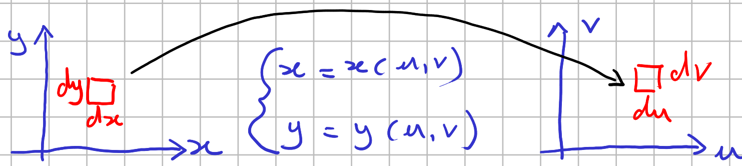
12.8 Jacobian



$dx\,dy = J\,du\,dv$
↑ Jacobian

Theorem:

Let $x(u,v)$ and $y(u,v)$ be the transformation from the xy-reference sytem to the uv-reference system. Then $dx\,dy = J\,du\,dv$, where

J is called the Jacobian of the transformation and it is equal to the absolute value of the determinan of the Jacobian matrix given by:

$$J_m = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

What is the Jacobian of the transformation from rectangular to polar coordinates? $dx\,dy = r\,dr\,d\theta$

↖ r is the Jacobian

In sec 12.3 we used a geometric argument to relate $dx\,dy$ and $dr\,d\theta$. Now we could do that algebraically:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad J_m = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

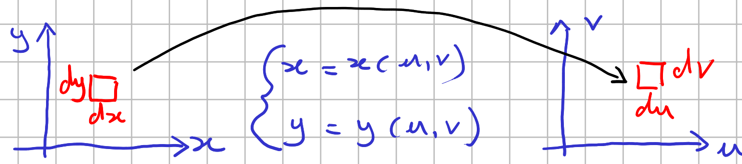
$$J_m = \begin{bmatrix} x_\theta & x_r \\ y_\theta & y_r \end{bmatrix} = \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}$$

$$J = |(\cos\theta)(r\cos\theta) - (-r\sin\theta)\sin\theta| = r(\cos^2\theta + \sin^2\theta) = r \quad \text{for } r \geq 0$$

$$J = |-r| = r$$

Jacobian inverse J^{-1}

Theorem :



$J^{-1}\,dx\,dy = du\,dv$

↑ Jacobian Inverse

Let J be the Jacobian From (x,y) to (u,v), then the Jacobian of the transformation From (u,v) to (x,y) is given by

$$J^{-1} = \frac{1}{J}$$

Ex. what is the Jacobian of the transformation From polar to rectangular?

$$J = r \Rightarrow J^{-1} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}} \Rightarrow dr\,d\theta = \frac{dx\,dy}{\sqrt{x^2+y^2}}$$

Ex. Let $\begin{cases} x = u - v \\ y = u + v \end{cases}$ Find J and J^{-1}

$$J_m = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$J = \left| (1)(1) - (-1)(1) \right| = 2$$

$$J^{-1} = \frac{1}{2}$$

Ex. Find the area enclosed by the ellipse of semiaxis a and b

D : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1$: transformation (r,θ) $\begin{cases} x = a r\cos\theta \\ y = b r \sin\theta \end{cases}$ For $\begin{cases} 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{cases}$

then $\dfrac{\cancel{a^2} r^2\cos^2\theta}{\cancel{a^2}} + \dfrac{\cancel{b^2} r^2\sin^2\theta}{\cancel{b^2}} \le 1$ or $r^2(\cos^2\theta+\sin^2\theta)\le 1$ or $\boxed{r\le 1}$

In the new (r,θ) reference system the ellipse becoms the unit circle

the Jacobian matrix is $J_m = \begin{bmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{bmatrix} \Rightarrow J = abr(\cos^2\theta+\sin^2\theta) = abr$

$$\text{Area} = \iint_{\text{ellipse}} dA = \iint_{\text{UNIT DISK}} abr\,dr\,d\theta = ab\iint_{\text{UNIT DISK}} \overbrace{r\,dr\,d\theta}^{dA} = ab\,\overset{\text{Area of the UNIT DISK}}{\pi(1^2)} = ab\pi$$

Jacobian in 3D

Theorem:

Let $x(u,v,w)$, $y(u,v,w)$ and $z(u,v,w)$ be a transformation from (x,y,z) to (u,v,w). The Jacobian J of the transformation is the absolute value of the determinant of the Jacobian matrix J_m given by

$$J_m = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}.$$

Namely, $dx\,dy\,dz = J\,du\,dv\,dw$

Ex. Prove that the Jacobian of the transformation from rectangular to spherical is

$$J = \rho^2 \sin\phi$$

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$$\begin{array}{cccc} & \partial_\rho & \partial_\phi & \partial_\theta \\ J_m = & \left[\begin{matrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{matrix}\right. & \begin{matrix} \rho\cos\phi\cos\theta \\ \rho\cos\phi\sin\theta \\ -\rho\sin\phi \end{matrix} & \left.\begin{matrix} -\rho\sin\phi\sin\theta \\ \rho\sin\phi\cos\theta \\ 0 \end{matrix}\right] \end{array}$$

$$J = \left|\sin\phi\cos\theta\left(0-(-)\rho^2\sin^2\phi\cos\theta\right) - \rho\cos\phi\cos\theta\left(0-\rho\sin\phi\cos\phi\cos\theta\right) + (-)\rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)\right|$$

$$= \left|\rho^2\sin^3\phi\cos^2\theta + \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta + \rho^2\cos^2\phi\sin\phi\sin^2\theta\right|$$

$$= \left|\rho^2\sin\phi\left(\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta\right)\right| = \left|\rho^2\sin\phi\right| \text{ for } 0 \le \phi \le \pi$$

$$(\sin^2\phi + \cos^2\phi)\cos^2\theta + \sin^2\theta(\sin^2\phi + \cos\phi^2) \qquad = \rho^2\sin\phi$$

$$\cos^2\theta + \sin^2\theta = 1$$