Sec 12.1 Double integrals on a rectangular region R.

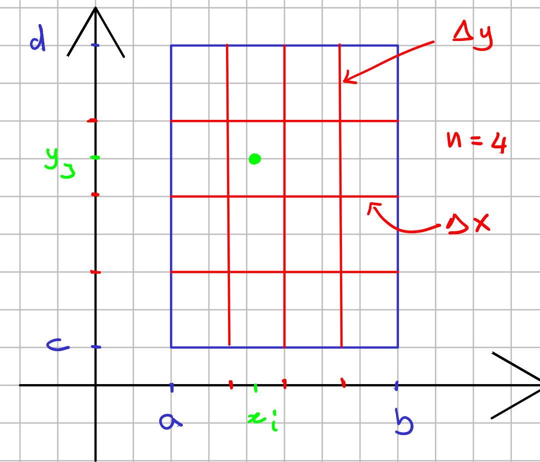
$$\iint_R f(x,y)\,dA = \lim_{n\to+\infty} S_n \qquad S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

$$R : \begin{cases} a \leq x \leq b \\ c \leq y \leq d \end{cases}$$

$$\Delta A = \Delta x \, \Delta y$$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta y = \frac{d-c}{n}$$



if $f(x,y) > 0$ for all (x,y) in R $\quad \iint_R f(x,y)\,dA =$ Volume inside R between $z=0$ and $z = f(x,y)$

Fubini-Tonelli Theorem:

DOUBLE INTEGRAL

ITERATED INTEGRALS

$$\iint_R f(x,y)\,dA = \overbrace{\int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\text{inner integral}}\,dx}^{\text{outer integral}} = \overbrace{\int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\text{inner integral}}\,dy}^{\text{outer integral}}$$

$$\int_a^b h(x)\,dx = XX = \int_c^d l(y)\,dy$$

Useful identity.

$$\int_a^b \int_c^d f(x)\, g(y)\, \underbrace{dA}_{dy\,dx} = \int_a^b f(x)\, dx \int_c^d g(y)\, dy$$

Proof:

because it is a constant → linearly rule

$$\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx = \int_a^b f(x) \left(\int_c^d g(y)\, dy \right) = C_1 \int_a^b f(x)\, dx$$

※ C_1

$$\Rightarrow \int_c^d g(y)\, dy \int_a^b f(x)\, dx$$

Ex.

$$\int_0^3 \int_0^2 (2-y)\, dy\, dx = \int_0^3 f(x)\, dx \int_0^2 g(y)\, dy = \int_0^3 dx \int_0^2 (2-y)\, dy$$

$f(x) = 1 \qquad g(y) = 2-y$

$$= \left(x \Big|_0^3 \right) \left(2y - y^2 \Big|_0^2 \right)$$

$$= (3-0)\left((4-2)-0\right) \boxed{= 6}$$

ex. $\iint_R x^2 y^5 \, dA$ $\qquad R = \begin{cases} 1 \leq x \leq 2 \\ 0 \leq y \leq 1 \end{cases}$

Without using the identity:

$$\int_1^2 \int_0^1 x^2 y^5 \, dy \, dx = \int_1^2 x^2 \frac{y^6}{6} \Big|_0^1 dx = \frac{1}{6} \int_1^2 x^2 dx = \frac{1}{6} \frac{x^3}{3} \Big|_1^2 = \frac{1}{6}\left(\frac{8}{3} - \frac{1}{3}\right)$$

$$\boxed{\Rightarrow \frac{7}{18}}$$

Using the identity:

$$\int_0^1 \int_1^2 \overset{f(x)\,g(y)}{x^2 y^5} \, dx \, dy = \int_0^1 y^5 \, dy \int_1^2 x^2 dx = \frac{y^6}{6} \Big|_0^1 \frac{x^3}{3} \Big|_1^2 = \frac{1}{6}\left(\frac{8}{3} - \frac{1}{3}\right)$$

$$\boxed{\Rightarrow \frac{7}{18}}$$

ex.

$$\underbrace{\int_0^1 \int_0^{\frac{\pi}{2}} y \cos(xy)\, dy\, dx}_{I_1}$$

Chain rule... but we don't like this so we are going to try to reverse the integral using Fubini's Rule backwards...

→ REVERSE the order of integration:

$$\Rightarrow \iint_R f(x,y)\, dA = \int_0^2 \int_0^1 y \cos(xy)\, dx\, dy$$

$$xy = u \qquad \cancel{y}\, dx = \frac{du}{y}$$

$$\Rightarrow \int_0^{\pi/2} y \left.\frac{\sin(xy)}{y}\right|_0^1 dy = \int_0^{\pi/2} \sin(y) - \underbrace{\cancel{\sin(0)}}_{0}\, dy = -\cos y \Big|_0^{\pi/2} = -0+1$$

$$\boxed{\Rightarrow 1}$$

Cannot apply this rule:

$$\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx$$

because

$g(y)$ $f(x,y)$ → are not just numbers