section 13.1 VECTOR FIELDS.

DEF. A 2D vector field $\vec{F}(x,y)$ is a rule that for all (x,y) in the domain $D \subseteq \mathbb{R}^2$ associates a unique vector $\langle f_1(x,y), f_2(x,y) \rangle$ in the range R.

DEF. A 3D vector field $\vec{F}(x,y,z)$ is a rule that for all (x,y,z) in the domain $D \subseteq \mathbb{R}^3$ associates a unique vector $\langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

Ex.

$$\vec{F}(x,y) = \langle 3x + y^2, \ x - y \rangle \quad \text{for all } (x,y) \text{ in } \mathbb{R}^2$$

$$\vec{F}(x,y,z) = \langle yz, \sqrt{1-x^2-z^2}, \ln(y) \rangle \quad \text{for } (x,y,z) \text{ in } \mathbb{R}^3 \text{ such that } \begin{cases} 1 - x^2 - z^2 \geq 0 \\ y > 0 \end{cases}$$

Remark. You can always think a 2D vector field as a 3D vector field that depends only on (x,y) and whose third component is zero.

$$\vec{F}_{2D}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle \Rightarrow \vec{F}_{3D}(x,y,z) = \langle f_1(x,y), f_2(x,y), 0 \rangle$$

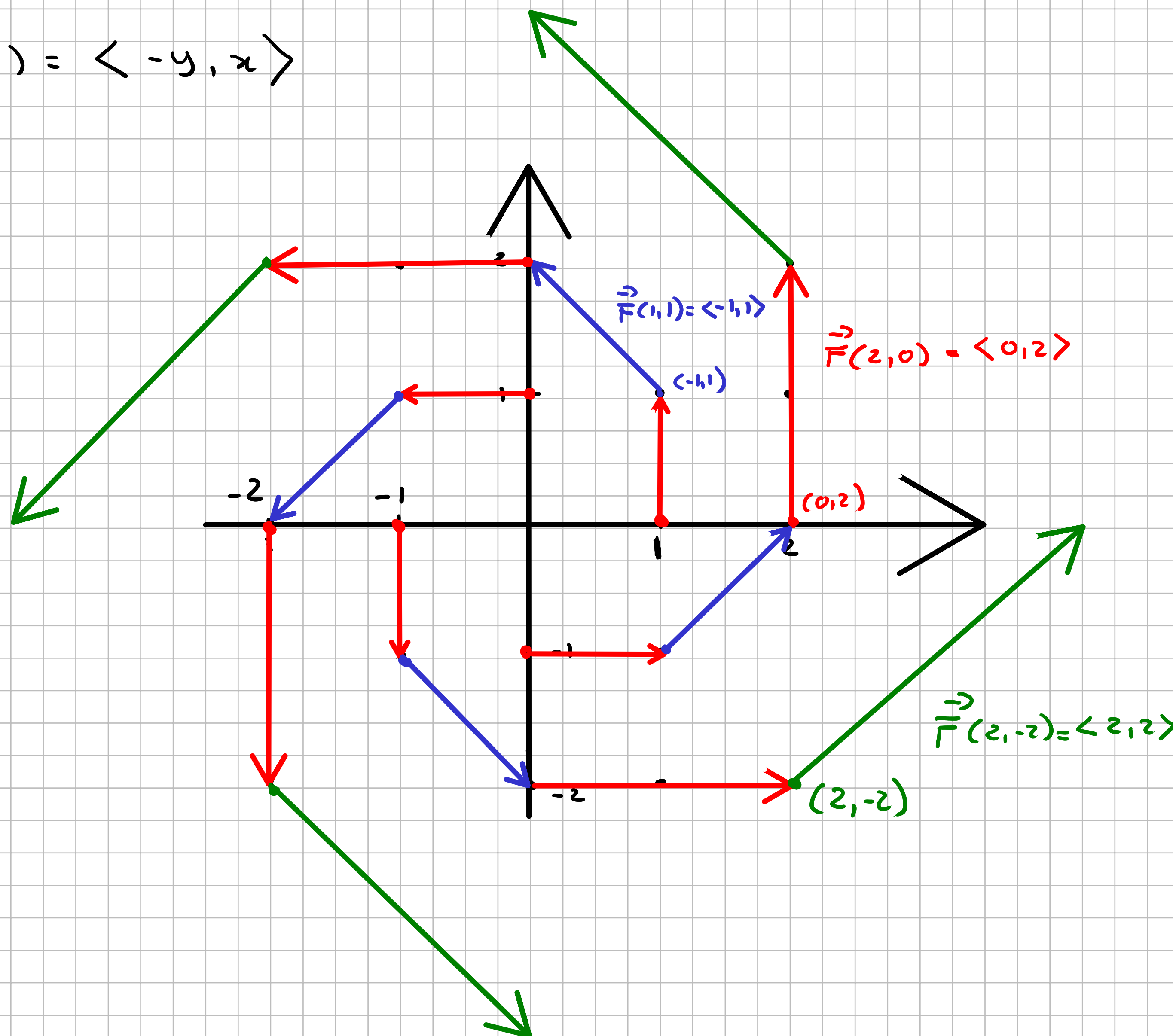
Ex.

$$\vec{F}(x,y) = \langle x^2+y, \ -xy \rangle \overset{3D}{\Longrightarrow} \vec{F}(x,y,z) = \langle x^2+y, -xy, 0 \rangle$$

Visualize a vector Field $\vec{F}$ as a Field of vectors:

use (x_0, y_0) as as Foot for the vector $\vec{F}(x_0, y_0)$

Ex. $\vec{F}(x,y) = \langle -y, x \rangle$



Have we ever seen a vector Field before. Yes!!

Think about the gradient of a Function F?

$$\nabla f(x,y) = \langle \partial_x f(x,y), \partial_y f(x,y) \rangle = \langle g_1(x,y), g_2(x,y) \rangle = \vec{G}(x,y)$$

We use the same algebra From vectors:

Let $\vec{F}(x,y)$ on D_1 and $\vec{G}(x,y)$ on D_2 then

$(a\vec{F} + b\vec{G})(x,y,z) := a\vec{F}(x,y,z) + b\vec{G}(x,y,z)$ For all (x,y,z) on $D_1 \cap D_2$

$(\vec{F} \cdot \vec{G})(x,y,z) := \vec{F}(x,y,z) \cdot \vec{G}(x,y,z)$ For all (x,y,z) on $D_1 \cap D_2$

$(\vec{F} \times \vec{G})(x,y,z) := \vec{F}(x,y,z) \times \vec{G}(x,y,z)$ For all (x,y,z) on $D_1 \cap D_2$

Recall the definition of the Del operator ∇

$$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$$

DeF: Divergece of $\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\rangle$

Function

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle\cdot\langle f_1, f_2, f_3\rangle = f_{1_x} + f_{2_y} + f_{3_z} = g(x,y,z)$$

DeF: Curl of $\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\rangle$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\,(f_{3_y} - f_{2_z}) - \hat{j}\,(f_{3_x} - f_{1_z}) + \hat{k}\,(f_{2_x} - f_{1_y}) = \vec{G}(x,y,z)$$

vector Field

Ex. Find the divergence and the curl of $\vec{F}(x,y,z) = \langle e^{xy}, \cos z, yz\rangle$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle\cdot\langle e^{xy}, \cos z, yz\rangle = y e^{xy} + 0 + y = y(1+e^{xy})$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ e^{xy} & \cos z & yz \end{vmatrix} = \hat{i}\,(z-(-\sin z)) - \hat{j}\,(0-0) + \hat{k}\,(0 - x e^{xy})$$

$$= \langle z + \sin z, 0, -x e^{xy}\rangle$$

Special cases

Let $\vec{F}(x,y,z) = \langle f_1(y,z), f_2(x,z), f_3(x,y) \rangle$ then

$$\nabla \cdot \vec{F} = \partial_x f_1(y,z) + \partial_y f_2(x,z) + \partial_z f_3(x,y) = 0$$

Let $\vec{F}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle$ then

$$\nabla \times \vec{F} = \nabla \times \langle f_1(x,y), f_2(x,y), 0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_1(x,y) & f_2(x,y) & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0,0) + \hat{k}(f_{2_x} - f_{1_y})$$

$$= \langle 0, 0, f_{2_x} - f_{1_y} \rangle$$

Let $\vec{F}(x,y,z) = \langle f_1(x), f_2(y), f_3(z) \rangle$ then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_1(x) & f_2(y) & f_3(z) \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{0}$$