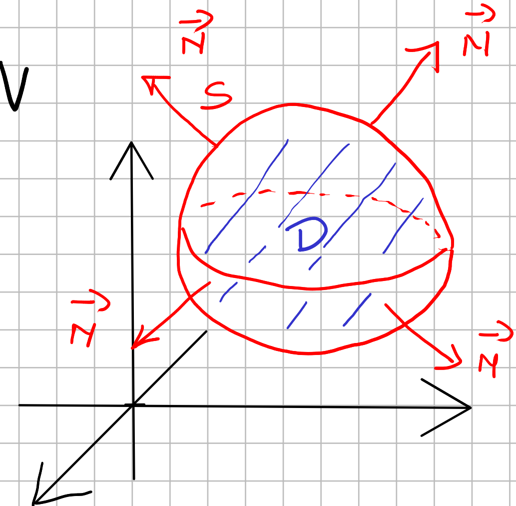
13.7 Divergence Theorem

Let $\vec{F} = \langle f_1, f_2, f_3 \rangle$ a vector field in $V \subseteq \mathbb{R}^3$. Let S a closed surface in V, oriented outward. Let D be the interior of S.

then:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D (\nabla \cdot \vec{F}) \, dV$$

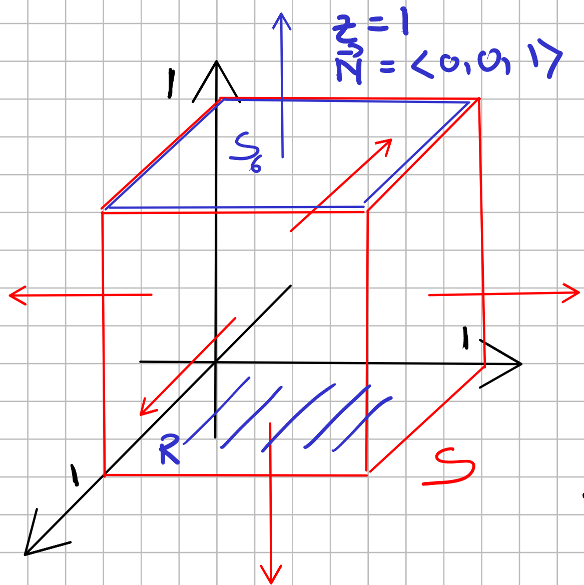


Ex. Evaluate $\iint_S \langle xy^2, yz^2, zx^2 \rangle \cdot d\vec{S}$, where S is the sphere $x^2 + y^2 + z^2 = 4$ oriented outward.

By divergence theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D (y^2 + z^2 + x^2) \, dV$ where D is the ball of radius 2 centered at the origin.

$$\iiint_D x^2+y^2+z^2 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^2 \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{\rho^5}{5} \Big|_0^2 \sin\phi \, d\phi \, d\theta = \frac{32}{5}(2)(2\pi) = \frac{128\pi}{5}$$

Ex. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = \langle x^2, y^2, z^2\rangle$ and S is the open unit box (no top) in the Figure oriented toward the outside.



The surface S is open. In order to use div. theorem we have to add and subtract the integral on the top Face S_6 oriented upward.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S+S_6} \vec{F}\cdot d\vec{S} - \iint_{S_6} \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV - \iint_{S_6} \vec{F}\cdot d\vec{S}$$

Instead of solving 5 Flux integrals we solve a flux integral and a triple integral.

$$\iiint_D \operatorname{div}\vec{F}\, dV = \int_0^1\int_0^1\int_0^1 2x+2y+2z\, dz\,dy\,dx \overset{?}{=} 6\int_0^1\int_0^1\int_0^1 x\, dx\, dy\, dz = 6\int_0^1\int_0^1 \left.\frac{x^2}{2}\right|_0^1 dy\,dx = 6\left(\frac{1}{2}\right) 1 = 3$$

$$\iint_{S_6} \vec{F}\cdot d\vec{S} = \iint_R \langle x^2, y^2, 1^2\rangle\cdot\langle 0,0,1\rangle\, dA = \iint_R 1\, dA = 1$$

$\vec{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_y, 1\rangle\, dA$

$f(x,y) = 1$

$$\iint_S \vec{F}\cdot dA = 3 - 1 = 2$$