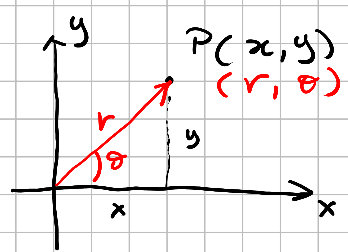
Sec 12.3 Integration in polar coordinates

Quick review



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

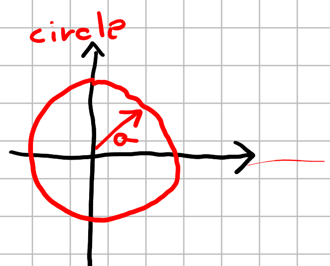
generally we take

$$\begin{cases} r \geq 0 \\ 0 \leq \theta \leq 2\pi \end{cases}, \quad \text{but also} \quad \begin{cases} r \geq 0 \\ -\pi \leq \theta \leq \pi \end{cases},$$

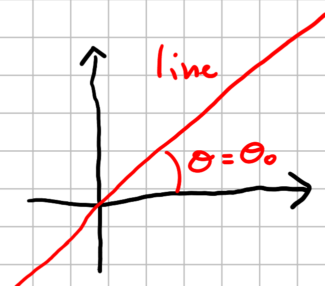
but also $\begin{cases} r \text{ in } \mathbb{R} \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{cases}$.

Eqs to remember

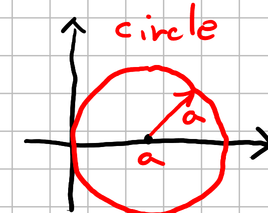
$$\begin{cases} r = a \\ 0 \leq \theta \leq 2\pi \end{cases}$$



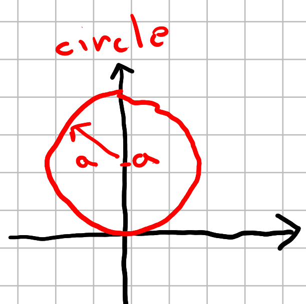
$$\begin{cases} \theta = \theta_0 \\ r \text{ in } \mathbb{R} \end{cases}$$



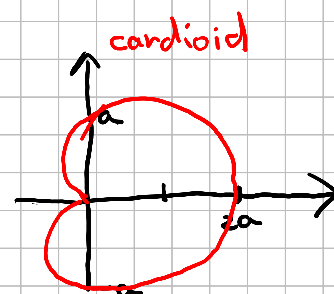
$$\begin{cases} r = 2a\cos\theta \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{cases}$$



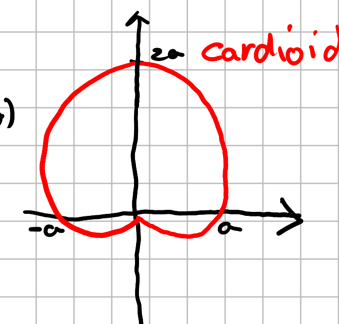
$$\begin{cases} r = 2a\sin\theta \\ 0 \leq \theta \leq \pi \end{cases}$$



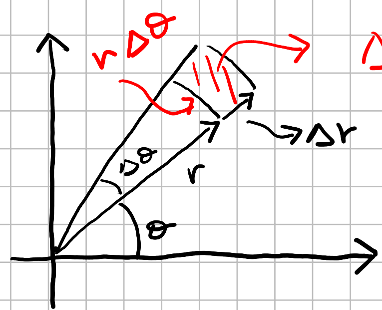
$$\begin{cases} r = a(1+\cos\theta) \\ 0 \leq \theta \leq 2\pi \end{cases}$$



$$\begin{cases} r = a(1+\sin\theta) \\ 0 \leq \theta \leq 2\pi \end{cases}$$



Element of area dA in polar coordinates



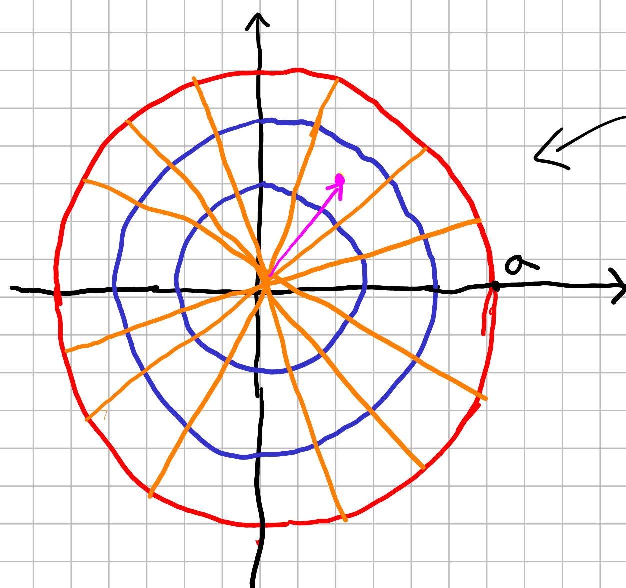
$\Delta A \simeq r\Delta\theta\,\Delta r$

In the limit $\Delta r \to 0$, $\Delta\theta \to 0$

$$dA = r\,d\theta\,dr$$

Definition of polar integral on a Disk D centered at the origin with radius a

$$D: \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \end{cases}$$



Partition the Disk in sectors.

Ex. $n_r = 3$ $\quad n_\theta = 16$

Let $\Delta r = \dfrac{a}{n_r}$ $\quad \Delta\theta = \dfrac{2\pi}{n_\theta}$

For each sector i, j identify a caractheristic point (r_i, θ_j)

Then

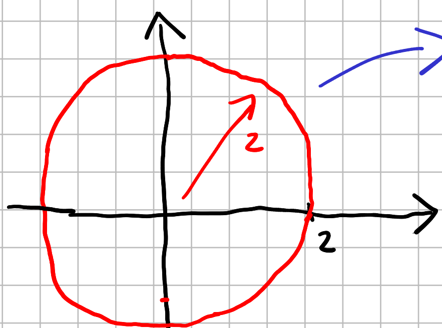
$$\iint_D f(r,\theta)\,dA := \lim_{n_r \to +\infty} \lim_{n_\theta \to +\infty} \sum_i^{n_r} \sum_j^{n_\theta} f(r_i, \theta_j)\; r_i\,\Delta r\,\Delta\theta$$

Fubini-Tonelli theorem in polar coordinates (on the Disk)

$$\iint_D f(r,\theta)\,dA = \int_0^{2\pi}\int_0^a f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dA}$$

Ex. use polar coordinates to solve

$$I = \iint_D (x+y)\,dA \quad \text{where } D: \ 0 \le x^2+y^2 \le 4$$



$$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$

Disk in polar coordinates

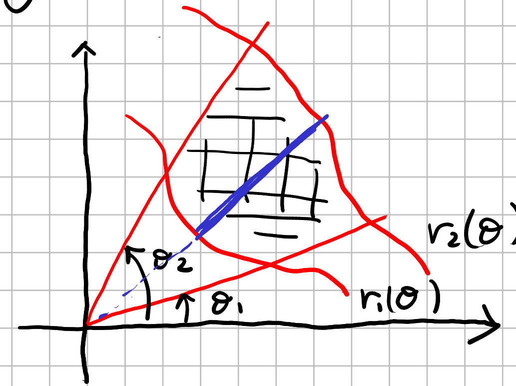
$$I = \iint_D (\overset{x}{\cos\theta} \overset{+}{+} \overset{y}{r\sin\theta})\,dA = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta)\,\overbrace{r\,dr\,d\theta}^{dA}$$

$$= \int_0^{2\pi} (\cos\theta + \sin\theta)\,\underbrace{\frac{r^3}{3}\Big|_0^2}_{8/3}\,d\theta = \frac{8}{3}\left(\sin\theta - \cos\theta\right)\Big|_0^{2\pi} = 0$$

Not surprising: x and y are odd functions on a symmetric domain

Fubini-Tonelli theorem on a generic region.

Let $D: \begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ r_1(\theta) \leq r \leq r_2(\theta) \end{cases}$



Then $$\iint_D f(r,\theta)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

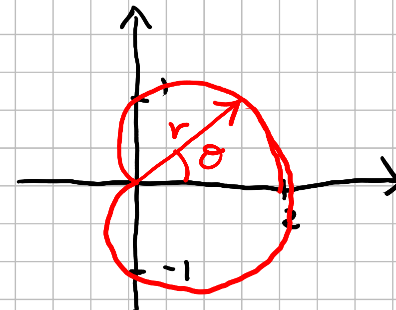
Let $f(r,\theta) \geq 0$ for all (r,θ) in D. Then

$\iint_D f(r,\theta)\,dA$ is the volume of the body between the polar plane (bottom) the function $f(r,\theta)$ (top) and inside the cylinder D (sides)

$$\text{Area of } D = \iint_D 1\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta$$

ex. Find the area inside the cardioid $r = 1 + \cos\theta$.

$$\text{Area} = \iint_D 1 \, dA \quad \text{where} \quad D: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1+\cos\theta \end{cases}$$



$$\text{Area} = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{2}\left(1+2\cos\theta+\cos^2\theta\right)d\theta = \frac{1}{2}\int_0^{2\pi} 1+2\cos\theta+\frac{1}{2}\left(1+\cos 2\theta\right)d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \, d\theta = \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Bigg|_0^{2\pi} = \frac{3}{2}\pi$$

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ex. Find the volume of the ball of radius a.

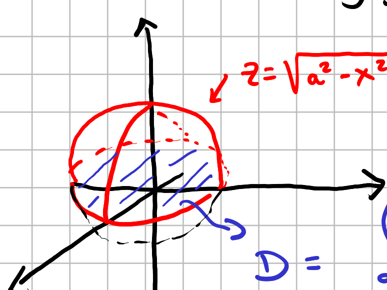
Place the ball with center at the origin. Then the equation of the sphere is given by $x^2+y^2+z^2=a^2$, or $z = \pm\sqrt{a^2-x^2-y^2}$. We can evaluate the volume of the positive half ball and double it:

$$\text{Volume} = 2\iint_D \sqrt{a^2-x^2-y^2}\,dA = 2\iint_D \sqrt{a^2-r^2}\,dA = -\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,(-)2r\,dr\,d\theta =$$

$a^2-(x^2+y^2)$
a^2-r^2

$u = a^2 - r^2$
$du = -2r\,dr$

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$$



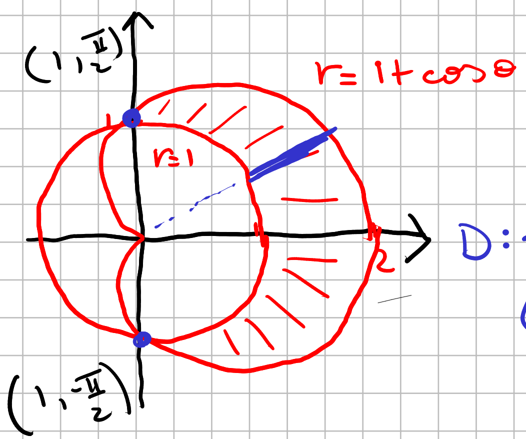
$z = \sqrt{a^2-x^2-y^2} \geq 0$

$$D = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq a \end{cases}$$

$$= -\int_0^{2\pi} \frac{2}{3}(a^2-r^2)^{3/2}\Big|_0^a d\theta = -\frac{2}{3}\left(0-a^3\right)\,\theta\Big|_0^{2\pi} = \frac{4}{3}\pi a^3$$

Hw 8 #4. Find the area bounded inside by $r=1$ and outside by $r = 1+\cos\theta$

(How do we find the limits in θ?
They are given by the intersections between two curves.



$$D: \begin{cases} -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 1 \leq r \leq 1+\cos\theta \end{cases}$$

$$\text{Area} = \iint_D 1\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} r\,dr\,d\theta = \cdots$$

Hw 8 #6

A. Convert to polar coordinates and integrate of the disk of infinite radius :

$$\mathbb{R}^2 : \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq a \quad \text{with } a \to +\infty \end{cases}$$

why is this? Prove it

B. Use this property $\displaystyle\int_a^b \int_c^d f(x) g(y)\, dy\, dx = \int_a^b f(x)\, dx \int_c^d g(y)\, dy$

Box of infinite side

then $\displaystyle\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \lim_{a\to+\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2} e^{-y^2}\, dy\, dx = \left(\lim_{a\to+\infty} \int_{-a}^{a} e^{-x^2} dx\right)\left(\lim_{a\to+\infty} \int_{-a}^{a} e^{-y^2} dy\right) = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2$