} \int_{-a}^{a} \frac{\partial}{\partial x} F(x,t)\,dx \\ &= -\lim_{a\to\infty} F(x,t)\Big|_{x=-a}^{x=a} \\ &= \lim_{a\to\infty} \left(F(-a,t) - F(a,t)\right) = 0 \end{aligned}$$

where this limit is zero due to our assumption. This means that the total heat energy over the whole line is a constant since

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\,dx = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} u(x,t)\,dx = \text{ constant.}$$

Thus we see that the total energy at any time t is the same as it is at time $t = 0$, i.e. the total energy in the initial condition: For every $t > 0$ we have

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Let us now consider an initial temperature confined to a small area around $x = 0$ by taking the initial condition to be the function considered in (A.2) where we replace r by n

$$f_n(x) = \begin{cases} n, & |x| \le 1/(2n) \\ 0, & |x| > 1/(2n) \end{cases}.$$

We know that for every $x \neq 0$ $f_n(x) \to 0$ as $n \to \infty$ while the amount of heat energy in $f_n(x)$ is

$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1 \quad \forall \quad n.$$

Assuming that a solution exists, let us denote the solution of the IVP (3.6)-(3.8) with this initial condition $f_n(x)$ by $u_n(x,t)$. Thus we have

$$\frac{\partial u_n}{\partial t} = \frac{\partial^2 u_n}{\partial x^2}, \quad u_n(x,0) = f_n(x). \tag{B.1}$$

Assuming a limiting function exists we define

$$S(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{B.2}$$

Let us consider some of the properties this function would need to have.

1. Since the initial functions form a delta-sequence (i.e. $f_n \to \delta$) we see that the initial condition for $S(x,t)$ is the delta function. So we have $S(x,0) = \delta(x)$.
2. Also since $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$ for all n we would expect $\int_{-\infty}^{\infty} S(x,t)\,dx = 1$ for all $t > 0$.

Let us collect what we expect the properties of S to be

$$\begin{aligned} \frac{\partial S}{\partial t}(x,t) &= \frac{\partial^2 S}{\partial x^2}(x,t) \\ S(x,0) &= \delta(x), \\ \int_{-\infty}^{\infty} S(x,t)\,dx &= 1 \qquad \text{for all } t > 0. \end{aligned} \tag{B.3}$$

With these three properties in mind we want to try to determine $S(x,t)$ assuming these properties to hold. To do this our game plan is to reduce finding S to solving an ODE. This requires quite a few manipulations.

Step 1. First we want to simplify matters by removing k from the picture. We do this by setting

$$\widetilde{S}(x,t) = S(x,t/k)$$

so that (by the chain rule)

$$\frac{\partial \widetilde{S}}{\partial t}(x,t) = \frac{\partial S}{\partial t}(x,t/k)\cdot(1/k) = k\frac{\partial^2 S}{\partial x^2}(x,t/k)\cdot(1/k) = \frac{\partial^2 \widetilde{S}}{\partial x^2}(x,t).$$

From this we see that $\widetilde{S}$ is a solution of the heat equation with $k = 1$.

Step 2. Now suppose that $v(x,t)$ is any solution of $v_t = v_{xx}$ and define $v_\lambda(x,t)$ for any $\lambda > 0$ by

$$v_\lambda(x,t) = v(\lambda x, \lambda^2 t).$$

Again appealing to the chain rule we have

$$\begin{aligned} (v_\lambda)_t - (v_\lambda)_{xx} &= \lambda^2\left(v_t(\lambda x,\lambda^2 t)\right) - \lambda^2\left(v_{xx}(\lambda x,\lambda^2 t)\right) \\ &= \lambda^2\left(v_t(\lambda x,\lambda^2 t) - v_{xx}(\lambda x,\lambda^2 t)\right) = 0. \end{aligned}$$

So v_λ solves the heat equation and if $v(x,0) = f(x)$ it follows that

$$v_\lambda(x,0) = v(\lambda x, 0) = f(\lambda x).$$

Step 3. Thus, in particular, we can apply the above for the solution $v(x,t) = \widetilde{S}(x,t)$ to conclude that $\widetilde{S}_\lambda(x,t) = \widetilde{S}(\lambda x, \lambda^2 t)$ satisfies the heat equation and since for all $x \neq 0$ we have $\widetilde{S}(x,0) = 0$ we must also have $\widetilde{S}_\lambda(x,0) = 0$ for $x \neq 0$.

Step 4. Next we make a claim.

Claim B.1. There exists a constant $C(\lambda)$ depending only on λ so that $\widetilde{S}_\lambda(x,t) = C(\lambda)\widetilde{S}(x,t)$. Indeed we will show that $C(\lambda) = 1/\lambda$ so that

$$\widetilde{S}_\lambda(x,t) = \frac{1}{\lambda}\,\widetilde{S}(x,t).$$

Proof of Claim: In order that this be true we would need the following to hold.

$$\begin{aligned}
1 = \int_{-\infty}^{\infty} \widetilde{S}(x,t)\,dx &= \frac{1}{C(\lambda)}\int_{-\infty}^{\infty} \widetilde{S}(\lambda x, \lambda^2 t)\,dx \\
(\text{ set } \ y = \lambda x \quad &\Rightarrow dy = \lambda dx) \\
&= \frac{1}{\lambda C(\lambda)}\int_{-\infty}^{\infty} \widetilde{S}(y, \lambda^2 t)\,dy \\
&= \frac{1}{\lambda C(\lambda)}
\end{aligned}$$

since $\displaystyle\int_{-\infty}^{\infty} \widetilde{S}(y,\lambda^2 t)\,dy = 1$ for all $t > 0$ and $\lambda > 0$. Therefore we would have

$$C(\lambda) = \frac{1}{\lambda}.$$

□

Thus we conclude

$$\widetilde{S}(\lambda x, \lambda^2 t) = \frac{1}{\lambda}\widetilde{S}(x,t).$$

Step 5. Now we select a particular value for λ. Namely we set $\lambda = t^{-1/2}$. Then we have

$$\widetilde{S}(x,t) = \frac{1}{\sqrt{t}}\widetilde{S}\left(\frac{x}{\sqrt{t}}, 1\right).$$

If we now define the scalar function of one variable $\psi(s)$ by

$$\psi(s) = \widetilde{S}\,(s,1)\,, \tag{B.4}$$

then we have

$$\widetilde{S}(x,t) = \frac{1}{\sqrt{t}}\widetilde{S}\left(\frac{x}{\sqrt{t}}, 1\right) = \frac{1}{\sqrt{t}}\psi\left(\frac{x}{\sqrt{t}}\right).$$

Step 6. Next we will use the fact that $\widetilde{S}_t = \widetilde{S}_{xx}$ to derive an ODE satisfied by ψ. By the chain rule, product rule, etc we have

$$\begin{aligned}\widetilde{S}_t(x,t) &= \left[t^{-1/2}\psi(t^{-1/2}x)\right]_t \\ &= (-1/2)\,t^{-3/2}\psi(t^{-1/2}x) + t^{-1/2}(-1/2)\,t^{-3/2}x\psi'(t^{-1/2}x) \\ &= (-1/2)\,t^{-3/2}\psi(t^{-1/2}x) + (-1/2)\,t^{-2}x\psi'(t^{-1/2}x),\end{aligned}$$

$$\widetilde{S}_x(x,t) = \left(t^{-1/2}\right)\left(t^{-1/2}\right)\psi'(t^{-1/2}x) = \left(t^{-1}\right)\psi'(t^{-1/2}x)$$

and

$$\widetilde{S}_{xx}(x,t) = \left(t^{-3/2}\right)\psi''(t^{-1/2}x).$$

So we must have

$$\begin{aligned}0 &= \widetilde{S}_{xx}(x,t) - \widetilde{S}_t(x,t) \\ &= t^{-3/2}\psi''(t^{-1/2}x) + (1/2)t^{-2}x\psi'(t^{-1/2}x) + (1/2)t^{-3/2}\psi(t^{-1/2}x).\end{aligned}$$

Introduce the new variable $\xi = t^{-1/2}x$ and the above formula becomes

$$\psi''(\xi) + \frac{1}{2}\xi\psi'(\xi) + \frac{1}{2}\psi(\xi) = 0. \tag{B.5}$$

Here recall that ψ is defined in (B.4).

Step 7. This is an ODE for the unknown function ψ but it is unlike any that we studied in Math 3354. It is second order but not constant coefficient. A method for solving (B.5) is to look for a solution in the form of an infinite power series and to use the equation itself to help determine the coefficients. So we look for

$$\psi(\xi) = \sum_{j=0}^{\infty} a_j \xi^j.$$

The we have

$$\psi'(\xi) = \sum_{j=1}^{\infty} j a_j \xi^{j-1}, \qquad \psi''(\xi) = \sum_{j=2}^{\infty} j(j-1) a_j \xi^{j-2}.$$

Plugging these relations in (B.5) gives

$$\sum_{j=2}^{\infty} j(j-1)a_j\xi^{j-2} + \frac{1}{2}\xi\sum_{j=1}^{\infty} j a_j \xi^{j-1} + \frac{1}{2}\sum_{j=0}^{\infty} a_j\xi^j = 0.$$

Shifting the indices in the first term gives

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}\xi^j + \frac{1}{2}\sum_{j=1}^{\infty} j a_j\xi^j + \frac{1}{2}\sum_{j=0}^{\infty} a_j\xi^j = 0.$$

The first term corresponding to $j=0$ gives

$$2\cdot 1\cdot a_2 \frac{1}{2} a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{-1}{2\cdot 2} a_0.$$

The sum of the rest of the terms can be written as

$$\sum_{j=1}^{\infty} \left[(j+2)(j+1)a_{j+2} + \frac{1}{2}\, j a_j + \frac{1}{2}\, a_j \right] \xi^j = 0.$$

From this we conclude

$$\left[(j+2)(j+1)a_{j+2} + \frac{1}{2}\, j a_j + \frac{1}{2}\, a_j \right] = 0 \quad j = 1, 2, \cdots .$$

This can also be rewritten as

$$a_{j+2} = -\frac{(j+1)}{2(j+2)(j+1)} a_j = -\frac{1}{2(j+2)} a_j \quad j = 1, 2, \cdots .$$

Notice that these terms can be grouped into j and odd or even integer. All the odd terms are zero if $a_1 = 0$ and all the even terms are then determined in terms of a_0. So let us take $a_1 = 0$ and let $a_0 = A$ be arbitrary. Then we have

$$a_1 = a_3 = \cdots = a_{2k+1} = 0 \quad k = 2, 3, \cdots .$$

and for $j = 2, 4, \cdots, 2k$, $k = 1, 2, \cdots$ we compute

$$\begin{aligned}
k = 1: &\quad a_{2\cdot 1} = a_2 = \frac{-1}{2\cdot 2} A, \\
k = 2: &\quad a_{2\cdot 2} = a_4 = \frac{-1}{4\cdot 2} a_2 = \frac{(-1)^2}{4\cdot 2\cdot 2^2} A, \\
k = 3: &\quad a_{2\cdot 3} = a_6 = \frac{-1}{6\cdot 2} a_4 = \frac{(-1)^3}{6\cdot 4\cdot 2\cdot 2^3} A.
\end{aligned}$$

Similarly we find

$$k = 4: \quad a_{2\cdot 4} = \frac{(-1)^4}{8\cdot 6\cdot 4\cdot 2\cdot 2^4} A = \frac{(-1)^4}{4!\cdot 4^4} A.$$

From this we can see the pattern

$$a_{2k} = \frac{(-1)^k}{k! 4^k}$$

and we obtain

$$\psi(\xi) = A \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x^2}{4} \right)^k .$$

Recall that the Taylor series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$$

and we can then observe that ψ is

$$\psi(\xi) = Ae^{-\xi^2/4}.$$

Finally we can compute A from the condition

$$\int_{-\infty}^{\infty} \widetilde{S}(x,t)\,dx = 1 \qquad \text{for all } t > 0.$$

Namely , using the fact that $\xi = x/\sqrt{t}$ we have

$$\widetilde{S}(x,t) = \frac{1}{\sqrt{t}}\psi\left(\frac{x}{\sqrt{t}}\right) = \frac{A}{\sqrt{t}}e^{-x^2/(4t)}$$

and using the substitution $y = x/(2\sqrt{t})$

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} \widetilde{S}(x,t)\,dx &= \frac{A}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-x^2/(4t)}\,dx \\ &= 2A\int_{-\infty}^{\infty} e^{-y^2}\,dy = 2A\sqrt{\pi}. \end{aligned}$$

So we have

$$A = \frac{1}{2\sqrt{\pi}}.$$

Next we want to involve k by converting this formula back to $S(x,t)$ which just requires us to replace t by kt to get

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}. \tag{B.6}$$

As a final note we see that if (B.6) is interpreted as a function of x parametrized by t then it describes a delta family in $r = 1/(4kt)$ where $r \to \infty$ as $t \to 0$. So we have

$$\lim_{t\downarrow 0} S(x,t) = \delta(x).$$

C Appendix: Solution Using Fourier Transforms

In this section we present an alternative approach to solving the IVP for the heat equation on $\mathbb{R}$. A completely rigorous development of this material would be well beyond the scope of this class so what I plan to do is to present the main points of this method and state results without proof. For a deeper understanding of the validity of this work please graduate- go to graduate school and take “Classical Applied math” (Math 5310 and 5311) and Graduate Partial Differential Equations (Math 5332).

Definition C.1. 1. We say a function $f(x)$ defined on $\mathbb{R}$ is in $L^p(\mathbb{R})$ if

$$\int_{\mathbb{R}} |f(x)|^p \, dx < \infty.$$

For $f \in L^p(\mathbb{R})$ we define the *norm* or length of f by

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}.$$

2. For $f \in L^1(\mathbb{R})$ we define the Fourier transform of f by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx \tag{C.1}$$

The variable ξ is called the transform variable.

The inverse Fourier transform (for $g \in L^1(\mathbb{R})$) is given by

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} g(\xi) \, d\xi \tag{C.2}$$

For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have the *Fourier Inversion Formula*

$$f(x) = \mathcal{F}^{-1}\left(\mathcal{F}(f)\right)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) \, d\xi. \tag{C.3}$$

We have

Theorem C.1. *(a) For $f \in L^1(\mathbb{R})$, $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ and*

$$\sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| = \|\widehat{f}\|_\infty \le \|f\|_1 = \int_{\mathbb{R}} |f(x)| \, dx.$$

Indeed, $\widehat{f}(\xi)$ is a continuous function on $\mathbb{R}$ and

$$\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0.$$

This is a famous result called the ***Riemann-Lebesque Lemma****.*

(b) (Plancherel Theorem)

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \, d\xi, \tag{C.4}$$

$$\int_{\mathbb{R}} f(x) \, \overline{g(x)} \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \, \overline{\widehat{g}(\xi)} \, d\xi, \tag{C.5}$$

where we allow functions to have complex values and if $z = a + bi$ is a complex number then $\overline{z} = a - bi$ is the complex conjugate*. Furthermore $|z| = \sqrt{a^2 + b^2}$ is the absolute value of the complex number.*

(c) *Assuming that f is sufficiently differentiable and decays to zero as $|x| \to \infty$ sufficiently rapidly we have the following formulas for the Fourier transform of the first and second derivative of f*

$$\widehat{f'}(\xi) = i\xi \widehat{f}(\xi) \quad \text{and} \quad \widehat{f''}(\xi) = \xi^2 \widehat{f}(\xi) \tag{C.6}$$

(d) *If $f, g \in L^1$, then we define the convolution of f and g by*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\, dy.$$

Then we have $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

(e) *If we define the function $f_a(x) = f(ax)$ then we have*

$$\widehat{f_a}(\xi) = \frac{1}{a}\widehat{f}\left(\frac{\xi}{a}\right) \tag{C.7}$$

(f) *Define the translation of f by $\tau_a(f)(x) = f(x-a)$ then we have*

$$\widehat{\tau_a(f)}(\xi) = e^{-ia\xi}\, \widehat{f}(\xi). \tag{C.8}$$

Proof. 1. The proof of the first part is easy once you realize that $|e^{i\theta}| = 1$ which follows from the Euler formula

$$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

So we have

$$|\widehat{f}(\xi)| = \left|\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx\right| \le \int_{\mathbb{R}} |f(x)|\, dx.$$

We will not prove the Reimann-Lebesque Lemma here.

2. We only need to verify (C.5) since (C.4) follows from (C.5) by setting $g = f$. We will use the Fourier Inversion Formula (C.3) which allows us to write

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\widehat{g}(\xi)\, d\xi$$

which implies

$$\overline{g(x)} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\overline{\widehat{g}(\xi)}\, d\xi$$

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\, \overline{g(x)}\, d\xi &= \frac{1}{2\pi}\int\int_{\mathbb{R}} f(x)\overline{(e^{ix\xi}\widehat{g}(\xi))}\, d\xi\, dx \\
&= \frac{1}{2\pi}\int\int_{\mathbb{R}} f(x)\left(e^{-ix\xi}\overline{\widehat{g}(\xi)}\right)\, d\xi\, dx \\
&= \frac{1}{2\pi}\int\left(\int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx\right)\overline{\widehat{g}(\xi)}\, d\xi \\
&= \frac{1}{2\pi}\int \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi.
\end{aligned}$$

3. This is the most important property which simply says that the Fourier transform changes differentiation into multiplication. To see this we apply integration by parts as follows

$$\widehat{(f')}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f'(x)\,dx = e^{-ix\xi} f(x)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} (-i\xi) e^{-ix\xi} f(x)\,dx = i\xi \widehat{f}(\xi).$$

Here we have assumed that f decays to zero as x tends to infinity. In order to see the second formula we could apply integration by parts twice or simply apply the formula we just derived twice, namely,

$$\widehat{(f'')}(\xi) = i\xi \widehat{f'}(\xi) = (i\xi)^2 \widehat{f}(\xi).$$

4. To verify the convolution formula we compute

$$\begin{aligned}
\widehat{(f*g)}(\xi) &= \iint e^{-ix\xi} f(x-y) g(y)\,dy\,dx \\
&= \iint e^{-i(x-y)\xi} f(x-y)\; e^{-iy\xi} g(y)\,dx\,dy \\
&(\text{ set } z = x-y \quad \Rightarrow \quad dz = dx) \\
&= \left(\int e^{-iz\xi} f(z)\,dz\right)\left(\int\; e^{-iy\xi} g(y)\,dy\right) \\
&= \widehat{f}(\xi)\,\widehat{g}(\xi).
\end{aligned}$$

5. To verify the scaling formula for $f_a(x) = f(ax)$ we have

$$\begin{aligned}
\widehat{f_a}(\xi) &= \int e^{-ix\xi} f_a(x)\,dx = \int e^{-ix\xi} f(ax)\,dx \\
&(\text{ set } z = ax \quad \Rightarrow \quad dz = adx) \\
&= \frac{1}{a}\int e^{-i(z/a)\xi} f(z)\,dz \\
&= \frac{1}{a}\widehat{f}\left(\frac{\xi}{a}\right)
\end{aligned}$$

6. Finally we have $\tau_a(f)(x) = f(x-a)$ and we compute

$$\begin{aligned}
\widehat{\tau_a(f)}(\xi) &= \int e^{-ix\xi}\, f(x-a)\,dx \\
&(\text{ set } z = x-a \quad \Rightarrow \quad dz = dx) \\
&= \int e^{-i(z+a)\xi}\, f(z)\,dx \\
&= e^{-ia\xi} \int e^{-iz\xi}\, f(z)\,dx = e^{-ia\xi}\, \widehat{f}(\xi).
\end{aligned}$$

□

Next we present a few examples of Fourier transforms.

I.	$f(x)$	$\widehat{f}(\xi)$
II.	$\chi_a(x) = \begin{cases} 1 & \text{if } \ \lvert x\rvert < a \\ 0 & \text{otherwise} \end{cases}$	$2\ \dfrac{\sin(a\xi)}{\xi}$
III.	$e^{-ax^2/2}$	$\sqrt{\dfrac{2\pi}{a}} e^{-\xi^2/(2a)}$
IV.	$(x^2+a^2)^{-1}$	$\dfrac{\pi}{a} e^{-a\lvert\xi\rvert}$
V.	$\dfrac{\sin(ax)}{x}$	$\pi\chi_a(\xi) = \begin{cases} \pi & \text{if } \ \lvert\xi\rvert < a \\ 0 & \text{otherwise} \end{cases}$
VI.	$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$

Table 1: Fourier Transforms of Common Functions

We can use the Fourier transform to obtain the formula in (3.9) for the solution of the initial value problem (3.6)-(3.8) as follows:

Apply the Fourier transform to the equations (3.6) and (3.7) using Part (c) of Theorem (C.1) to obtain

$$\frac{\partial \widehat{u}}{\partial t}(\xi,t) = -k\xi^2\widehat{u}(\xi,t).$$

For each fixed ξ this is a first order ODE in t with general solution

$$\widehat{u}(\xi,t) = A(\xi)e^{-k\xi^2 t}$$

where $A(\xi)$ is an arbitrary function of ξ. But we obtain a unique solution using the initial condition from the Fourier transform of (3.7) which gives

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-k\xi^2 t}$$

and it remains to compute the inverse Fourier transform.

We have

$$u(x,t) = \mathcal{F}^{-1}\left(\widehat{f}(\xi)e^{-k\xi^2 t}\right) = \left[\mathcal{F}^{-1}\left(e^{-k\xi^2 t}\right) * f\right](x)$$

but we know from Table I. item III. with $a = 1/(2kt)$ that the inverse Fourier transform of $\exp(-k\xi^2 t)$ is

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}}\, e^{-x^2/(4kt)}. \tag{C.9}$$

Namely

$$\mathcal{F}^{-1}\left(e^{-\xi^2/(2a)}\right) = \sqrt{\frac{a}{2\pi}} e^{-ax^2/2}$$

and now let $a = 1/(2kt)$ to obtain (C.9). Furthermore the inverse Fourier transform of $\widehat{f}(\xi)$ is $f(x)$. Therefore we have

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int e^{-(x-y)^2/(4kt)} f(y)\, dy. \tag{C.10}$$

D Appendix: Proof of Maximum Principle

To prove Theorem 3.1 we first prove a version of the result for a rod of finite length. This result will also be of use to us in later notes on boundary value problems. For $-\infty < a < b < \infty$ and $T > 0$ let us define

$$Q \equiv Q(a,b,T) = \{(x,t) \,:\, a < x < b, \quad 0 < t < T\},$$

$$\Gamma \equiv \Gamma(a,b,T) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where

$$\Gamma_1 = \{(x,t) \,:\, x = -a, \quad 0 \le t \le T\},\ \Gamma_2 = \{(x,t) \,:\, x = a, \quad 0 \le t \le T\},$$

$$\Gamma_3 = \{(x,t) \,:\, a \le x \le b, \quad t = 0\}, \quad \Gamma_T = \{(x,t) \,:\, a \le x \le b, \quad t = T\}$$

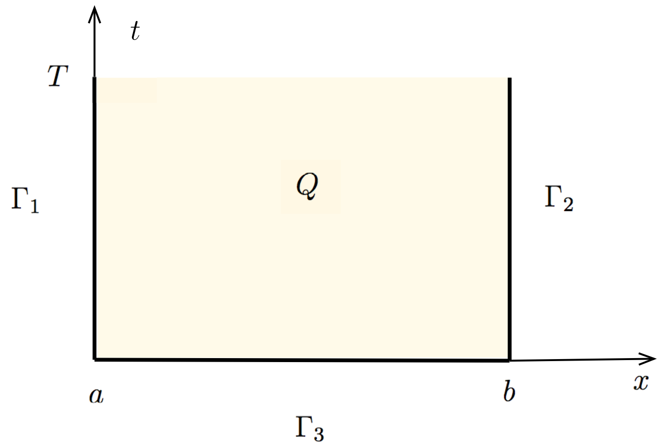
and finally

$$\overline{Q} \equiv \overline{Q(a,b,T)} = Q(a,b,T) \cup \Gamma(a,b,T) \cup \Gamma_T.$$

(See the figure below)

Theorem D.1. *Assume that u is a strict solution to* (3.6)-(3.8) *with $|f(x)| \le M_0$ for all $x \in \mathbb{R}$ and let $T > 0$ be arbitrary. Then*

$$\max_{(x,t)\in\overline{Q}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \tag{D.1}$$



In other words, the temperature on a piece $a \leq x \leq b$ (of a perhaps infinite rod) can never exceed the larger of the maximum of the initial temperature $f(x)$ and the temperature at the ends of the rod (i.e. when $x = a$ or $x = b$ and $0 \leq t \leq T$).

Proof. Let u be a strict solution and $\epsilon > 0$ arbitrary. Define

$$v^\epsilon(x,t) = u(x,t) - \epsilon t.$$

Then u and v^ϵ are continuous on $\overline{Q}$ and therefore must have a maximum on $\overline{Q}$ and also on Γ. Also v^ϵ satisfies

$$v_t^\epsilon - k v_{xx}^\epsilon = (u(x,t) - \epsilon t)_t - k\,(u(x,t) - \epsilon t)_{xx} = -\epsilon < 0. \tag{D.2}$$

If the maximum of v^ϵ occurs at (x_0, t_0), i.e.,

$$u(x,t) \leq u(x_0,t_0) \quad \text{for all} \quad (x,t) \in \overline{Q}.$$

Now, if also, $a < x_0 < b$ and $0 < t_0 \leq T$, (i.e. $(x_0,t_0) \in Q \backslash \Gamma$) then by the second derivative test from Calculus we must have

$$v_{xx}^\epsilon(x_0,t_0) \leq 0.$$

This implies $-k v_{xx}^\epsilon(x_0,t_0) \geq 0$ (note we use $\geq$ since the maximum may occur over an x interval not just at the single point x_0) so by (D.2) we would have

$$v_t^\epsilon(x_0,t_0) \leq v_t^\epsilon(x_0,t_0) - k v_{xx}^\epsilon(x_0,t_0) < -\epsilon.$$

But from calculus again, at a maximum, $v_t^\epsilon(x_0,t_0) = 0$ which is a contradiction. Therefore (x_0,t_0) cannot be in Q (nor on $t = T$) so it must be in Γ. □

Now we can apply Theorem D.1 to help prove Theorem 3.1 which we restate here for convenience

Assume the u is a bounded strict solution to (3.6)-(3.8) with $|f(x)| \le M_0$ for all $x \in \mathbb{R}$ and let $T > 0$ be arbitrary. Define

$$B_T = \{(x,t) \,:\, x \in \mathbb{R}, \quad 0 < t < T\}, \qquad \overline{B_T} = \{(x,t) \,:\, x \in \mathbb{R}, \quad 0 \le t \le T\}.$$

Since u is bounded there is an M so that $u(x,t) \le M$ for all $(x,t) \in \overline{B_T}$. Then we have

$$u(x,t) \le M_0 \quad \textit{for all} \quad (x,t) \in \overline{B_T}.$$

Proof of Theorem 3.1. Fix $\epsilon > 0$ and set

$$v^\epsilon(x,t) = u(x,t) - \epsilon\left(kt + \frac{x^2}{2}\right).$$

Then v^ϵ satisfies the heat equation.

For $-\infty < a < \infty$ and $T > 0$ let us define

$$Q \equiv Q(a,T) = \{(x,t) \,:\, -a < x < a, \quad 0 < t < T\},$$

$$\Gamma \equiv \Gamma(a,T) = \{(x,t) \,:\, |x| = a, \quad 0 \le t \le T\} \cup \{(x,t) \,:\, |x| \le a, \quad t = 0\}.$$

Then by Theorem D.1 we have

$$\max_{\overline{Q}} v^\epsilon(x,t) = \max_{\Gamma} v^\epsilon(x,t)$$

But we have

$$\begin{aligned}\max_{\Gamma} v^\epsilon(x,t) &= \max_{\Gamma}\left[u(x,t) - \epsilon\left(kt + \frac{x^2}{2}\right)\right] \\ &\le \max_{\Gamma} u(x,t) + \max_{\Gamma}\left[-\epsilon\left(kt + \frac{x^2}{2}\right)\right].\end{aligned}$$

We have assumed that $\max_\Gamma u(x,t) \le M$ and we need to compute

$$\max_{\Gamma}\left[-\epsilon\left(kt + \frac{x^2}{2}\right)\right]$$

To do this we simply compute the maximum on each of Γ_1, Γ_2, and Γ_3.

1. $\mathbf{\Gamma_1}$: On Γ_1 we have $x = -a$ and $0 \le t \le T$ so
$$\max_{\Gamma_1}\left[-\epsilon\left(kt + \frac{a^2}{2}\right)\right] = -\epsilon\frac{a^2}{2}$$

2. $\mathbf{\Gamma_2}$: On Γ_2 we have $x = a$ and $0 \le t \le T$ so
$$\max_{\Gamma_2}\left[-\epsilon\left(kt + \frac{a^2}{2}\right)\right] = -\epsilon\frac{a^2}{2}$$

3. $\mathbf{\Gamma_3}$: On Γ_3 we have $-a \leq x \leq a$ and $t = 0$ so

$$\max_{\Gamma_3} \left[-\epsilon\left(kt + \frac{x^2}{2}\right)\right] = 0$$

So we have

$$\max_{\Gamma} v^\epsilon(x,t) \leq \max\{M - \epsilon\frac{a^2}{2}, M_0\}$$

where we have used that

$$\max_{\Gamma_3} v^\epsilon(x,t) = \max_{\Gamma_3} v^\epsilon(x,0) = \max_{|x|\leq a} f(x) - \epsilon\left(\frac{x^2}{2}\right) \leq M_0.$$

Now choose a_0 so that

$$M - \epsilon\frac{a^2}{2} = M_0.$$

Then for every $a > a_0$ we have

$$\max_{\Gamma_3} v^\epsilon(x,t) \leq M_0$$

But now we notice that

□