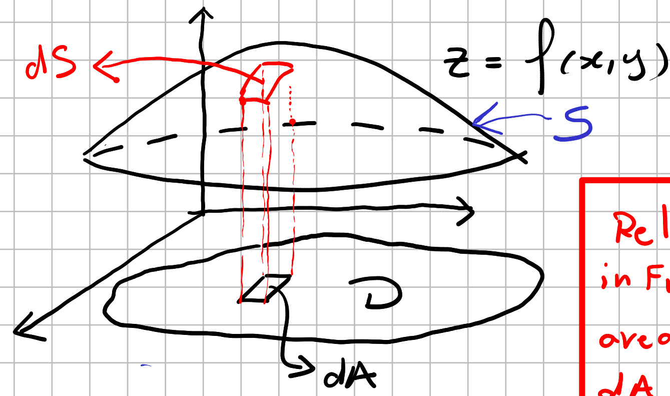
Sec 12.4 Surface Area of S



Let S be the part of the graph $z = f(x,y)$ on top of the region D in the xy-plane

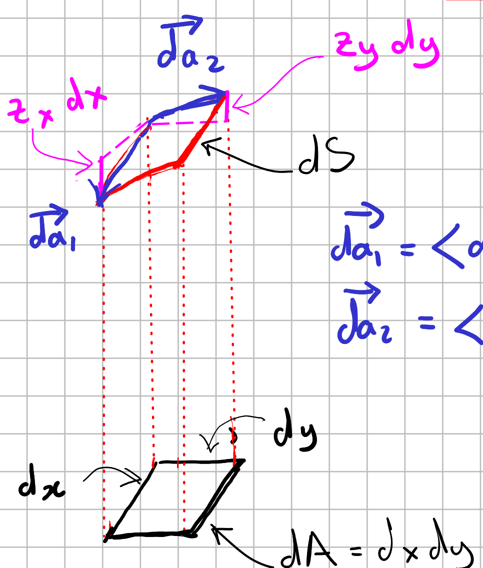
Relation between the infinitesimal element of area dS and its projection dA on the xy-plane

$$\boxed{dS = \sqrt{1+z_x^2+z_y^2}\, dA}$$

The surface area of S is given by

$$S.A. = \iint_S dS = \iint_D \sqrt{1+Z_x^2+Z_y^2}\; dA$$

$$\boxed{\text{Why is } dS = \sqrt{1+z_x^2+z_y^2}\; dA \text{ ?}}$$



$\vec{da_1} = \langle dx, 0, z_x\, dx \rangle$

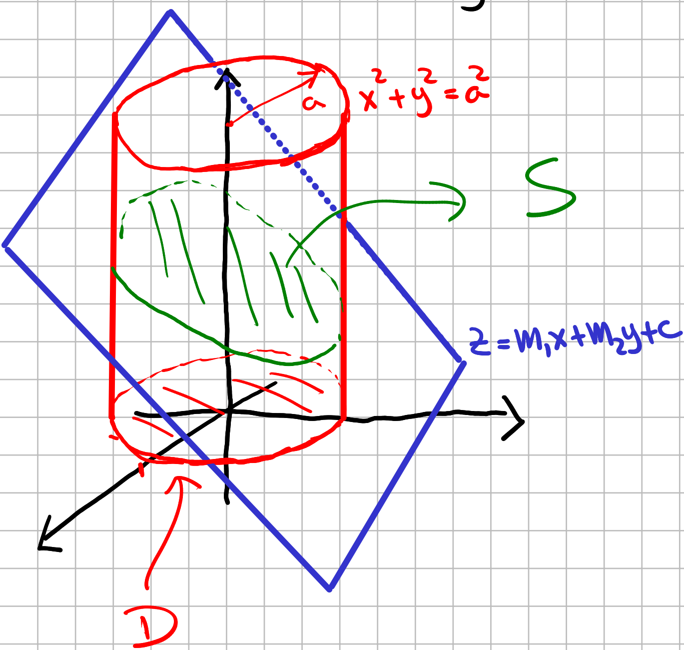
$\vec{da_2} = \langle 0, dy, z_y\, dy \rangle$

From Sec 9.4 $dS = \|\vec{da_1} \times \vec{da_2}\|$

$$\vec{da_1} \times \vec{da_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & z_x dx \\ 0 & dy & z_y dy \end{vmatrix} = \left\langle -z_x\, dx dy, -z_y\, dx dy, dx dy \right\rangle$$

$$dS = \|\vec{da_1} \times \vec{da_2}\| = \sqrt{z_x^2 + z_y^2 + 1}\; \underbrace{dx\,dy}_{dA}$$

Ex. Find the surface area of the portion of the plane $z = m_1 x + m_2 y + c$ inside the cylinder $x^2 + y^2 = a^2$.



$$S.A. = \iint_S dS = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dA$$

$z_x = m_1$

$z_y = m_2$

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + m_1^2 + m_2^2}$$

$$S.A. = \iint_D \sqrt{1+m_1^2+m_2^2}\, dA = \sqrt{1+m_1^2+m_2^2} \iint_D dA =$$

$$= \sqrt{1+m_1^2+m_2^2}\,(\text{Area of } D) = \sqrt{1+m_1^2+m_2^2}\,\pi a^2$$

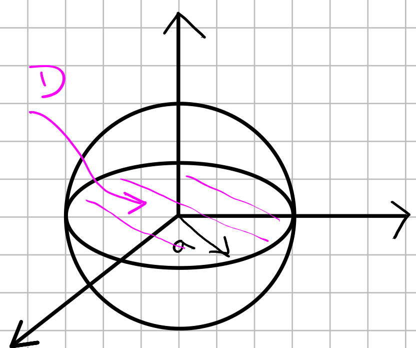
For a plane $dS = c\, dA$ with $c \geq 1$

When is $c = 1$?

Can it be $dS < dA$?

Ex. Find the surface Area of the sphere of radius a.

Let us double the surface area of the positive half sphere with center at the origin: $x^2+y^2+z^2=a^2$



$$dS=\sqrt{1+z_x^2+z_y^2}\;dA$$

Easy way to find dS. Use implicit differentiation for

$F(x,y,z)=x^2+y^2+z^2=a^2$ $\boxed{z_x=-\frac{F_x}{F_z}=-\frac{2x}{2z}=-\frac{x}{z}}$ $\boxed{z_y=-\frac{F_y}{F_z}=-\frac{y}{z}}$

Then $dS=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\,dA=\sqrt{\frac{z^2+x^2+y^2}{z^2}}\,dA=\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dA=\frac{a}{\sqrt{a^2-x^2-y^2}}\,dA$

half sphere

$$S.A.=2\iint_D\frac{a}{\sqrt{a^2-x^2-y^2}}\,dA=-\int_0^{2\pi}\int_0^a\frac{a}{\sqrt{a^2-r^2}}(-)2\,r\,dr\,d\theta=-a\int_0^{2\pi}2\sqrt{a^2-r^2}\Big|_0^a\,d\theta=-2a\left(0-\sqrt{a^2}\right)\int_0^{2\pi}d\theta$$

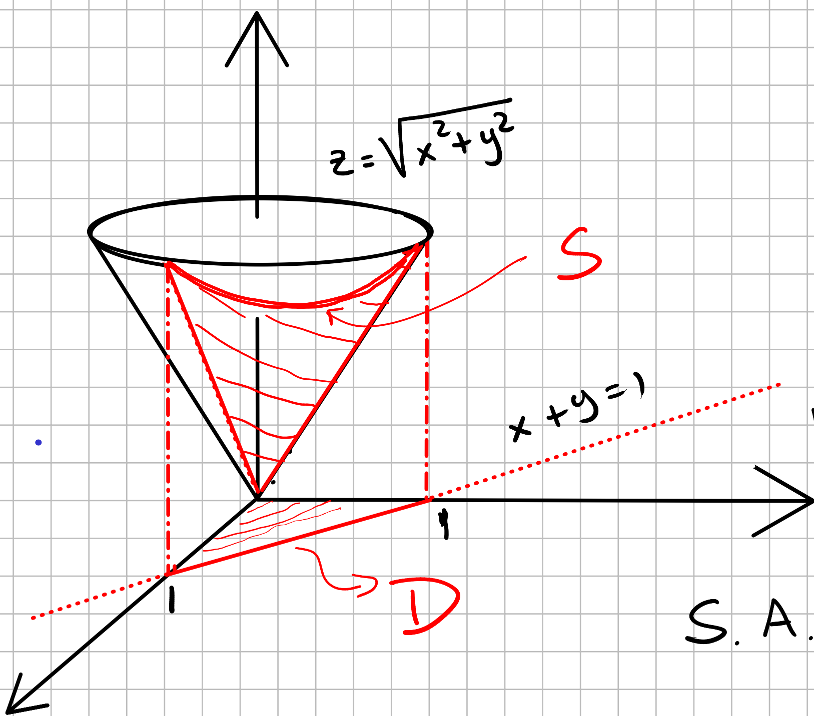
$$=(2a^2)(2\pi)=4\pi a^2$$

$u=a^2-r^2$

$du=-2r\,dr$

$\int\frac{1}{\sqrt{u}}\,du=2\sqrt{u}$

Find the surface area of the portion of the cone $z=\sqrt{x^2+y^2}$ above the triangle D bouded by the x-axis, the y-axis and $x+y=1$.



$$dS=\sqrt{1+z_x^2+z_y^2}\,dA$$

$$z_x=\frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{x^2+y^2}} \qquad z_y=\frac{1}{\cancel{2}}\frac{\cancel{2}y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=\sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}=\sqrt{2}$$

$$S.A.=\iint_S dS=\iint_D \sqrt{2}\,dA=\sqrt{2}\,(\text{Area of } D)=\frac{\sqrt{2}}{2}$$

$$\frac{(1)(1)}{2}=\frac{1}{2}$$

$$\boxed{\text{For a cone } dS=c\,dA \text{ with } c>1}$$