Section 13.2 Line Integrals

In section 10.4 we studied arclength of a curve in 3D:

Let c be a smooth curve parametrized by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ $a \leq t \leq b$

then the arclength is given by:

$$S = \int_c ds = \int_a^b \|\vec{R}'(t)\| \, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt$$

Line integral of $f(x,y,z)$

Let $f(x,y,z)$ be a function of several variables on $D \subseteq \mathbb{R}^3$. Let c be a smooth curve within D parametrized by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ $a \leq t \leq b$

then the line integral of f on c is given by

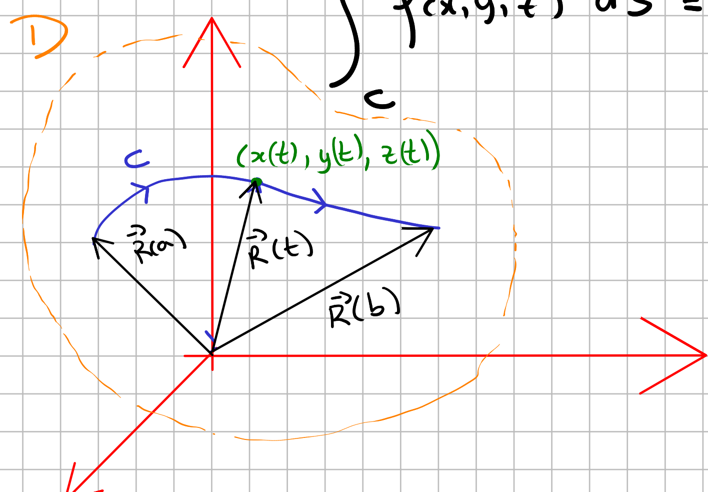
$$\int_c f(x,y,z)\, ds = \int_a^b f(\vec{R}(t)) \underbrace{\|\vec{R}'(t)\| \, dt}_{ds} = \int_a^b f(x(t), y(t), z(t)) \underbrace{\sqrt{(x')^2+(y')^2+(z')^2}\, dt}_{\text{as for the arclength}}$$

f evaluated on c

ds

Replace (x,y,z) with the points on the curve $x(t), y(t), z(t)$

as for the arclength



Remark. The arclength formula is a particular case of line integral with $f(x,y,z)=1$.

Ex. Evaluate the line integral of $f(x,y,z) = x^2 z$ on the curve c given by
$\vec{R}(t) = \langle \cos(t), 2t, \sin(t) \rangle \quad 0 \leq t \leq \pi$.

$D = \mathbb{R}^3$, c is a smooth curve inside D

$\vec{R}'(t) = \langle -\sin(t), 2, \cos(t) \rangle$, $\quad \|\vec{R}'(t)\| = \sqrt{(-\sin(t))^2 + 2^2 + (\cos t)^2} = \sqrt{5}$, then

$$L.I. = \int_c f\,ds = \int_0^{\pi} f(\cos(t), 2t, \sin(t))\ \sqrt{5}\,dt = -\int_0^{\pi} \cos^2(t)(-)\sin(t)\ \sqrt{5}\,dt = -\sqrt{5}\ \frac{\cos^3(t)}{3}\Big|_0^{\pi}$$

$u = \cos(t)$

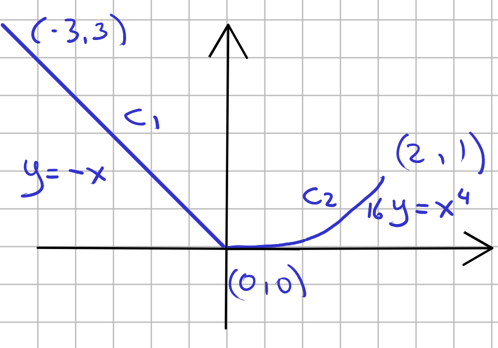
$du = -\sin(t)dt$

$-\sqrt{5}\int u^2\,du = -\sqrt{5}\ \frac{u^3}{3}$

$$= -\frac{\sqrt{5}}{3}\left((-1)^3 - (1)^3\right) = \frac{2}{3}\sqrt{5}$$

Ex. Evaluate $\int_c xy\,ds$ where C is the union of the line segment from $(-3,3)$ to $(0,0)$ and the function $16y = x^4$ from $(0,0)$ to $(2,1)$.

c is not given as a parametrization but as functions in the xy-plane
we have to find a suitable parametrization for c.



c is piecewise defined to parametrize it we need to split it in two curves and parametrize each individually

c_1 : $\vec{R}_1(t) = \langle t, -t \rangle \quad -3 \leq t \leq 0$, $\vec{R}_1' = \langle 1, -1 \rangle$, $\|\vec{R}_1'\| = \sqrt{2}$

c_2 : $\vec{R}_2(t) = \langle 2t, t^4 \rangle \quad 0 \leq t \leq 1$, $\vec{R}_2' = \langle 2, 4t^3 \rangle$, $\|\vec{R}_2'\| = \sqrt{4 + 16t^6}$
$= 2\sqrt{1+4t^6}$

$16y = x^4$ or $y = \frac{x^4}{16} = \left(\frac{x}{2}\right)^4$ set $x = 2t$ then $y = \left(\frac{2t}{2}\right)^4 = t^4$
$0 \leq x \leq 2 \Rightarrow 0 \leq t \leq 1$

$$L.I. = \int_C xy\,ds = \int_{C_1} xy\,ds + \int_{C_2} xy\,ds = \int_{-3}^{0} t(-t)\sqrt{2}\,dt + \int_0^1 (2t)\,t^4\, 2\sqrt{1+4t^6}\,dt$$

$$= -\sqrt{2}\,\frac{t^3}{3}\Big|_{-3}^{0} + \frac{1}{9}\left(1+4t^6\right)^{3/2}\Big|_0^1 = -\sqrt{2}\left(0-\left(\frac{-27}{3}\right)\right) + \frac{1}{9}\left(5^{3/2}-1\right)$$

$$= -9\sqrt{3} + \frac{1}{9}\left(5^{3/2}-1\right)$$

$u = 1+4t^6$

$du = 4(6)\,t^5\,dt$

$\frac{du}{6} = 4t^5\,dt$

$\int \sqrt{u}\,\frac{du}{6} = \frac{1}{6}\,\frac{2}{3}\,u^{3/2} = \frac{1}{9}\,u^{3/2}$

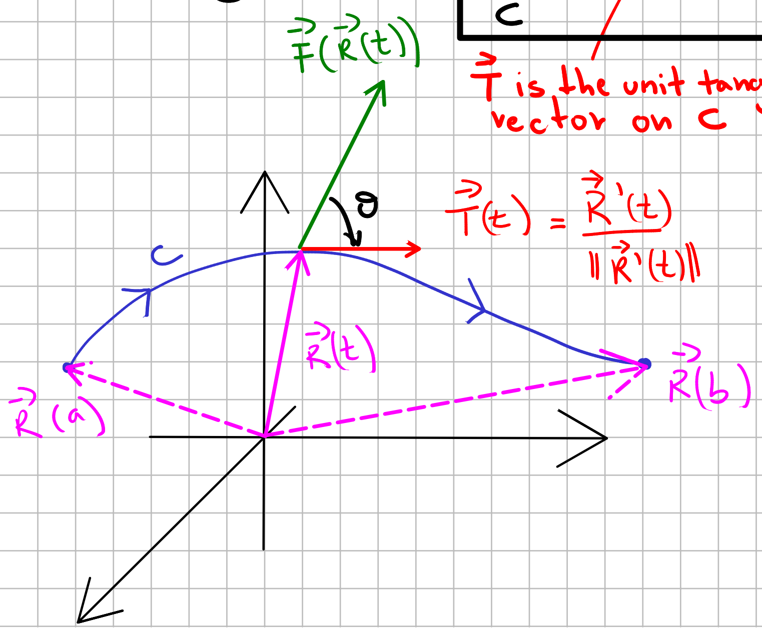
Line integral of a vector field $\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\rangle$

Let $\vec{F}(x,y,z)$ be a vector field on $D \subseteq \mathbb{R}^3$. Let C be smooth curve inside D parametrized by $\vec{R}(t)$ $a \le t \le b$. The line integral of $\vec{F}$ on C is given by

$$\int_C \vec{F}\cdot d\vec{R} = \boxed{\int_C \vec{F}\cdot\vec{T}\,ds} = \int_a^b \vec{F}(\vec{R}(t))\cdot\vec{T}(t)\,\|\vec{R}'(t)\|\,dt = \int_a^b \vec{F}(\vec{R}(t))\cdot\frac{\vec{R}'(t)}{\cancel{\|\vec{R}'(t)\|}}\,\cancel{\|\vec{R}'(t)\|}\,dt$$

$\vec{T}$ is the unit tangent vector on C

$$= \boxed{\int_a^b \vec{F}(x(t),y(t),z(t))\cdot\langle x'(t), y'(t), z'(t)\rangle\,dt}$$



Ex. Evaluate $\int_C (y^2-z^2)\,dx + (2yz)\,dy - x^2\,dz$ where C is given by $\vec{R}(t) = \langle t^2, 2t, t \rangle$ for $0 \le t \le 1$.

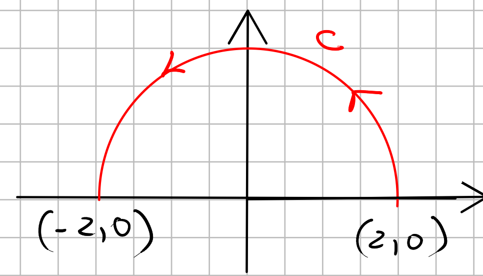
$$\int_C (y^2-z^2)\,dx + (2yz)\,dy - x^2\,dz = \int_C \underbrace{\langle y^2-z^2,\ 2yz,\ -x^2 \rangle}_{\vec{F}(x,y,z)} \cdot \underbrace{\langle dx, dy, dz \rangle}_{d\vec{R}}$$

$$\int_C \vec{F} \cdot d\vec{R} = \int_a^b \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\,dt = \int_0^1 \overbrace{\langle (2t)^2 - t^2,\ 2(2t)t,\ -(t^2)^2 \rangle}^{\langle 3t^2,\ 4t^2,\ -t^4 \rangle} \cdot \overbrace{\langle 2t, 2, 1 \rangle}^{\vec{R}'(t)}\,dt$$

$$= \int_0^1 6t^3 + 8t^2 - t^4\,dt = 6\frac{t^4}{4} + 8\frac{t^3}{3} - \frac{t^5}{5}\Big|_0^1 = \frac{3}{2} + \frac{8}{3} - \frac{1}{5} = \frac{45+80-6}{30} = \frac{119}{30}$$

Ex. Let $\vec{F} = \langle y, x \rangle$ and C be the top of the half circle $x^2+y^2=4$ traversed from $(2,0)$ to $(-2,0)$. Evaluate

$$\int_C \vec{F} \cdot d\vec{R}$$



There are infinite many ways to parametrize a curve C. We must chose the most suitable to get a easy integral to solve.

Poor choice : $y=\sqrt{4-x^2}$, $\vec{R}(t) = \langle -t, \sqrt{4-t^2} \rangle$ for $-2 \le t \le 2$

$$\int_C \vec{F} \cdot d\vec{R} = \int_{-2}^{2} \underbrace{\langle \sqrt{4-t^2}, -t \rangle}_{\vec{F}(\vec{R}(t))} \cdot \underbrace{\langle (-1), \tfrac{1}{2}\tfrac{-2t}{\sqrt{4-t^2}} \rangle}_{\vec{R}'(t)} dt = \int_{-2}^{2} -\sqrt{4-t^2} + \frac{t^2}{\sqrt{4-t^2}}\, dt = \int_{-2}^{2} \frac{-4+2t^2}{\sqrt{4-t^2}}\, dt$$

$$= -t\sqrt{4-t^2}\Big|_{-2}^{2} = 0$$

I cheated : I looked in the book to solve!!

Good choice : $\vec{R}(t) = \langle 2\cos t, 2\sin t \rangle$ for $0 \le t \le \pi$

Much simpler!!

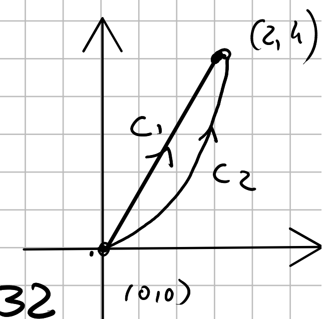
$$\int_C \vec{F} \cdot d\vec{R} = \int_0^{\pi} \underbrace{\langle 2\sin t, 2\cos t \rangle}_{\vec{F}(\vec{R}(t))} \cdot \underbrace{\langle -2\sin t, 2\cos t \rangle}_{\vec{R}'(t)} dt = \int_0^{\pi} -4\sin^2 t + 4\cos^2 t\, dt = \int_0^{\pi} 4\cos(2t)\, dt$$

$$= 2\sin(2t)\Big|_0^{\pi} = 0 - 0 = 0$$

Ex. Evaluate $\int_c \langle xy^2, x^2y\rangle \cdot d\vec{R}$ where c is either the line segment From (0,0) to (2,4) or the parabola $y = x^2$ From (0,0) to (2,4)

a) $y = 2x$, $\vec{R}(t) = \langle t, 2t\rangle$ $0 \le t \le 2$, $\vec{R}' = \langle 1, 2\rangle$

$$\int_{c_1} \vec{F}\cdot d\vec{R} = \int_0^2 \langle t(2t)^2, t^2(2t)\rangle \cdot \langle 1,2\rangle dt = \int_0^2 4t^3 + 4t^3\, dt = 8\,\frac{t^4}{4}\Big|_0^2 = 2(16-0) = 32$$



b) $y = x^2$, $\vec{R}(t) = \langle t, t^2\rangle$ $0 \le t \le 2$, $\vec{R}' = \langle 1, 2t\rangle$

$$\int_{c_2} \vec{F}\cdot d\vec{R} = \int_0^2 \langle t(t^2)^2, t^2(t^2)\rangle \cdot \langle 1, 2t\rangle dt = \int_0^2 t^5 + 2t^5\, dt = 3\,\frac{t^6}{6} = \frac{1}{2}(64-0) = 32$$

For 2 different trajectories we got the same value 32.
Is this a coincidence? No, $\vec{F} = \langle xy^2, x^2y\rangle$ is a special vector Field. We call it conservative. Any curve c From (0,0) to (2,4) would give the same result 32. We will study conservative vector Fields in 13.3.

From Physics. If $\vec{F}(x,y,z)$ is a Force Field on D. The work done by $\vec{F}$ on a object moving on a trajectory c in D is given by

$$W = \int_c \vec{F}\cdot d\vec{R}$$