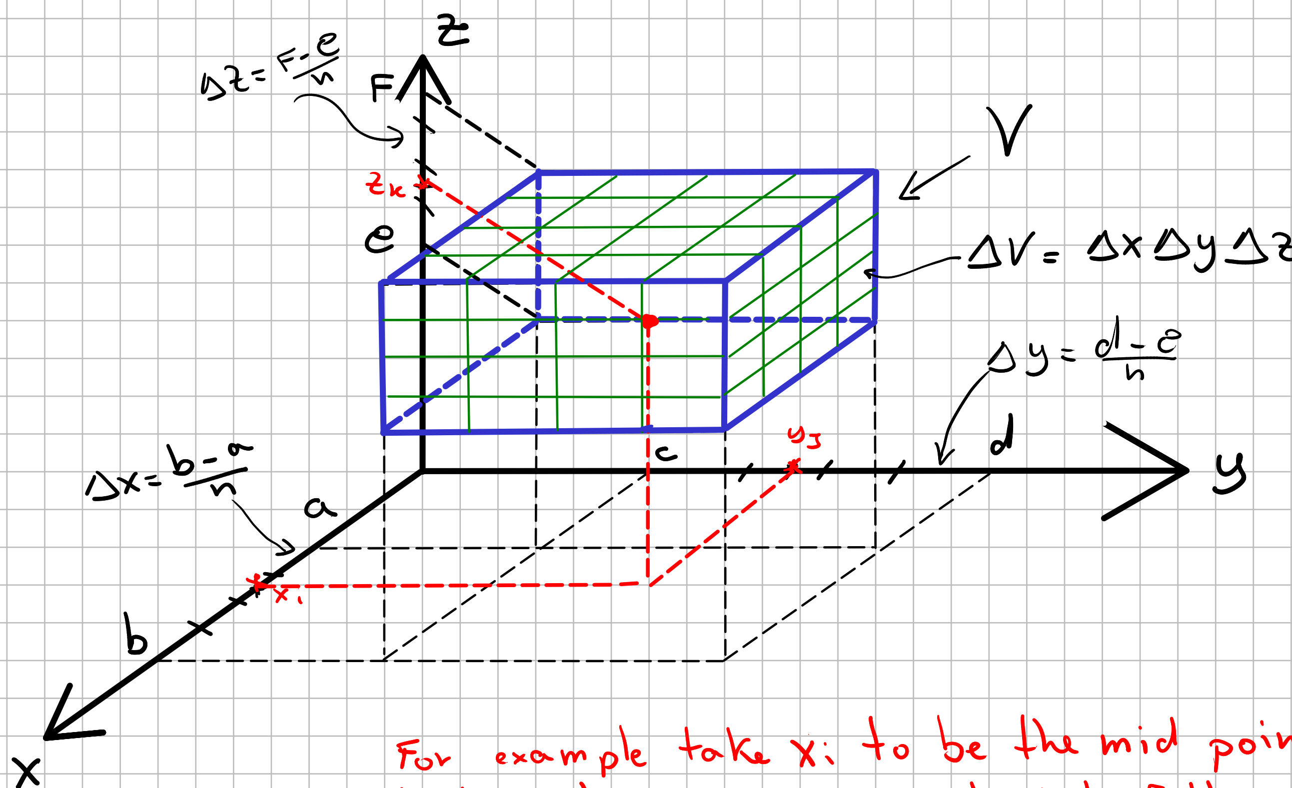
12.5 Triple integrals in rectangular coordinates

Let $f(x,y,z)$ be a Function of 3 variables deFined on the domain $V: \begin{cases} a \leq x \leq b \\ c \leq y \leq d \\ e \leq z \leq F \end{cases}$

then $\iiint_V f(x,y,z)\, dV := \lim_{n \to +\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta z \Delta y \Delta x$, where

$\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{n}$, $\Delta z = \frac{F-e}{n}$ and (x_i, y_j, z_k) is any caracktheristic point within the cell i,j,k built as below.



For example take x_i to be the mid point of the i interval on x, y_j the mid point of the j-interval on y and z_k the mid point of the k interval on z.

Then (x_i, y_j, z_k) is the center of the box (i,j,k) with sides Δx, Δy, Δz.

Fubini-Tonelli Theorem on $V: \begin{cases} a \leq x \leq b \\ c \leq y < d \\ e \leq z \leq F \end{cases}$

$$\iiint_V f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^F f(x,y,z)\,dz\,dy\,dx = \int_a^b \int_e^F \int_c^d f(x,y,z)\,dy\,dz\,dx$$

$$= \int_c^d \int_a^b \int_e^F f(x,y,z)\,dz\,dx\,dy = \int_c^d \int_e^F \int_a^b f(x,y,z)\,dx\,dz\,dy$$

$$= \int_e^F \int_a^b \int_c^d f(x,y,z)\,dy\,dy\,dz = \int_e^F \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$\boxed{\text{6 possible permutations}}$

Ex. $I = \iiint_V x y^2 z\,dV$ where $V: \begin{cases} 1 \leq x < 2 \\ 0 < y \leq 1 \\ 0 \leq z \leq 3 \end{cases}$

$$I = \int_1^2 \int_0^1 \int_0^3 x y^2 z\,dz\,dy\,dx = \int_1^2 \int_0^1 x y^2 \left.\frac{z^2}{2}\right|_0^3 dy\,dx = \frac{9}{2} \int_1^2 x \left.\frac{y^3}{3}\right|_0^1 dx = \frac{9}{2}\,\frac{1}{3} \left.\frac{x^2}{2}\right|_1^2 = \frac{3}{2}\overset{3/2}{\left(2 - \frac{1}{2}\right)} = \frac{9}{4}$$

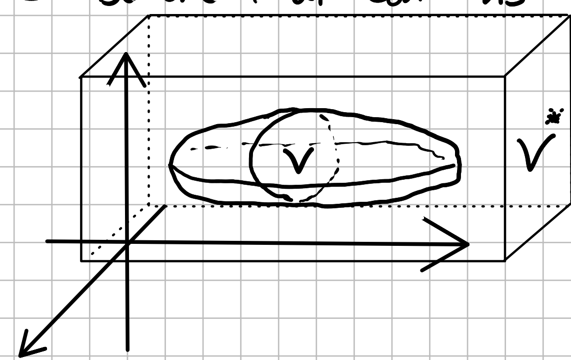
but also

$$I = \int_0^3 \int_1^2 \int_0^1 x y^2 z\,dy\,dx\,dz = \int_0^3 \int_1^2 x z \left.\frac{y^3}{3}\right|_0^1 dx\,dz = \frac{1}{3} \int_0^3 z \left.\frac{x^2}{2}\right|_1^2 dz = \frac{1}{3}\overset{3/2}{\left(2 - \frac{1}{2}\right)} \left.\frac{z^2}{2}\right|_0^3 = \frac{1}{2}\,\frac{9}{2} = \frac{9}{4}$$

but also ...

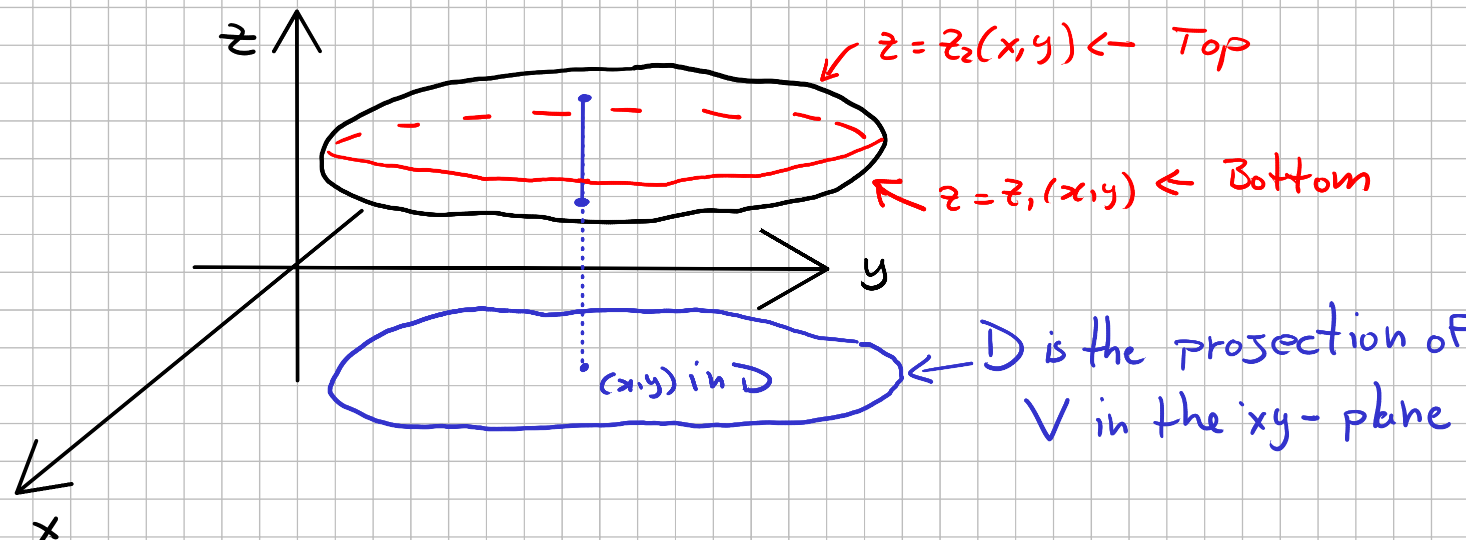
Triple integral on a generic region V.

$$\iiint_V f(x,y,z)\,dV := \iiint_{V^*} g(x,y,z)\,dV$$ where V^* is a box as before that contains V



and $g(x,y,z) = \begin{cases} f(x,y,z) & \text{For all } (x,y,z) \text{ in } V \\ 0 & \text{otherwise} \end{cases}$

V is z-simple integrable if can be described as $z_1(x,y) \le z \le z_2(x,y)$ For all (x,y) in D.



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There are regions that are y-simple $y_1(x,z) \le y \le y_2(x,y)$ For all (x,z) in D

There are regions that are x-simple $x_1(y,z) \le x \le x_2(y,z)$ For all (y,z) in D

Front back

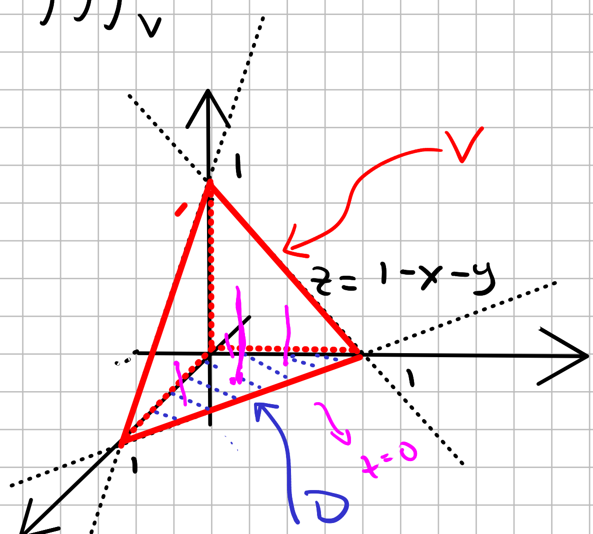
Fubini-Tonelli theorem on a z-simple integrable region V.

Let V be z-simple, then

$$\iiint_V f(x,y,z)\,dV = \iint_D \left(\underbrace{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz}_{g(x,y)} \right) dA$$

$\boxed{\text{The same theorem holds For x-simple and y-simple domains}}$

$\iiint_V x\,dV$ where V is the region bounded by the reference planes and $x+y+z=1$



V is z-simple integrable. But also x-simple and y-simple

$$\iiint_V x\,dV = \iint_D \int_{z=0}^{z=1-x-y} x\,dz\,dA = \iint_D x\,z\Big|_0^{1-x-y} dA = \iint_D \underbrace{x(1-x-y)}_{g(x,y)}\,dA$$

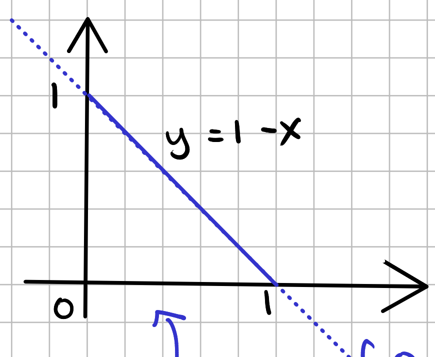
double integral

$$= \int_0^1 \int_0^{1-x} x(1-x-y)\,dy\,dx = \int_0^1 x\left[(1-x)y - \frac{y^2}{2}\right]\Big|_0^{1-x} dx$$

$$= \int_0^1 x\left((1-x)^2 - \frac{(1-x)^2}{2}\right)dx = \frac{1}{2}\int_0^1 x - 2x^2 + x^3\,dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right)\Big|_0^1 =$$

$\frac{1}{2}(1-2x+x^2)$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2}\,\frac{6-8+3}{12} = \frac{1}{24}$$

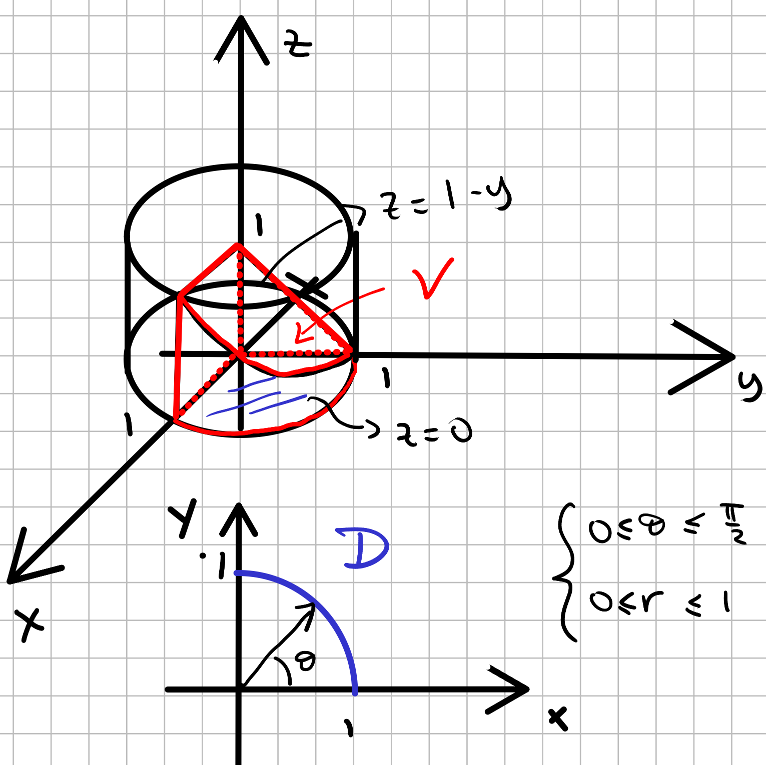


$$D = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

$\boxed{\text{This volume is also y-simple and x-simple}}$

$$\iiint_V 1 \, dV = \text{Volume of } V$$

Find the Volume of V inside the cylinder $x^2+y^2=1$, in the First octant and below $z = 1-y$.



V is z-simple integrable

$$\text{Volume} = \iiint_V 1\,dV = \iint_D \int_{z=0}^{z=1-y} dz\, dA = \iint_D z \Big|_0^{1-y} dA =$$

$$= \iint_D (1-y)\, dA = \int_0^{\frac{\pi}{2}} \int_0^1 (1 - r\sin\theta)\, r\, dr\, d\theta =$$

Polar coordinates

$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 1 \end{cases}$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^2}{2} - \frac{r^3}{3}\sin\theta \Big|_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{3}\sin\theta \, d\theta =$$

$$= \frac{1}{2}\theta + \frac{1}{3}\cos\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}\frac{\pi}{2} + \frac{1}{3}(0-1) = \frac{\pi}{4} - \frac{1}{3}$$

$$\boxed{\text{This Volume is also } x\text{-simple}}$$