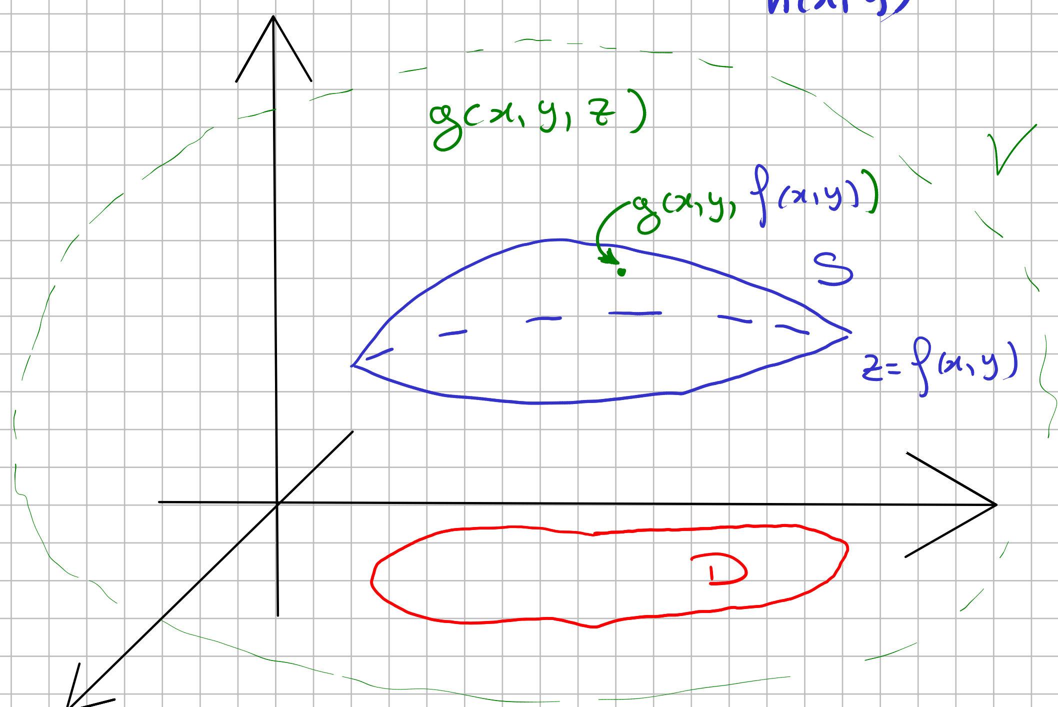
13.5 Surface Integrals.

In 12.4 we studied surface area for a function $z=f(x,y)$ over D.

$$S.A. = \iint_S dS = \iint_D \sqrt{1+z_x^2+z_y^2}\, dA \qquad dS = \sqrt{1+z_x^2+z_y}\ dA$$

Def. Let $g(x,y,z)$ a function on $V \subseteq \mathbb{R}^3$. Let S be a surface described by $z=f(x,y)$ on D. Let S be inside V. Then

$$\iint_S g(x,y,z)\, dS = \iint_D \underbrace{g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}}_{h(x,y)}\ dA$$



Evaluate $\iint_S z^2\, dS$ where S is the positive emisphere $z=\sqrt{4-x^2-y^2}$.

$$g(x,y,z) = z^2$$

$$f(x,y) = \sqrt{4-x^2-y^2}$$

$$dS = \sqrt{1+f_x^2+f_y^2} = \frac{2}{z} = \frac{2}{\sqrt{4-x^2-y^2}}$$

implicit differentiation of the sphere $x^2+y^2+z^2=4$

$$z_x = -\frac{F_x}{F_z} = -\frac{x}{z} \qquad z_y = -\frac{F_y}{F_z} = -\frac{y}{z}$$

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{x^2+y^2+z^2}{z^2}} = \sqrt{\frac{4}{z^2}} = \frac{2}{z}$$

$$g(x,y,f(x,y)) = \left(\sqrt{4-x^2-y^2}\right)^2 = 4-x^2-y^2$$

$D: \quad x^2+y^2 \le 4$

$$\iint_S g(x,y,z)\,dS = \iint_D g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA = \iint_D (4-x^2-y^2)\frac{2}{\sqrt{4-x^2-y^2}}\,dA$$

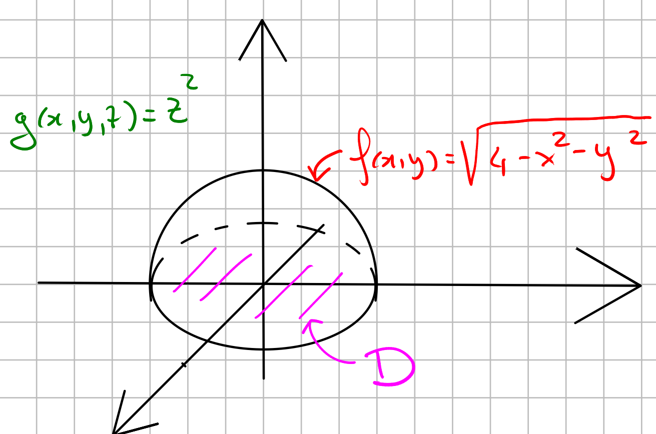
$$= \iint_D 2\sqrt{4-x^2-y^2}\,dA = -\int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\,(-)2r\,dr\,d\theta = -\frac{2}{3}\int_0^{2\pi}(4-r^2)^{3/2}\Big|_0^2 d\theta = -\frac{2}{3}(0-4^{3/2})\,2\pi$$

$$= \frac{32}{3}\pi$$

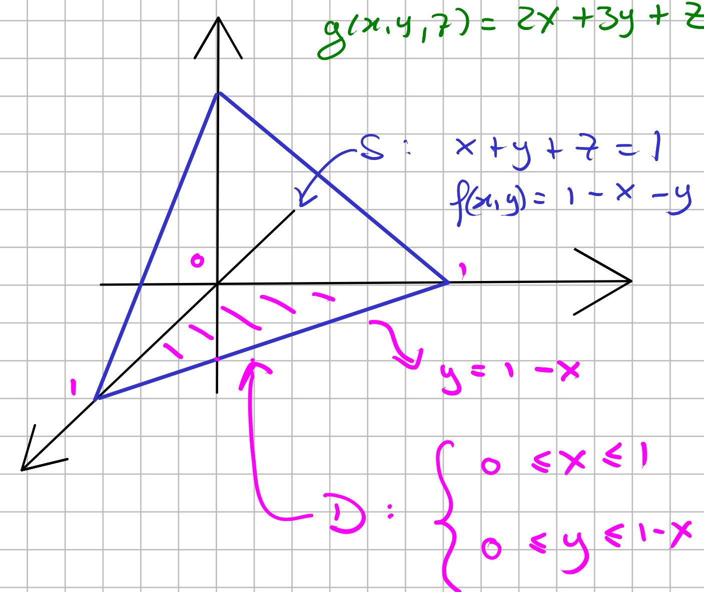
$u = 4-r^2$

$du = -2r\,dr$

$-\int \sqrt{u}\,du = -\frac{2}{3}u^{3/2}$



Ex. Evaluate $\displaystyle\iint_S 2x + 3y + z \; dS$ where S is the portion of $x+y+z = 1$ in the first octant.



$f(x,y) = 1 - x - y,$

$f_x = -1 , \quad f_y = -1$

$$ds = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}$$

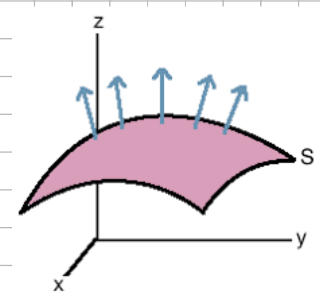
$$g(x,y,f(x,y)) = 2x + 3y + (1 - x - y) = x + 2y + 1$$

$$\iint_S g(x,y,z)\, dS = \iint_D g(x,y,f(x,y)) \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_D (x + 2y + 1)\sqrt{3}\, dA$$

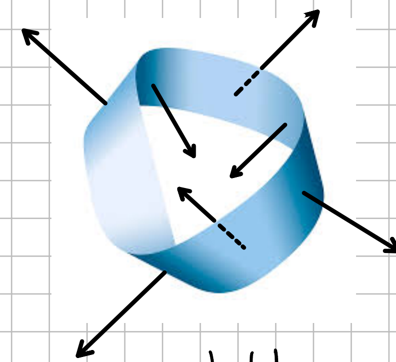
$$= \sqrt{3}\int_0^1 \int_0^{1-x} (x + 2y + 1)\; dy\, dx = \sqrt{3}\int_0^1 (x+1)y + y^2 \Big|_0^{1-x} dy = \sqrt{3}\int_0^1 (1 - x^2) + (1-x)^2 dx$$

$$= \sqrt{3}\int_0^1 2 - 2x \; dx = \sqrt{3}\; (2x - x^2)\Big|_0^1 = \sqrt{3}\; (2 - 1) = \sqrt{3}$$

Def. An orientable surface is a surface where is possible to make a consisten choice for the surface normal vector



orientable



non-orientable

mobius strip

Def. Flux Integral. Let $\vec{F}(x,y,z)$ be a vector field in $V \subseteq \mathbb{R}^3$.

Let S be an orientable surface inside V with unit normal $\vec{N}$. Then the flux integral of $\vec{F}$ in S is given

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N}\, dS$$

scalar function $g(x,y,z)$

Remark. If S_1 is a surface with opposite orientation respect to S

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \vec{N}_1\, dS = \iint_S \vec{F} \cdot (-\vec{N})\, dS = -\iint_S \vec{F} \cdot d\vec{S}$$

Theorem. Let $\vec{F}(x,y,z)$ be a vector Field in $V \subseteq \mathbb{R}^3$. Let S, defined by $z = f(x,y)$ for (x,y) in D, be a surface in V oriented upward. Then the Flux integral of $\vec{F}$ though S is given by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N}\, dS = \iint_D \vec{F}(x,y,f(x,y)) \cdot \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+\cancel{f_x^2+f_y^2}}} \underbrace{\sqrt{1+\cancel{f_x^2+f_y^2}}\, dA}_{dS}$$

From 11.5 $\vec{N} = \dfrac{\langle -f_x, -f_x, 1\rangle}{\sqrt{1+f_x^2+f_y^2}}$

upward normal to the graph $z = f(x,y)$

$$\boxed{= \iint_D \vec{F}(x,y,f(x,y)) \cdot \langle -f_x, -f_x, 1\rangle\, dA}$$

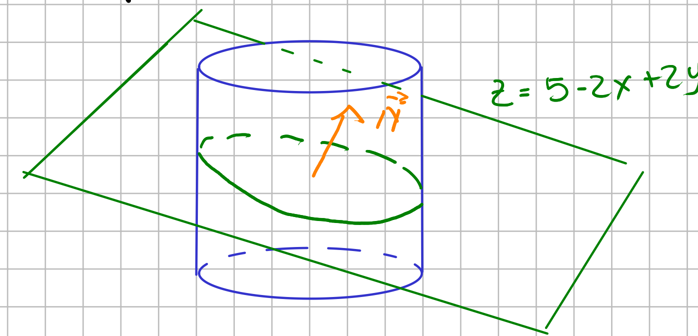
Remark if S is oriented downward

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(x,y,f(x,y)) \cdot \langle f_x, f_x, -1\rangle\, dA$$

What if S is oriented leftward, righward, frontward or backward?

$y = g(x,z)$ $\qquad$ $x = h(x,y)$

Ex. Evaluate $\displaystyle\iint_S \langle 3x, 2y, z\rangle \cdot d\vec{S}$ where S is the portion of $2x-2y+z=5$ inside the cylinder $x^2+y^2=1$ oriented upward



$$f(x,y) = 5-2x+2y$$

$$f_x = -2, \quad f_y = 2$$

$$\langle -f_x, -f_y, 1\rangle = \langle 2,-2,1\rangle$$

$$\vec{F}(x,y,f(x,y)) = \langle 3x, 2y, 5-2x+2y\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle 3x, 2y, 5-2x+2y\rangle \cdot \langle 2,-2,1\rangle \, dA = \iint_D 6x-4y+5-2x+2y \, dA$$

$$= \iint_D 4x-2y+5 \, dA = \int_0^{2\pi}\int_0^1 (4r\cos\theta - 2r\sin\theta)\, r\,dr\,d\theta + 5\,\text{Area}(D) =$$

$$= \int_0^{2\pi} (4\cos\theta - 2\sin\theta) \underbrace{\frac{r^3}{3}\Big|_0^1}_{\left(\frac{1}{3}-0\right)} d\theta + 5\pi(1)^2 = \frac{1}{3}\underbrace{(4\sin\theta + 2\cos\theta)\Big|_0^{2\pi}}_{4(0-0)+2(1-1)} + 5\pi = 5\pi$$