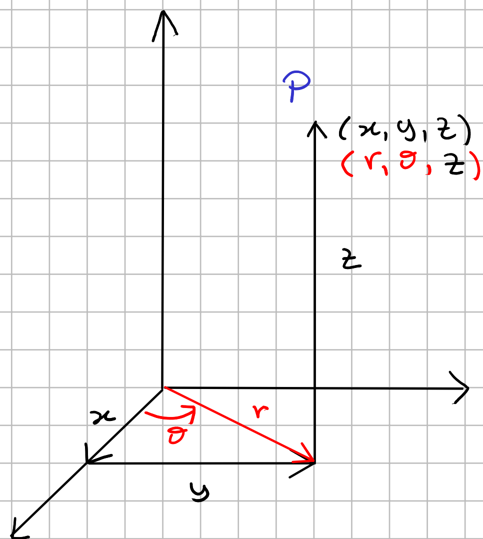
12.7 CYLINDRICAL COORDINATES

Extension to 3D of polar coordinates.

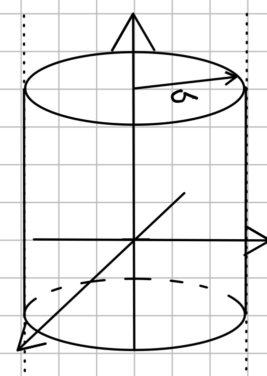


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

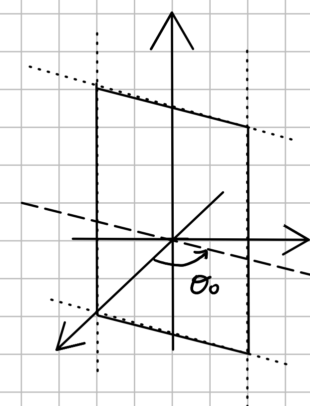
$$\begin{cases} x^2+y^2 = r^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

simple equations :

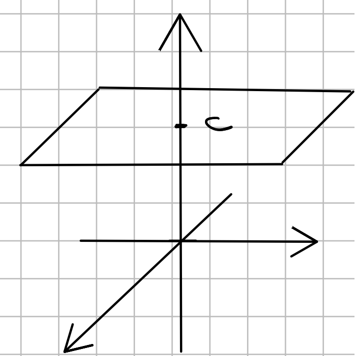
$r = a$



$\theta = \theta_0$



$z = c$



more simple equations: all equations containg only the group x^2+y^2.

circular parabolid

$z = 3x^2+3y^2$

$z = 3r^2$

circular cone

$z = 2\sqrt{x^2+y^2}$

$z = 2r$

sphere

$x^2+y^2+z^2 = 4$

$r^2+z^2 = 4$

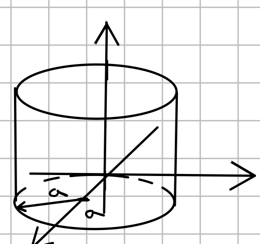
circular ellipsoid

$\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$

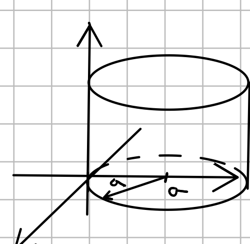
$\frac{r^2}{4} + \frac{z^2}{9} = 1$

more simple equations : all simple equations in polar coordinates

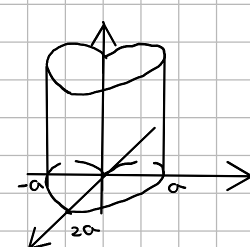
$r = 2a\cos\theta$



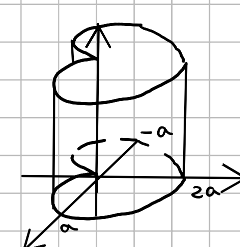
$r = 2a\sin\theta$



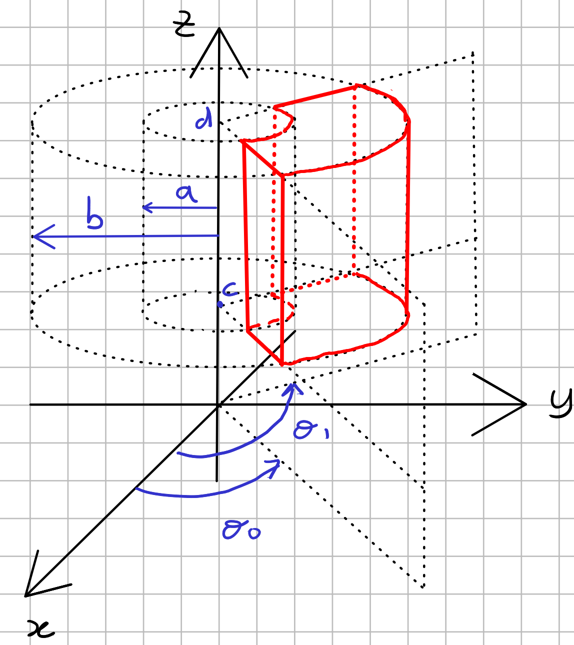
$r = a(1+\cos\theta)$



$r = a(1+\sin\theta)$



Box in cylindrical coordinates: $\begin{cases} a \le r \le b \\ \theta_0 \le \theta \le \theta_1 \\ c \le z \le d \end{cases}$



Elementary volume $dV = dA\,dz = r\,dz\,dr\,d\theta$

triple integral on the cylindrical box as above $\begin{cases} a \le r \le b \\ \theta_0 \le \theta \le \theta_1 \\ c \le z \le d \end{cases}$

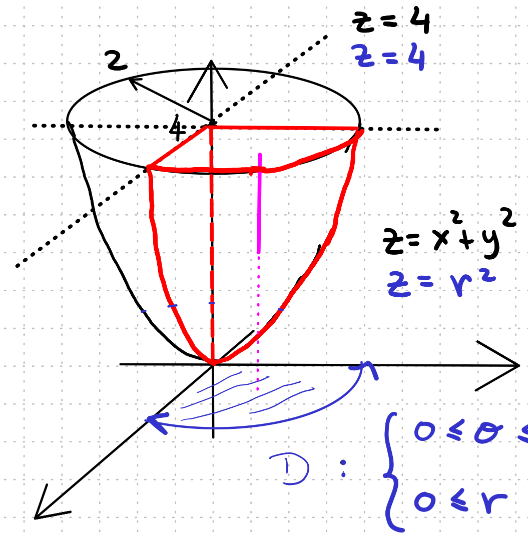
$$\iiint_V f(x,y,z)\,dV = \int_{\theta_0}^{\theta_1}\int_a^b\int_c^d f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

z-simple integrable region: For all (r,θ) in D $\quad z_1(r,\theta) \le z \le z_2(r,\theta)$

Triple integral on z-simple integrable region:

$$\iiint_V f(r,\theta,z)\,dV = \iint_D \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)\,dz\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)\, r\,dz\,dr\,d\theta$$

Ex. $\iiint_V x^2+y^2\,dV$, where V is the Volume in the first octant above the paraboloid $z=x^2+y^2$ and below $z=4$.



$$D:\begin{cases}0\le\theta\le\pi/2\\ 0\le r\le 2\end{cases}$$

Find the intersection between the plane $z=4$ and the paraboloid $z=x^2+y^2$

$$\begin{cases}z=4\\ z=x^2+y^2\end{cases}\qquad\begin{cases}z=4\\ x^2+y^2=4\end{cases}$$

$$I=\int_0^{\frac{\pi}{2}}\int_0^2\int_{r^2}^4 r^2\; r\,dz\,dr\,d\theta=\int_0^{\frac{\pi}{2}}\int_0^2 r^3\, z\Big|_{r^2}^4 dr\,d\theta=\int_0^{\frac{\pi}{2}}\int_0^2 4r^3-r^5\,dr\,d\theta=\int_0^{\frac{\pi}{2}} 4\frac{r^4}{4}-\frac{r^6}{6}\Big|_0^2 d\theta=\left(16-\frac{32}{3}\right)\frac{\pi}{2}=\frac{8}{3}\pi$$

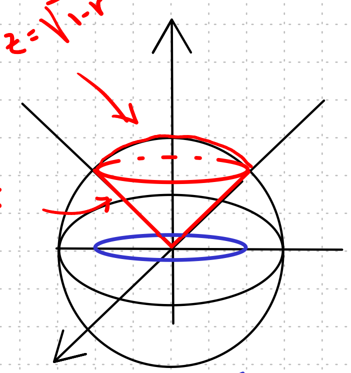
$r^3(4-r^2)$

Ex. Find the Volume of the region enclosed by the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$.

$z=\sqrt{1-x^2-y^2}$

$z=\sqrt{1-r^2}$

$z=r$



$$D:\begin{cases}0\le\theta\le 2\pi\\ 0\le r\le\frac{\sqrt{2}}{2}\end{cases}$$

Intersection

$$\begin{cases}x^2+y^2+z^2=1\\ z=\sqrt{x^2+y^2}\end{cases}\qquad\begin{cases}x^2+y^2+(x^2+y^2)=1\\ z=\sqrt{x^2+y^2}\end{cases}\qquad\begin{cases}x^2+y^2=\frac{1}{2}=\left(\sqrt{\frac{1}{2}}\right)^2=\left(\frac{\sqrt{2}}{2}\right)^2\\ z=\frac{\sqrt{2}}{2}\end{cases}$$

$$\text{Vol}=\iiint_V 1\,dV=\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}\int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}\frac{-2}{-2}r\sqrt{1-r^2}-r^2\,dr\,d\theta$$

$r z\Big|_r^{\sqrt{1-r^2}}$

$u=1-r^2$

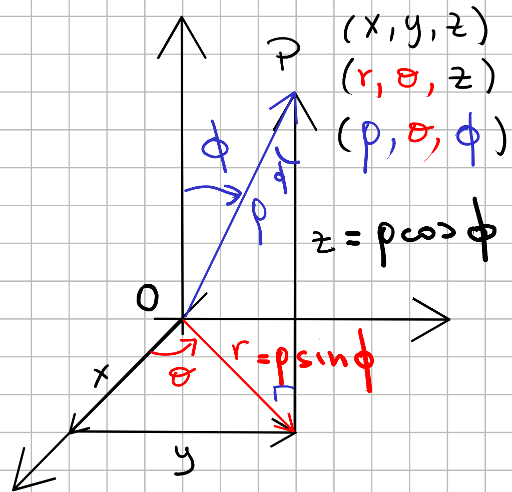
$du=-2r\,dr$

$\int\sqrt{u}=\frac{2}{3}u^{3/2}$

$$=\int_0^{2\pi}-\frac{1}{2}\left(\frac{2}{3}(1-r^2)^{3/2}\right)-\frac{r^3}{3}\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{1}{3}\left[-\left(\left(\frac{1}{2}\right)^{3/2}-1\right)-\left(\frac{1}{\sqrt{2}}\right)^3\right]2\pi=$$

$$=\frac{2}{3}\pi\left[1-2\left(\frac{1}{\sqrt{2}}\right)^3\right]=\frac{2}{3}\pi\left(1-\frac{\sqrt{2}}{2}\right)$$

SPHERICAL COORDINATES



$$\begin{cases} \rho \text{ is the lenght of } \vec{OP} \\ \theta \text{ is the polar angle} \\ \phi \text{ is the azimutal angle} \end{cases}$$

unique representation

$$\begin{cases} \rho \geq 0 \\ 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

TRANSFORMATIONS

Rectangular to spherical

$$\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

cylindrical to spherical

$$\begin{cases} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases}$$

spherical to rectangular

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = \frac{y}{x} \\ \cos\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

spherical to cylindrical

$$\begin{cases} \rho^2 = r^2 + z^2 \\ \theta = \theta \\ \cos\phi = \frac{z}{\rho} = \frac{z}{\sqrt{r^2+z^2}} \end{cases}$$

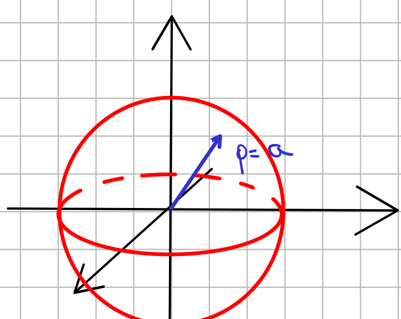
Easy surfaces in spherical coordinates:

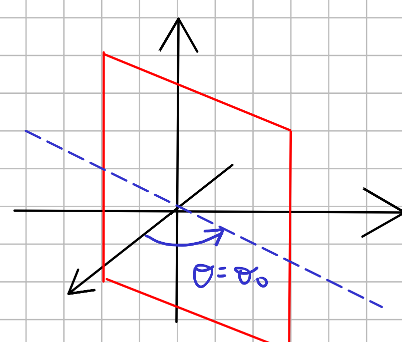
$\rho = a$ or $x^2+y^2+z^2 = a^2$

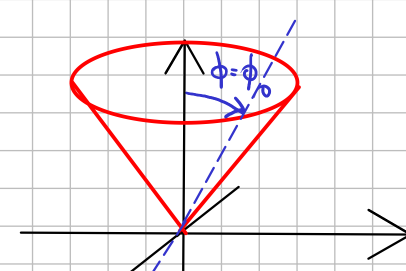
$\theta = \theta_0$

$\phi = \phi_0$ $\quad \tan\phi = \tan\phi_0$ $\quad \frac{r}{z} = \tan\phi_0$

$z = \cot\phi_0 \, r$ $\quad z = \cot\phi_0 \sqrt{x^2+y^2}$ ← cone







$\left(\phi = \frac{\pi}{4} \quad z = \sqrt{x^2+y^2}\right)$

Particular cases

1) $\phi = 0$ positive z-axis

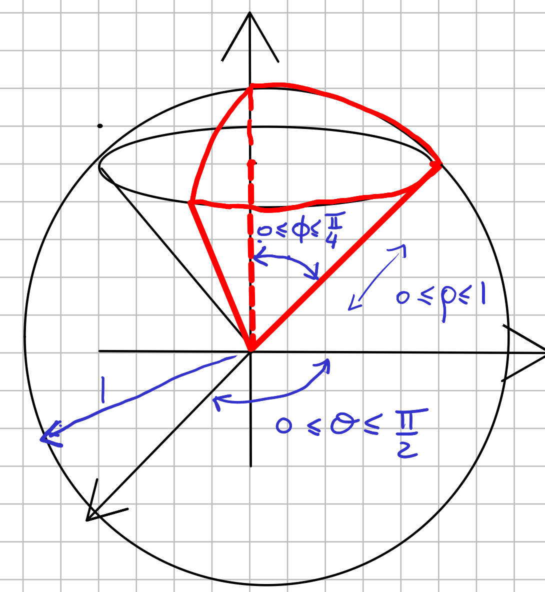
2) $\phi = \pi$ negative z-axis

3) $\phi = \frac{\pi}{2}$ xy-plane

The box in spherical coordinates

$$\begin{cases} a \le \rho \le b \\ \theta_0 \le \theta \le \theta_1 \\ \phi_0 \le \phi \le \phi_1 \end{cases}$$

Ex. $\begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le \frac{\pi}{4} \end{cases}$



The element of volume $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. It can be proved geometrically but it is messy. We will show this in sec 12.8 algebraically.

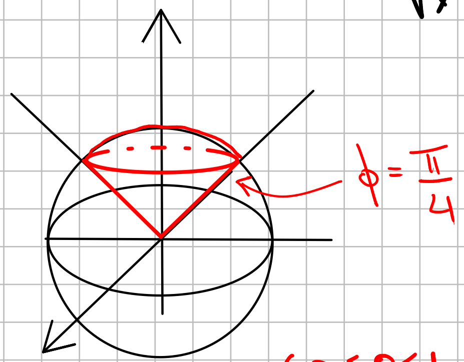
Ex. Find the volume of the ball with radious a.

$$V = \begin{cases} 0 \le \rho \le a \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{cases}$$

$$\text{Vol} = \iiint_V 1 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \overbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}^{dV} = \int_0^{2\pi} \int_0^{\pi} \sin\phi \left. \frac{\rho^3}{3} \right|_0^a d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \left. -\cos\phi \right|_0^{\pi} d\theta$$

$$= \frac{a^3}{3} \left(-(-1) + (1) \right) 2\pi = \frac{4}{3} \pi a^3$$

Ex. Find the volume of the region enclosed by the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$



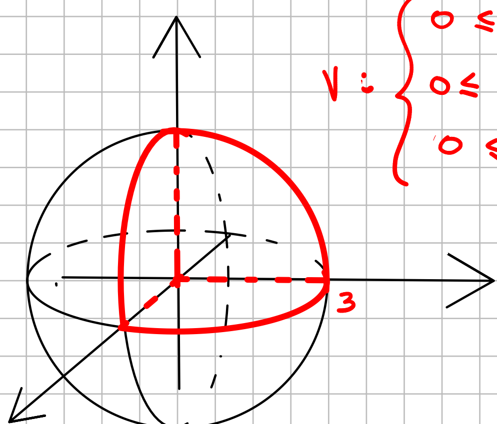
$$V : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{4} \end{cases}$$

$$\text{Vol} = \iiint_V dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\Big|_0^1 \sin\phi \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} \frac{1}{3}\left(-\cos\phi\right)\Big|_0^{\frac{\pi}{4}} d\theta = \frac{1}{3}\left(1-\frac{\sqrt{2}}{2}\right) 2\pi = \frac{2}{3}\pi\left(1-\frac{\sqrt{2}}{2}\right)$$

This computation is much easier than using cylindrical coordinates!!

Ex. Find $\iiint_V x^2+y^2+z^2 \, dV$ where V is the portion of the ball $x^2+y^2+z^2 \le 9$ in the first octant $(x \ge 0, y \ge 0, z \ge 0)$



$$V : \begin{cases} 0 \le \rho \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le \frac{\pi}{2} \end{cases}$$

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\rho^5}{5}\Big|_0^3 \sin\phi \, d\phi \, d\theta$$

$$= \frac{3^5}{5}\int_0^{\frac{\pi}{2}} -\cos\phi \Big|_0^{\frac{\pi}{2}} d\theta = \frac{3^5}{3}(1-0)\,\frac{\pi}{2} = \frac{3^5}{6}\pi$$