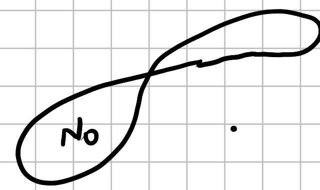
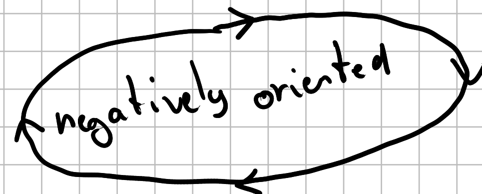
Section 13.4   Green's Theorem, 2D only

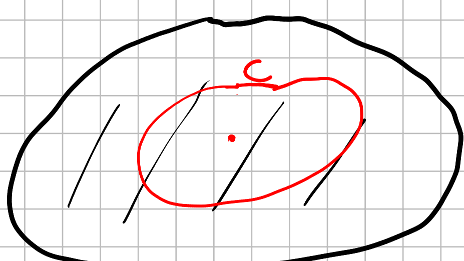
DEF. A Jordan curve is a closed curve that does not self intersect.



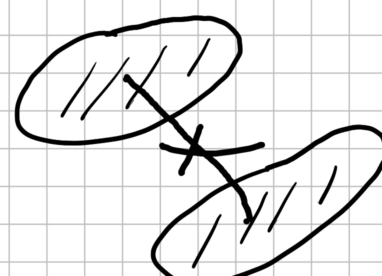
DEF. A positively oriented Jordan curve is an oriented Jordan curve where the inside is always on the left.



DEF. A connected domain D is simply connected if any closed curve in D can be shrunk continuosly to a point without leaving D.



simply connected



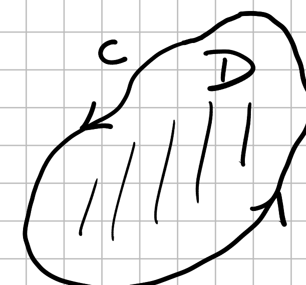
no connected

connected, but no simply connected

THEOREM. Let $\vec{F} = \langle f_1, f_2 \rangle$ be a vector field defined on a simply connected domain. Let C be a positively oriented Jordan curve in the domain of $\vec{F}$ and let D be the region inside C.

Then

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_D \left(f_{2_x} - f_{1_y} \right) dA$$



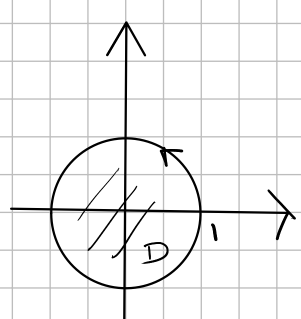
Symbol for positively oriented Jordan curve

If $\vec{F}$ is conservative $f_{2_x} - f_{1_y} = 0$

Equality between a line integral and a double integral

Ex. Find $\int_C \vec{F} \cdot d\vec{R}$, where $\vec{F} = \langle y^3, -x^3 \rangle$ and C is the unit circle centered at the origin traversed counterclockwise.

The Domain of $\vec{F}$ is $\mathbb{R}^2$, C is a posititely oriented Jordan curve, then



$$\oint_C \vec{F} \cdot d\vec{R} = \iint_D -3x^2 - 3y^2 \, dA = -3 \iint_D x^2 + y^2 \, dA = -3 \int_0^{2\pi} \int_0^1 r^2 \, r \, dr \, d\theta$$

$$= -3 \int_0^{2\pi} \left. \frac{r^4}{4} \right|_0^1 d\theta = -\frac{3}{4} (2\pi) = -\frac{3}{2}\pi$$

Application. We can use Green's theorem in both ways.

Consider the 3 vector fields $\frac{1}{2}\langle -y, x\rangle$, $\langle -y, 0\rangle$, $\langle 0, x\rangle$

Each of them satisfies $f_{2_x} - f_{1_y} = 1$.

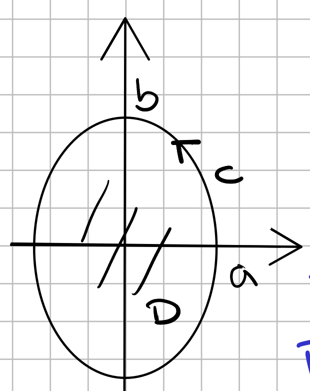
Then for any simple connected domain D bounded by the positively oriented Jordan curve C we have:

$$\text{Area}(D) = \iint_D 1\, dA = \oint_C \frac{1}{2}\langle -y, x\rangle \cdot d\vec{R} = \oint_C \langle -y, 0\rangle \cdot d\vec{R} = \oint_C \langle 0, x\rangle \cdot d\vec{R}$$

Ex. Find the area of the ellipse with semiaxis a and b centered at the origin.

$$\text{Area}(D) = \iint_D 1\, dA = \oint_C \frac{1}{2}\langle -y, x\rangle \cdot d\vec{R} = \int_0^{2\pi} \frac{1}{2} \underbrace{\langle -b\sin t, a\cos t\rangle}_{\vec{F}(\vec{R}(t))} \cdot \underbrace{\langle -a\sin t, b\cos t\rangle}_{\vec{R}'(t)} dt$$

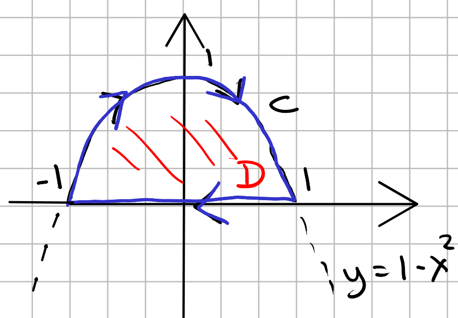
$$= \frac{1}{2} ab \int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1}\, dt = \frac{1}{2} ab(2\pi) = ab\pi$$



$\vec{R}(t) = \langle a\cos t, b\sin t\rangle \quad 0 \le t \le 2\pi$

$\vec{R}'(t) = \langle -a\sin t, b\cos t\rangle$

Ex. Find $\oint_C \langle x^3 + xy^2 + y^2, -\cos y + x^2y + x \rangle \cdot d\vec{R}$, where c is the closed piecewise defined curve defined by the posite arc of parabola $y = 1 - x^2$ and the x-axis traversed clockwise.



c is negatively oriented: $\oint_C \vec{F} \cdot d\vec{R} = \iint_D f_{1_y} - f_{2_x} \, dA$.

then

Everything conservative simplifies

$$\oint_C \langle x^3 + xy^2 + y^2, -\cos y + x^2y + x \rangle \cdot d\vec{R} = \iint_D 0 + 2xy + 2y - (0 + 2xy + 1) \, dA$$

$$= \iint_D 2y - 1 \, dA = \int_{-1}^{1} \int_0^{1-x^2} 2y - 1 \, dy\, dx = \int_{-1}^{1} y^2 - y \Big|_0^{1-x^2} dx = \int_{-1}^{1} (1-x^2)^2 - (1-x^2) \, dx$$

$1 - 2x^2 + x^4 - 1 + x^2$

$$= \int_{-1}^{1} -x^2 + x^4 \, dx = -\frac{x^3}{3} + \frac{x^5}{5} \Big|_{-1}^{1} = \left(-\frac{2}{3} + \frac{2}{5}\right) = -\frac{4}{15}$$