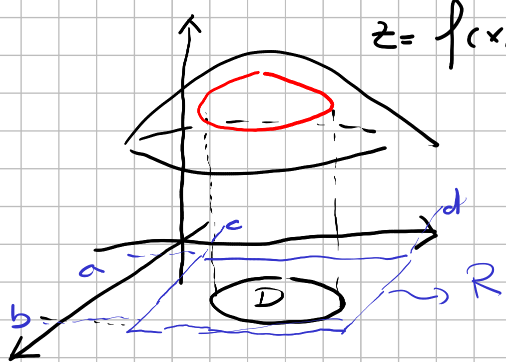
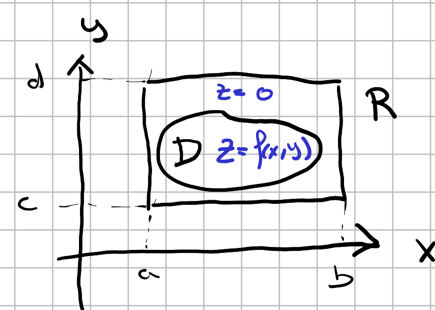
Chapter 12, Sec 2 : DOUBLE INTEGRALS ON GENERIC DOMAIN D

$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA$$

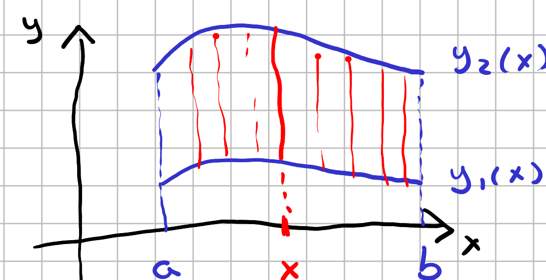




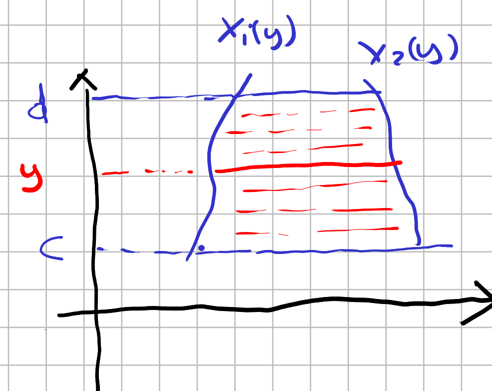
$$g(x,y) = \begin{cases} f(x,y) & \text{in } D \\ 0 & \text{in } R \setminus D \end{cases}$$

DEF.

D is type I if $\begin{cases} a < x < b \\ y_1(x) < y < y_2(x) \end{cases}$



D is type II if $\begin{cases} c < y < d \\ x_1(y) < x < x_2(y) \end{cases}$



Fubini-Tonelli Theorem

1) If D is type I as above

$$\iint_D f(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

we call the RHS iterated integrals

2) If D is type II as above

$$\iint_D f(x,y)\,dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

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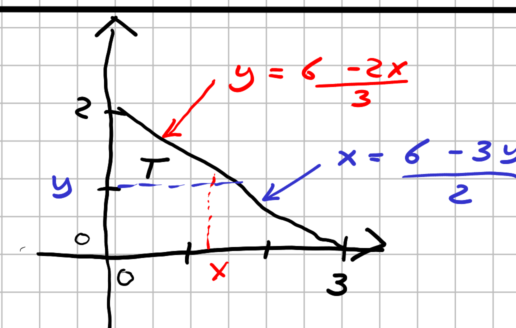
ex.

$$\iint_T xy\,dA \quad \text{where} \quad T = \begin{cases} 2x + 3y < 6 \\ x > 0 \\ y > 0 \end{cases}$$



T is either region of type I or II

Type I $\begin{cases} 0 < x < 3 \\ 0 < y < 2 - \frac{2}{3}x \end{cases}$

Type II $\begin{cases} 0 < y < 2 \\ 0 < x < 3 - \frac{3}{2}y \end{cases}$

Type I $\quad I = \int_0^3 \int_0^{2-\frac{2}{3}x} x\,y \; dy\,dx = \int_0^3 x \; \frac{y^2}{2}\Big|_0^{2-\frac{2}{3}x} dx = \frac{1}{2}\int_0^3 x \left(\left(2-\frac{2}{3}x\right)^2 - 0 \right) dx$

$$= \frac{1}{2}\int_0^3 x\left(4 - \frac{8}{3}x + \frac{4}{9}x^2\right) dx = \frac{1}{2}\int_0^3 4x - \frac{8}{3}x^2 + \frac{4}{9}x^3 \; dx =$$

$$= \frac{1}{2}\left(\cancel{4}^{2}\frac{x^2}{\cancel{2}} - \frac{8}{3}\,\frac{x^3}{3} + \frac{\cancel{4}}{9}\,\frac{x^4}{\cancel{4}} \right)\Big|_0^3 = \frac{1}{2}\left[2\; 3^2 - \frac{8}{9}\,3^3 + \frac{3^4}{9}\right] = \frac{1}{2}\left[18 - 24 + 9\right] = \boxed{\frac{3}{2}}$$

Type II $= \int_0^2 \int_0^{3-\frac{3}{2}y} x\,y \; dx\,dy = \int_0^2 y \; \frac{x^2}{2}\Big|_0^{3-\frac{3}{2}y} dy = \frac{1}{2}\int_0^2 y\left(\left(3-\frac{3y}{2}\right)^2 - 0\right) dy$

$$= \frac{1}{2}\int_0^2 y\left(9 - 9y + \frac{9}{4}y^2\right) dy = \frac{1}{2}\int_0^2 9y - 9y^2 + \frac{9}{4}y^3 \, dy =$$

$$\frac{1}{2}\left(9\frac{y^2}{2} - 9\frac{y^3}{3} + \frac{9}{4}\,\frac{y^4}{4} \right)\Big|_0^2 = \frac{1}{2}\left(9(2) - 3(8) + 9\right) = \boxed{\frac{3}{2}}$$

i) Linearity rule $\iint_D f(x,y) + g(x,y) \, dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

ii) Dominance rule: if $f(x,y) \geq g(x,y)$ for all (x,y) on D then

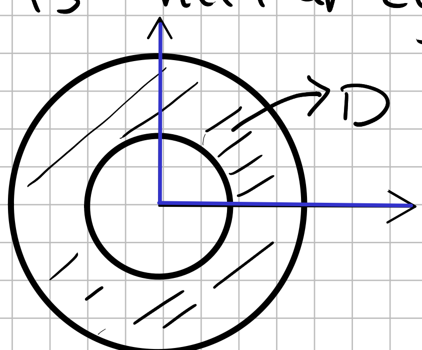
$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

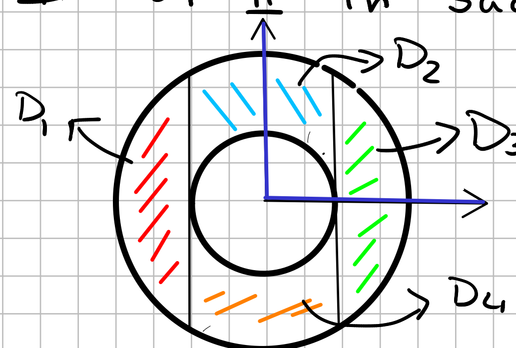
iii) Division rule: Let $D = D_1 + D_2$ then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Remark: Division rule is very useful to split a region that is neither type I nor II in such region types

Ex.





Type I or II

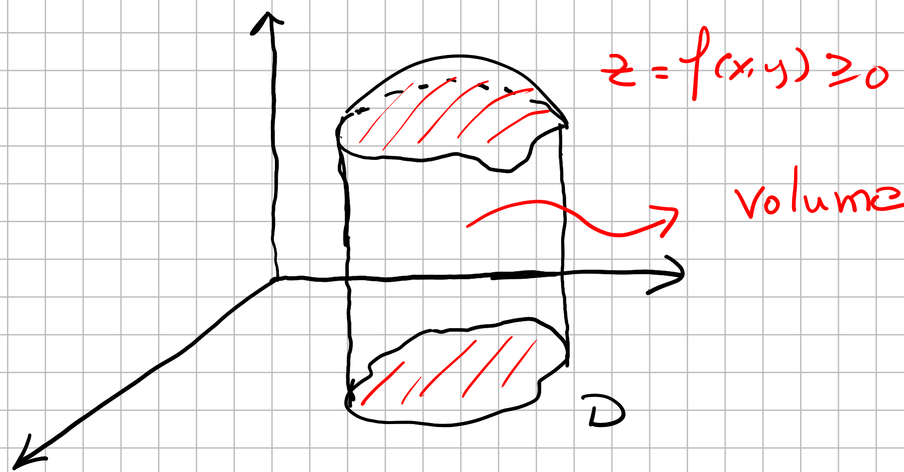
$D = D_1 + D_2 + D_3 + D_4$

neither

Type I

* Let $f(x,y) \geq 0$ For all (x,y) on D, then

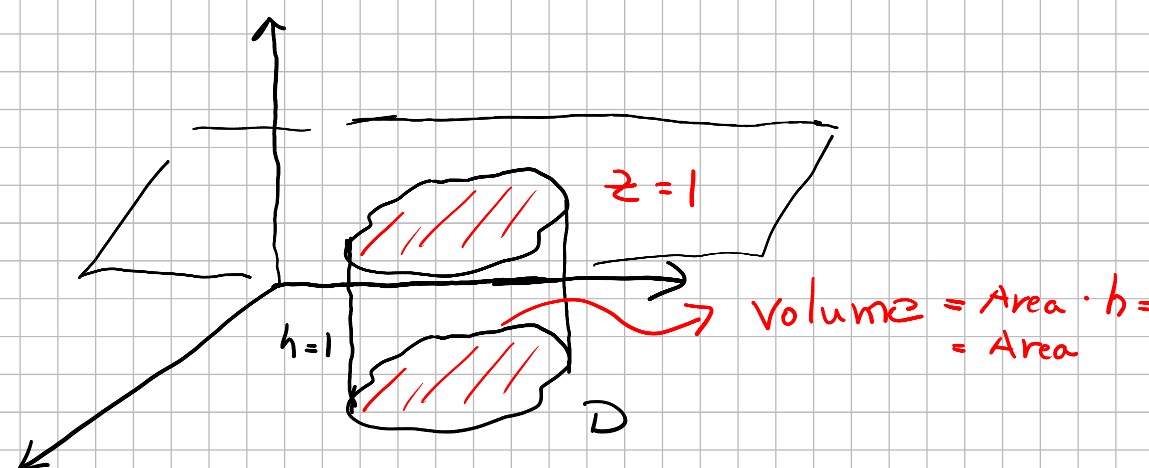
$\iint_D f(x,y)\,dA =$ Volume between $z=0$, $z=f(x,y)$, inside the cylinder D



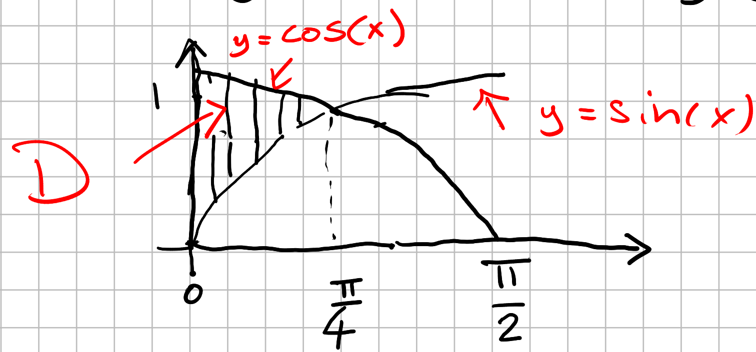
* $\iint_D 1\,dA =$ Area of D

z=1 plane

$f(x,y)=1 \geq 0$ then $\iint_D f(x,y)\,dA =$ Volume $=$ Area $* \, h =$ Area $(1) =$ Area



Find the area of the region D bounded by $y=\sin(x)$, $y=\cos(x)$, $x=0$ and $x=\frac{\pi}{4}$

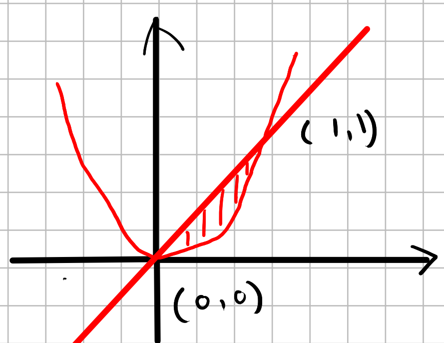


D is type I

$$\text{Area}(D) = \iint_D 1\, dA = \int_0^{\frac{\pi}{4}} \int_{\sin(x)}^{\cos(x)} 1 \, dy\, dx = \int_0^{\frac{\pi}{4}} y \Big|_{\sin(x)}^{\cos(x)} dx$$

$$= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)\, dx = \sin(x) + \cos(x) \Big|_0^{\frac{\pi}{4}} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 1 = \sqrt{2} - 1$$

Find the area of the region in between $y=x$ and $y=x^2$



intersections $\begin{cases} y = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} y = x \\ x = x^2 \end{cases}$ $\nearrow (0,0)$ $\searrow (1,1)$

$$\text{Area} = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 (x - x^2)\, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Remark. The region above is also type II so we could have also used

$$\text{Area} = \int_0^1 \int_y^{\sqrt{y}} 1\, dx\, dy = \int_0^1 \sqrt{y} - y\, dy = \frac{2}{3} y^{3/2} - \frac{y^2}{2} \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

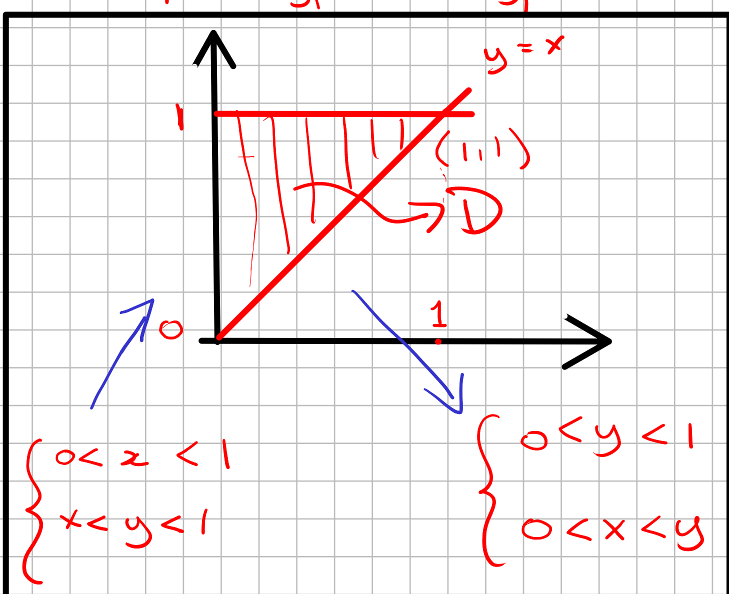
If D is either type I or II

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

ex. Solve $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$ reversing the order of integration

the antiderivative of e^{y^2} is something we do not know, then we try to solve the integral reversing the order

From Type I to type II



$$\int_0^1 \int_x^1 e^{y^2}\,dx\,dy = \iint_D e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dx\;dy$$

easy integral

$u = y^2$
$du = 2y\,dy$

$$= \int_0^1 e^{y^2}\, x\Big|_0^y\,dy = \int_0^1 e^{y^2}(y-0)\,dy = \int_0^1 e^{y^2}y\,dy =$$

$$= \frac{1}{2}\int_0^1 \underbrace{e^{y^2}(2y)\,dy}_{e^u\,du} = \frac{1}{2}\, e^{y^2}\Big|_0^1 = \frac{1}{2}\left(e^{(1)} - e^0\right) = \frac{1}{2}(e-1)$$