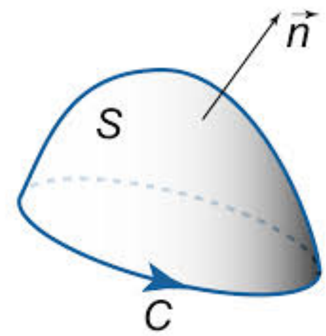
SEC. 13.6 STOKE'S THEOREM:

Let S be an oriented surface whose boundary is the curve C oriented according to $\hat{n}$

When the thumb points in the direction of $\hat{\mathbf{n}}$, the fingers curl in the forward direction around C



Let $\vec{F} = \langle f_1, f_2, f_3 \rangle$ be a vector field with domain V. Let S be in V. Then

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

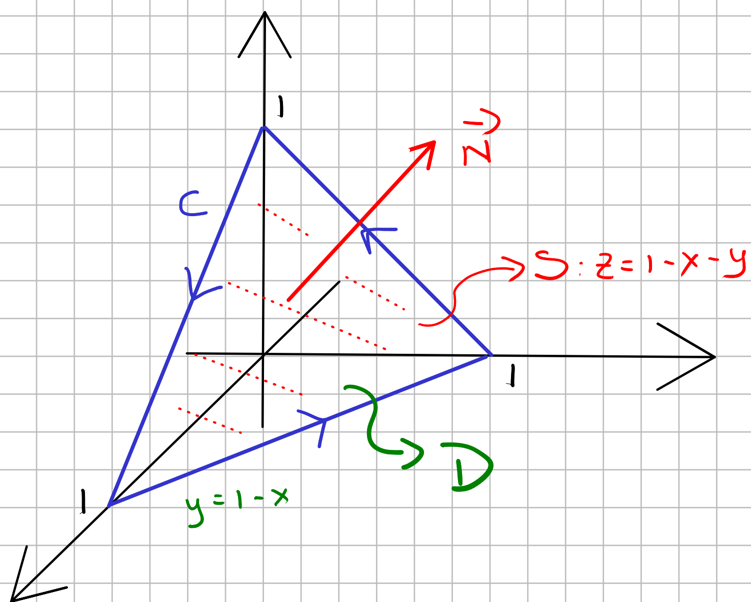
Remark:

Green's theorem is Stoke's theorem in 2D, where S is D (a surface on xy-plane) $\hat{n}$ is $\hat{k} = \langle 0,0,1 \rangle$, c is oriented with $\hat{k}$ (positively oriented Jordan curve) and $\vec{F} = \langle f_1, f_2, 0 \rangle$.

$$\int_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_D \langle 0, 0, f_{2_x} - f_{1_y} \rangle \cdot \langle 0,0,1 \rangle \, dA = \iint_D f_{2x} - f_{1y} \, dA$$

Ex. Let S be the portion of $x+y+z=1$ in the First octant and Let c be the boundary of S traversed counterclockwise when viewed From above. Let $\vec{F} = \langle -\frac{3}{2}y^2, -2xy, yz \rangle$. Use STOKE's Theorem to Find

$$\oint_C \vec{F} \cdot d\vec{R}$$



$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{3}{2}y^2 & -2xy & yz \end{vmatrix} = \hat{i}(z-0) - \hat{j}(0-0) + \hat{k}(-2y+3y)$$

$$= \langle z, 0, y \rangle$$

$$\int_C \vec{F} \cdot d\vec{R} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \langle z, 0, y \rangle \cdot d\vec{S}$$

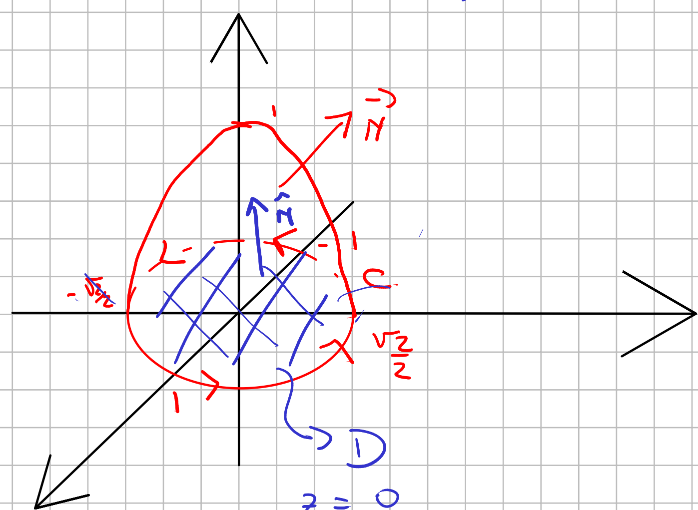
Now we solve the Flux integral according to 13.5

$f(x,y) = 1-x-y$ $\qquad \vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle 1, 1, 1 \rangle$

$$\iint_S \langle z, 0, y \rangle \cdot d\vec{S} = \iint_D \langle 1-x-y, 0, y \rangle \cdot \langle 1,1,1 \rangle \, dA = \iint_D 1-x-y+y \, dA = \iint_D (1-x) \, dA = \int_0^1 \int_0^{1-x} (1-x) \, dy \, dx$$

$$= \int_0^1 (1-x) y \Big|_0^{1-x} dx = \int_0^1 (1-x)^2 dx = \int_0^1 1-2x+x^2 \, dx = x - x^2 + \frac{x^3}{3} \Big|_0^1 = 1-1+\frac{1}{3} = \frac{1}{3}$$

Ex. Evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle x, y^2, z e^{xy} \rangle$ and S is the portion of the paraboloid $z = 1 - x^2 - 2y^2$ with $z \geq 0$ oriented upward.



$z = 0$

$\vec{N} = \langle 0, 0, 1 \rangle = \hat{k}$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{R} = \iint_D \nabla \times \vec{F} \cdot d\vec{A}$$

$d\vec{A} = \hat{k}\, dA$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y^2 & z e^{xy} \end{vmatrix} = \hat{i}(xze^{xy} - 0) - \hat{j}(yze^{xy} - 0) + \hat{k}(0 - 0)$$
$$= \langle xze^{xy}, -yze^{xy}, 0 \rangle$$

$$\iint_D \langle xze^{xy}, -yze^{xy}, 0 \rangle \cdot d\vec{A} = \iint_D \langle \overset{0}{0}, 0, 0 \rangle \cdot \langle 0, 0, 1 \rangle\, dA = 0$$

$z = 0$
xy-plane