My work with Chris Byrnes (at a glance)

Root-Locus Design,
Nonlinear Zeros,
Zero Dynamics and regulation

by David Gilliam

Memorial Workshop, September 2010
From David Gilliam to Chris Byrnes
Outline

1. **Zeros and Root-Locus Design**
   - Root Locus for Lumped Systems
   - Root Locus for DPS

2. **Zero Pole Dynamics**
   - High Gain Limit of Trajectories
   - Nonlinear Zeros: Convergence of Trajectories

3. **ZDI Design for Regulation of DPS**
   - Nonlinear Boundary Control System
   - Statement of Regulation Problem
   - The Zero Dynamics Controller
A Brief Review of Classical Automatic Control
Finite Dimensional Models

SISO finite dimensional linear system

\[ z_t = A z + b u, \quad z, b, c^T \in \mathbb{R}^n \]
\[ y = c z. \]

- Assume that the system has relative degree 1, i.e., \( cb \neq 0 \)
- The transfer function

\[ G_0(s) = c(sI - A)^{-1}b \]

has the form

\[ G_0(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \frac{b_0 + b_1 s + \cdots + s^{n-1}}{a_0 + a_1 s + \cdots + s^n}. \]

- The zeros of the numerator \( \mathcal{N}(s) \): open loop zeros.
- The zeros of the denominator \( \mathcal{D}(s) \): open loop poles.
For a proportional error feedback law

\[ u(t) = -ky(t) + v(t) \]

the closed loop transfer function becomes

\[ G_k(s) = \frac{N(s)}{D(s) + kN(s)} \]

For the closed loop system we have:

- The same zeros as the open loop system,
- The closed loop poles are given by the zeros of the denominator, (i.e., the return difference equation)

\[ D(s) + kN(s) = 0. \]
Variation of closed loop poles

A well known result from finite dimensional root locus theory is that:

“The closed loop poles vary from the open loop poles to the open loop zeros (and infinity) as the gain $k$ varies from 0 to $\infty$”
An important alternative statement to the variation is that trajectories of the closed loop system converge to trajectories of the zero dynamics as the gain tends to infinity.
Root Locus for Boundary Controlled DPS
For a distributed parameter Boundary Control System (BCS)

\[ z_t(x, t) = A z(x, t), \]
\[ B z(t) = u(t), \]
\[ z(x, 0) = \varphi(x) \in L^2(0, 1), \]
\[ y(t) = C z(t) \]

we have an \((A, B, C)\) triple in infinite dimensional state space.
CHRIS ASKED:
To what extent can we make a general root locus theory?

Some things to Note

- In this case we often have unbounded operators in an infinite dimensional Banach space.
- Consider, for example, an operator $\frac{d}{dx}$ with various domains in $C[a, b]$ corresponding to different restrictions.
- We denote the spectrum of an operator by $\sigma(\cdot)$.
Maximal operator

\[ T = \frac{d}{dx} \text{ with } D(T) = C^1[a, b] \]

\[ \Rightarrow \quad \sigma(T) = \mathbb{C} \text{ (Point Spectrum)}. \]

Minimal operator

\[ T_0 = \frac{d}{dx} \text{ with } D(T_0) = \{ \varphi \in C^1[a, b] : \varphi^{(j)}(a) = 0, \varphi^{(j)}(b) = 0, \ j = 0, 1 \} \]

\[ \Rightarrow \quad \sigma(T_0) = \mathbb{C} \text{ (Continuous Spectrum)} \]

Here \((\lambda I - T_0)^{-1}\) exists but \(\text{Ran}(\lambda I - T_0) \neq C[a, b]\).
BC at left end

\[ T_1 = \frac{d}{dx} \text{ with } D(T_1) = \{ \varphi \in C^1[a, b] : \varphi(a) = 0 \} \]

\[ \Rightarrow \sigma(T_1) = \emptyset. \]

BC at right end

\[ T_2 = \frac{d}{dx} \text{ with } D(T_2) = \{ \varphi \in C^1[a, b] : \varphi(b) = 0 \} \]

\[ \Rightarrow \sigma(T_2) = \emptyset. \]
Non-Separated BC

\[ T_3 = \frac{d}{dx} \text{ with} \]

\[ D(T_3) = \{ \varphi \in C^1[a, b] : \varphi(b) - k\varphi(a) = 0, \ k \neq 0 \} \]

\[ \Rightarrow \quad \sigma(T_3) = \left\{ \lambda_n = \frac{1}{b-a}(\ln(k) + 2\pi ni) \right\} \]

for \( n = 0, \pm 1, \pm 2, \cdots \).
Fortunately many differential operators behave better.

Consider a distributed parameter boundary control system

\[ z_t(x, t) = Az(x, t), \]
\[ Bz(t) = u(t), \]
\[ z(x, 0) = \varphi(x) \in L^2(0, 1), \]
\[ y(t) = Cz(t) \]

Here \( A \) is an even order differential operator with BCs.
In particular, $A = L$ where $L$ is an even order, $n = 2\mu$, ordinary differential operator in $Z = L^2(0, 1)$ with real $L^\infty$ coefficients $p_j(x)$

$$L = (-1)^{\mu-1} D^n + \sum_{j=0}^{n} p_j(x) D^j, \quad D = \frac{d}{dx}.$$
The domain of $A$ is defined in terms of $(n - 1)$ Birkhoff Regular boundary conditions

$$D(A) = \left\{ \varphi \in H^{(n)}(0, 1) : B_j \varphi = 0, \ j = 2, \cdots, n \right\}$$

The operator $A$ is not a generator
$\mathcal{B}$ is a boundary operator and $u$ a scalar control input.
$\mathcal{C}$ is another boundary operator and $y$ is the measured output.
For $i = 0, \cdots, \mu$ assume boundary operators of the form

$$B_i(\varphi) \equiv \alpha_i \varphi^{(m_i)}(0) + \sum_{j=0}^{m_i-1} \left\{ \alpha_{ij} \varphi^{(j)}(0) + \beta_{ij} \varphi^{(j)}(1) \right\}$$

Here $\alpha_i \neq 0$, $\beta_i \neq 0$, $\alpha_{ij}$, $\beta_{ij}$ are real.
For $i = 0, \ldots, \mu$ assume boundary operators of the form

$$B_i(\varphi) \equiv \alpha_i \varphi^{(m_i)}(0) + \sum_{j=0}^{m_i-1} \left\{ \alpha_{ij} \varphi^{(j)}(0) + \beta_{ij} \varphi^{(j)}(1) \right\}$$

and for $i = (\mu + 1), \ldots, n$ we have BCs

$$B_i(\varphi)(t) \equiv \beta_i \varphi^{(m_i)}(1) + \sum_{j=0}^{m_i-1} \left\{ \alpha_{(i)j} \varphi^{(j)}(0) + \beta_{(i)j} \varphi^{(j)}(1) \right\}$$

Here $\alpha_i \neq 0$, $\beta_i \neq 0$, $\alpha_{ij}$, $\beta_{ij}$ are real $m_1, m_{\mu+1} \leq (n - 1)$

$$m_1 > \cdots > m_\mu, \ m_{\mu+1} > \cdots > m_n$$
Proportional Error Feedback – High Gain Limit

High Gain Limit

For $u = -ky \Rightarrow \frac{u}{k} = -y \Rightarrow 0 = \lim_{k \to \infty} \frac{u}{k} = -y$ or $y = 0$

Plant

\[
\begin{align*}
  z_t(x, t) &= Az(x, t), \\
  Bz(t) &= u(t) \\
  y(t) &= Cz(t)
\end{align*}
\]

Zero Dynamics

\[
\begin{align*}
  z_t(x, t) &= Az(x, t) \\
  Cz(t) &= 0
\end{align*}
\]
High Gain Limit

With $B = B_1$, $C = B_0$, $u = -ky + v \Rightarrow (B + kC)z = v$.

Closed Loop System

$z_t(x, t) = Az(x, t)$,

$$(B + kC)z(t) = v(t)$$

$y(t) = Cz(t)$
The open loop transfer function (OLTF) has the form

\[ G_0(s) = \frac{N(s)}{D(s)} \]

where \( N, D \) are entire functions of \( s \) of order \((1/n)\) with infinitely many zeros diverging to infinity.

Moreover, the transfer function \( G_0 \) is real, i.e.,

\[ G_0(\bar{s}) = \overline{G_0(s)} \]

Hence, the poles and zeroes occur in conjugate pairs.
**Theorem**

The CLS transfer function for the feedback law

\[ u = -k y + v \]

is

\[ G_k(s) = \frac{N(s)}{D(s) + kN(s)} \]

and hence the closed loop poles (CLP) are the roots of the return difference equation

\[ D(s) + kN(s) = 0 \]
Theorem

The operator \( A_k = L \) with

\[
D(A_k) = \{ \varphi \in H^n : B_1(\varphi) + kB_0(\varphi) = 0, \\
B_j(\varphi) = 0, \; j = 2, \ldots, n \}.
\]

generates a \( C_0 \) semigroup and \( \sigma(A_k) \) is precisely the CLP, i.e., the roots of the return difference equation

\[
D(s) + kN(s) = 0.
\]
Theorem

The closed loop spatial operators $A_k$ form a holomorphic family in $k$ satisfying the “separation of the spectrum” condition for real $k$. 
Theorem

- The closed loop spatial operators $A_k$ form a holomorphic family in $k$ satisfying the “separation of the spectrum” condition for real $k$.
- In particular, the operators $A_k$ are discrete spectral operators with spectrum $\{s_j^k\}_{j=1}^{\infty}$ such that the eigenfunctions and associated functions form a Riesz basis in $L^2(0, 1)$. 
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In particular, the operators $A_k$ are discrete spectral operators with spectrum $\{s_{kj}^k\}_{j=1}^\infty$ such that the eigenfunctions and associated functions form a Riesz basis in $L^2(0, 1)$. 
Definition

For DPS there is no high frequency gain since \( G_k \) always has an essential singularity at \( \infty \). There is an “instantaneous gain”

Instantaneous Gain

The “instantaneous gain” \( \mathcal{K} \) is defined by \( \mathcal{K} = (-1)^p \) where

\[
p = \arg(\alpha_0/\alpha_1)/\pi + (\ell_0 + \ell),
\]

with \( \ell = m_1 - m_0 > 0 \),

\[
\ell_0 = 1 + \max\{j : m_j > m_0, \ j = 1, \ldots, \mu\},
\]

and \( \alpha_0, \alpha_1 \) and \( m_0, m_1 \) are the coefficients of the highest order terms and orders of the output and input operators respectively.
Theorem

For all $k$, all but a finite number of the open loop poles $\{s^0_j\}_{j=1}^{\infty}$ and zeroes $\{s^\infty_j\}_{j=1}^{\infty}$ are real, interlace on the negative real axis and tend to minus infinity as $j \to \infty$. If the system is minimum phase and the sign of the gain is chosen so that $k \cdot K > 0$, then there exists a $k_0$ such that for $k \cdot K > k_0 \cdot K$ the closed loop system is exponentially stable.
Theorem

For all $k$, all but a finite number of the open loop poles $\{s_j^0\}_{j=1}^\infty$ and zeroes $\{s_j^\infty\}_{j=1}^\infty$ are real, interlace on the negative real axis and tend to minus infinity as $j \to \infty$.

Choosing $k \cdot \mathcal{K} > 0$, the CLP $\{s_j^k\}_{j=1}^\infty$ vary from the open loop poles to the open loop zeroes as the gain $k$ varies from 0 to $\infty$.

If the system is minimum phase and the sign of the gain is chosen so that $k \cdot \mathcal{K} > 0$, then there exists a $k_0$ such that for $k \cdot \mathcal{K} > k_0 \cdot \mathcal{K}$ the closed loop system is exponentially stable.
Root Locus for Birkhoff Regular systems
The spectrum of $L_0 = (d/dx)^{2\mu}$ with boundary conditions

$$B_i(f) = f^{(m_i)}(0) = 0, \quad B_{\mu+i}(f) = f^{(m_{\mu+i})}(1) = 0,$$

$i = 1, \ldots, \mu$, and

$$m_1 > \cdots > m_\mu, \quad m_{\mu+1} > \cdots > m_n$$

consists of only real points.
I have proven this for $n = 2, 4, 6, 8$. 
Convergence of Trajectories

Set $T_k(t) = e^{A_k t}$ for $0 \leq k \leq \infty$. For initial condition $\varphi$ define $z^k(\cdot, t) = T_k(t)\varphi$. A consequence of the root locus results is that for any $T > 0$

$$\sup_{t \in [t_0, T]} \|z^k(t) - z^\infty(t)\|_{H^1(0,1)} \xrightarrow{k \to \infty} 0.$$  

Due to the embedding $H^1(0,1) \subset C[0,1]$, this implies that for $T > t_0 > 0$

$$\max_{x \in [0,1]} \sup_{t \in [t_0, T]} |z^k(x, t) - z^\infty(x, t)| \xrightarrow{k \to \infty} 0.$$  

“Closed-loop trajectories converge to trajectories of the open-loop zero dynamics as $k \to \infty$”
Nonlinear Zeros for BCS
For nonlinear systems we cannot talk about system zeros.

Thus while the notions of zeros and poles no longer make sense for nonlinear systems, the notion of convergence of trajectories does.

In the nonlinear case the closed-loop trajectories asymptotically tend to trajectories of the open-loop zero dynamics as the gain parameter is increased to infinity.
Nonlinear BCS

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with $C^2$-boundary $\Gamma = \partial\Omega$, $x = (x_1, \cdots, x_n) \in \Omega$ and $t \geq 0$.

$$z_t - Lz + \text{div} \vec{F}(z) + G(z) = h, \quad \text{Dynamics}$$

$$\mathcal{B}(z) \equiv \sum a_{ij}(x) z_{x_i}(x, t) \eta_j(x) \bigg|_{x \in \Gamma} = u(x, t), \quad \text{B.C.}$$

$$\vec{\eta}(x) = (\eta_1(x), \cdots, \eta_n(x)) \quad \text{unit normal vector } x \in \Gamma,$$

$$z(x, 0) = \varphi(x), \quad \varphi \in L^2(\Omega), \quad \text{I.C.}$$

$$y(x, t) = z(x, t), \quad x \in \Gamma, \quad \text{MeasuredOutput}$$
Nonlinear BCS

\[ z(x, t) \quad \text{State of System} \]

\[ \text{div} \vec{F}(z) = \text{div} \vec{F}(x, t, z) \quad \text{Convection} \]

\[ G(z) = G(x, t, z) \quad \text{Reaction} \]

\[ h = h(x, t) \quad \text{Disturbance} \]

The convective term is given by

\[ \text{div} \vec{F}(x, t, w) = \sum (F_{ix_i} + F_{iw}w_{x_i}) \]

\[ \text{Burgers:} \quad \vec{F}(w) = \frac{w^2}{2} \text{ and } \text{div} \vec{F}(w) = ww_{x}. \]
$Q_{t_0, T} = \Omega \times [t_0, T]$ and $Q_T = \Omega \times [0, T]$
For \((x, t) \in Q_{0,T}\) and \(\xi \in \mathbb{R}\)

1. \(\xi G(x, t, \xi) \geq -c_1\xi^2 - c_2\)

2. For \(p, q\) and \(\alpha\)

\[|\mathcal{D}^p_t \mathcal{D}^\alpha_x \mathcal{D}^q_\xi G(x, t, \xi)| \leq c_3|\xi|^r + c_4, \ r < 2 + \frac{4}{n}\]

3. For \(p, q\) and \(\alpha\)

\[|\mathcal{D}^p_t \mathcal{D}^\alpha_x \mathcal{D}^q_\xi F_i(x, t, \xi)| \leq c_5|\xi|^s + c_6, \ s < 1 + \frac{2}{n}\]

4. For any \(t_0 \in (0, T)\), with \(Q_{t_0,T} = \Omega \times [t_0, T]\)

\(h \in L^\infty([0, T], L^2(\Omega)) \cap H^{2m,m}(Q_{t_0,T}).\)
Nonlinear Feedback Control

We consider a nonlinear feedback control law:

\[ u(x, t) = -k(x)y(x, t) \]

where \( k \in C^1(\Gamma) \).
Closed Loop System

\[
z_t - Lz + \text{div}\vec{F}(z) + G(z) = h,
\]

\[
\left.\left[B(z)(x, t) + k(x)z(x, t)\right]\right|_x \in \Gamma = 0,
\]

\[
z(x, 0) = \varphi(x), \quad \varphi \in L^2(\Omega),
\]
Dynamics of Nonlinear System

**Theorem**

For the CLS there is a globally defined dynamical system on the state space $L^2(\Omega)$. 
Theorem

For time independent $h$, $F_i$ and $G$, the dynamics define a semigroup $\{S_t, t \geq 0\}$.

* $S_t$ is continuous in $t$ and $\varphi \in L^2(\Omega)$.
* $S_t$ is compact for $t > 0$.
* the closed loop system is globally Lyapunov stable.
* there is a global attractor in $L^2(\Omega)$. 
Nonlinear Zeros

Zero and Pole Dynamics

- Denote the solution of the CLS by \( z^k(x, t) \) to emphasize that it depends on the (nonlinear) gain.
- Denote the solution of the zero dynamics by \( z^\infty(x, t) \), i.e., the CLS system with \( k = \infty \).

Zero Dynamics

\[
\begin{align*}
  z_t - Lz + \text{div}\tilde{F}(z) + G(z) &= h, \\
  \left. z(x, t) \right|_{x \in \Gamma} &= 0, \\
  z(x, 0) &= \varphi(x), \quad \varphi \in L^2(\Omega),
\end{align*}
\]
Consider a sequence of controls

\[ \{k_n(x), x \in \Gamma\} \]

such that

\[ \min_{x \in \Gamma} k_n(x) \xrightarrow{n \to \infty} +\infty \]

Then

\[ \max_{t \in [0,T]} \|z^k(t) - z^\infty(t)\| \xrightarrow{n \to \infty} 0 \]
Nonlinear Asymptotic Phase

Theorem

Moreover, for any \( T > t_0 > 0 \)

\[
\max_{t \in [t_0, T]} \| z^k(t) - z^\infty(t) \|_{H^q(\Omega)} \xrightarrow{n \to \infty} 0
\]

where \( q = \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

Due to the embedding \( H^q(\Omega) \subset C(\overline{\Omega}) \), this implies that

\[
\max_{t \in [t_0, T]} \| z^k(t) - z^\infty(t) \|_{C(\Omega)} \xrightarrow{n \to \infty} 0.
\]
Regulation of Boundary Controlled DPS
Nonlinear Boundary Control System

Nonlinear BCS:

\[ z_t = Az + f(z) + d, \quad z(0) = z_0 \]
\[ Bz(t) = u_{in}(t), \quad \text{boundary input} \]
\[ y(t) = Cz(t), \quad \text{boundary output} \]
Statement of Regulation Problem

- **Given a boundary control system and**
- **a Reference Signal (signal to be tracked) \( y_r(t) \)**

Define the error by 
\[ e(t) = y(t) - y_r(t) \]

**Main Problem**
Find a control \( u(t) \), so that 
\[ e(t) \xrightarrow{t \to \infty} 0 \]
while the state \( z(t) \) remains bounded.
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Main Problem

Find a control \( u(t) \), so that

\[
e(t) \xrightarrow{t \to \infty} 0
\]

while the state \( z(t) \) remains bounded.
Zero Dynamics Inverse Design
Given a nonlinear co-located BCS

\[ z_t = A_0 z + f(z) + d, \quad z(0) = z_0 \]

\[ Bz(t) = u_{in}(t), \quad \text{(boundary input)} \]

\[ y(t) = Cz(t), \quad \text{(boundary output)} \]

with signal to be tracked \( y_r(t) = Qw(t) \).
Zero Dynamics Inverse (ZDI)

Plant

Given a nonlinear co-located BCS

\[ z_t = A_0 z + f(z) + d, \quad z(0) = z_0 \]

\[ \mathcal{B} z(t) = u_{in}(t), \quad \text{(boundary input)} \]

\[ y(t) = \mathcal{C} z(t), \quad \text{(boundary output)} \]

with signal to be tracked \( y_r(t) = Q w(t) \).

We define the ZDI Controller

ZDI

\[ \begin{aligned}
\xi_t &= A_0 \xi + f(\xi) + d, \quad \xi(0) = \psi \\
\mathcal{C} \xi(t) &= Q w \quad \text{(boundary input)} \\
u_{out}(t) &= \mathcal{B} \xi(t) \quad \text{(boundary output)}
\end{aligned} \]
Zero Dynamics Inverse (ZDI)

\[
\begin{align*}
\dot{\xi}(t) &= A_0 \xi + f(\xi) + d, \quad \xi(0) = \psi \\
C \dot{\xi}(t) &= Qw \quad \text{(boundary input)} \\
u_{\text{out}}(t) &= B \xi(t) \quad \text{(boundary output)}
\end{align*}
\]

The ZDI has the same state equation as the original plant.

**BUT the boundary input has been replaced by the constraint that the error be identically zero,**

\[0 = e(t) = y(t) - y_r(t) = C \xi(t) - Qw.\]
The desired control law for regulation is the boundary input operator $B$ applied to the state of the ZDI, i.e.,

$$u_{out}(t) = B\xi(t).$$
**Theorem**

Under the Main Assumptions, and if $f = 0$, the problem of regulation is solved with $\hat{u}_{\text{in}} = \hat{u}_{\text{out}}$, i.e., for this control

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} (y(t) - y_r(t)) = 0.$$ 

**Proof**

With $\hat{u}_{\text{in}} = \hat{u}_{\text{out}}$ the vector $\eta = z - \xi$ satisfies

$$\eta_t = A\eta, \quad \eta(0) = z_0 - \xi_0 \equiv \eta_0$$

and

$$e(t) = \mathbf{C}\eta(t) \to 0 \quad t \to \infty$$

by our assumptions.
As would be expected, in the nonlinear case there is no general result but the same idea has been applied in numerous examples.
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In the Nonlinear Case

With \( \hat{u}_{\text{in}} = \hat{u}_{\text{out}} \) the vector \( \eta = z - \xi \) satisfies

\[
\eta_t = A\eta + F(\eta, \xi), \quad \eta(0) = z_0 - \xi_0 \equiv \eta_0
\]

where

\[
F(\eta, \xi) = f(\eta + \xi) - f(\xi).
\]

Show this system is (locally) exponentially stable in \( \mathcal{Z}_1 \) so that

\[
e(t) = y(t) - y_r(t) = Cz - C\xi = C\eta(t) \rightarrow 0 \quad t \rightarrow \infty.
\]

The proofs always depend on properties of the nonlinear term, the domains and the boundary conditions.
Thanks for the Great Times
Old Friend – Goodbye!