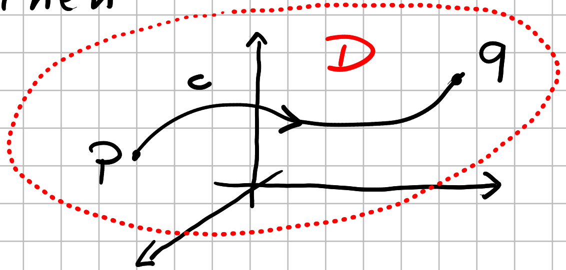
13.3 CONSERVATIVE VECTOR FIELDS.

Def. Let $\vec{F}$ be a vector field and f a scalar function both defined on D. If $\nabla f = \vec{F}$ we say that $\vec{F}$ is conservative and we call f the scalar potential of $\vec{F}$.

$$\nabla f = \langle \partial_x, \partial_y, \partial_z \rangle f = \langle f_x, f_y, f_z \rangle = \langle f_1, f_2, f_3 \rangle$$

THEOREM: Let $\vec{F}$ be conservative with scalar potential f in D. Let c be any curve in D with start point P and end point Q. Then

$$\int_c \vec{F} \cdot d\vec{R} = f(Q) - f(P)$$



Proof: Let c be parametrized by $\vec{R}(t)$ for $a \le t \le b$, then

$$\int_c \vec{F} \cdot d\vec{R} = \int_a^b \overset{\nabla f}{\vec{F}}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt = \int_a^b \nabla f(\vec{R}(t)) \cdot \vec{R}'(t)\, dt = \int_a^b \overbrace{f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}}^{\frac{df}{dt} \text{ by chain rule}}\, dt =$$

$$= \int_a^b \frac{df}{dt}(\vec{R}(t))\, dt \overset{\text{Fondamental theorem of calculus}}{=} f(\vec{R}(b)) - f(\vec{R}(a)) = f(Q) - f(P)$$

This theorem gives us an easy alternative way to solve line integrals if and only if $\vec{F}$ is conservative on C.

Ex. Evaluate $\int_c \langle xy^2, x^2y \rangle \cdot d\vec{R}$ where c is either the line segment From (0,0) to (2,4) or the parabola $y = x^2$ From (0,0) to (2,4)

This is the last problem we solved in 13.2.

$\vec{F} = \langle \overset{f_1(x,y)}{xy^2}, \overset{f_2(x,y)}{x^2y} \rangle$ is conservative with scalar potential $f = \frac{1}{2} x^2 y^2$

↑ Try it

$f_x = \frac{1}{2} 2 x y^2 = x y^2 = f_1$

$f_y = \frac{1}{2} 2 x^2 y = x^2 y = f_2$

Then $\int_c \vec{F} \cdot d\vec{R} = f(2,4) - f(0,0) = \frac{1}{2} 2^2 4^2 - 0 = 32$

↑ immediate answer

it holds For both curves since they share the same end points.

How do we know if $\vec{F}$ is conservative?

How do we find f ?

$\boxed{2D.}$ To test if $\vec{F} = \langle f_1, f_2 \rangle$ is conservative, check if $f_{1_y} = f_{2_x}$

Ex. $\vec{F} = \langle xy^2, x^2y \rangle$, $\begin{cases} f_{1_y} = 2xy \\ f_{2_x} = 2xy \end{cases}$, $f_{1y} = f_{2x} \Rightarrow \vec{F}$ is conservative

Why is that? If $\vec{F}$ is conservative $\vec{F} = \langle f_1, f_2 \rangle = \nabla f = \langle f_x, f_y \rangle$ then

$\begin{cases} f_1 = f_x \\ f_2 = f_y \end{cases}$, then $\begin{cases} f_{1_y} = f_{xy} \\ f_{2_x} = f_{yx} \end{cases}$, since $f_{xy} = f_{yx} \Rightarrow f_{1y} = f_{2x}$.

To find f solve the system $\begin{cases} f_x = f_1 \\ f_y = f_2 \end{cases}$

Ex. $\begin{cases} f_x = xy^2 \\ f_y = x^2y \end{cases}$ or $\begin{cases} \int f_x \, dx = \int xy^2 \, dx \\ \int f_y \, dy = \int x^2y \, dy \end{cases}$ or $\begin{cases} f = \frac{1}{2}x^2y^2 + g(y) + C \\ f = \frac{1}{2}x^2y^2 + h(x) + C \end{cases}$ >> combine with $C = 0$

$$f = \frac{1}{2}x^2y^2$$

How do we combine the solutions from the two integrals?

$$\begin{cases} f = \int f_1(x,y)\,dx \\ f = \int f_2(x,y)\,dy \end{cases}$$

$$\left\{\begin{array}{l} f = \\ \\ f = \end{array}\right. \boxed{\begin{array}{l} \left(h(x) + \ell(x,y)\right) \\ \\ \left(g(y) + \cancel{\ell(x,y)}\right) \end{array}} \begin{array}{l} + \; g(y) \\ \\ + \; h(x) \end{array}$$

↑ combine the two antiderivatives removing repetitions.

Ex. chek if the Following vector Field is conservative and if true Find the scalar potential.

$$\vec{F} = \langle x^3 y \, , \, x^2 y^2 \rangle$$

check $\begin{cases} f_{1y} = x^3 \\ f_{2x} = 2xy^2 \end{cases} \Rightarrow f_{1y} \neq f_{2x}$

the vector Field is not conservative
there is no scalar potential.

Ex. Verify if $\vec{F} = \langle y^2 e^{xy^2} + x^2 y,\ 2xy e^{xy^2} + \frac{x^3}{3} + y \rangle$ is conservative, and if true Find the scalar potential f.

$$\begin{cases} f_{1_y} = 2y e^{xy^2} + 2xy^3 e^{xy^2} + x^2 \\ f_{2_x} = 2y e^{xy^2} + 2xy^3 e^{xy^2} + x^2 + 0 \end{cases} \Rightarrow f_{1y} = f_{2x} \Rightarrow \vec{F} \text{ is conservative}$$

$$\begin{cases} f = \int \left(y^2 e^{xy^2} + x^2 y \right) dx = \boxed{e^{xy^2} + \frac{x^3}{3} y} + g(y) \\ f = \int \left(2x e^{xy^2} + \frac{x^3}{3} + y \right) dy = \boxed{\cancel{e^{xy^2}} + \cancel{\frac{x^3}{3} y} + \frac{y^2}{2}} + h(x) \end{cases}$$

$$f = e^{xy^2} + \frac{x^3}{3} y + \frac{y^2}{2}$$

3D.

$\vec{F}$ is conservative if curl $\vec{F} = \vec{0} = \nabla \times \vec{F}$

Find f solving the system

$$\begin{cases} f = \int f_1 dx = \\ f = \int f_2 dy = \\ f = \int f_3 dz = \end{cases} \boxed{\begin{matrix} g_1(x,y,z) \\ g_2(x,y,z) \\ g_3(x,y,z) \end{matrix}} \begin{matrix} + h_1(y,z) \\ + h_2(x,z) \\ + h_3(x,y) \end{matrix}$$

↓

combine the 3 antiderivatives removing repetitions!

Ex. Verify that $\vec{F} = \left\langle \frac{1}{2}x^2y^2z + xy^2, \frac{1}{3}x^3yz + x^2y + y^2, \frac{1}{6}x^3y^2 \right\rangle$ is conservative and find the scalar potential f.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ \frac{1}{2}x^2y^2z + xy^2 & \frac{1}{3}x^3yz + x^2y + y^2 & \frac{1}{6}x^3y^2 \end{vmatrix} = \hat{i}\left(\frac{1}{3}x^3y - \frac{1}{3}x^3y\right) - \hat{j}\left(\frac{1}{2}x^2y^2 - \frac{1}{2}x^2y^2\right)$$

$$+\hat{k}\left(x^2yz + 2xy - (x^2yz + 2xy)\right) = \langle 0, 0, 0 \rangle = \vec{0}$$

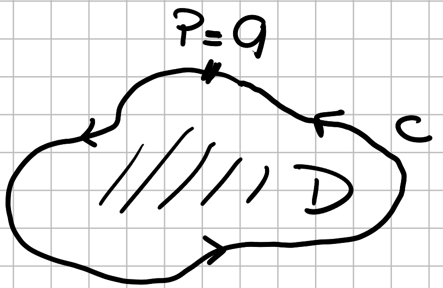
then $\vec{F}$ is conservative

$$\left\{\begin{array}{l} f = \int\left(\frac{1}{2}x^2y^2z + xy^2\right)dx = \frac{1}{2}\frac{x^3}{3}y^2z + \frac{x^2y^2}{2} + \dots \\ f = \int\left(\frac{1}{3}x^3yz + x^2y + y^2\right)dy = \frac{1}{3}\cancel{x^3\frac{y^2}{2}}z + \cancel{\frac{x^2y^2}{2}} + \frac{y^3}{3} + \dots \\ f = \int \frac{1}{6}x^3y^2\,dz = \frac{1}{6}\cancel{x^3y^2}z + \dots \end{array}\right\}$$

combine $f = \frac{1}{6}x^3y^2z + \frac{x^2y^2}{2} + \frac{y^3}{3}$

THEOREM: Let $\vec{F}$ be conservative in D, and c be any close curve whose interior is in D. Then

closed curve integral sign → $\oint_c \vec{F} \cdot d\vec{R} = 0$ ← We do not need to find the scalar potential



Proof: Cut the curve at any point P=q then

$$\oint_c \vec{F} \cdot d\vec{R} = \int_c \vec{F} \cdot d\vec{R} = f(q) - f(P) = 0$$

close ← artifically open