11.1 Function of several variables. $z = f(x,y)$ or $T = f(x,y,z)$

Def. A function of two variables $f(x,y)$ is a rule that associates for each point (x,y) in the domain D a unique value $f(x,y)$ in the range R.

$z = f(x,y)$

z: dependent variable

x, y: independent variables

If the domain is not given we take the largest set D for which the rule is defined.

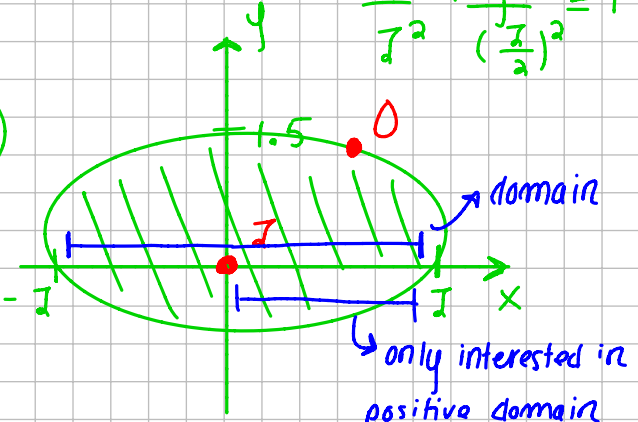
Ex. Find the domain and the range of $z = f(x,y) = \sqrt{9 - x^2 - 4y^2}$

$9 - x^2 - 4y^2 \geq 0$

$x^2 + 4y^2 \leq 9$

$\frac{x^2}{3^2} + \frac{y^2}{(\frac{3}{2})^2} \leq 1$

D: (x,y) such that $\frac{x^2}{3^2} + \frac{y^2}{(\frac{3}{2})^2} \leq 1$



R: $z \geq 0$ $\quad z \leq \sqrt{9} = 3$

$f(x,0) = \sqrt{9 - x^2} = g(x)$

$0 \leq x \leq 3$

R: $\boxed{0 \leq z \leq 3}$

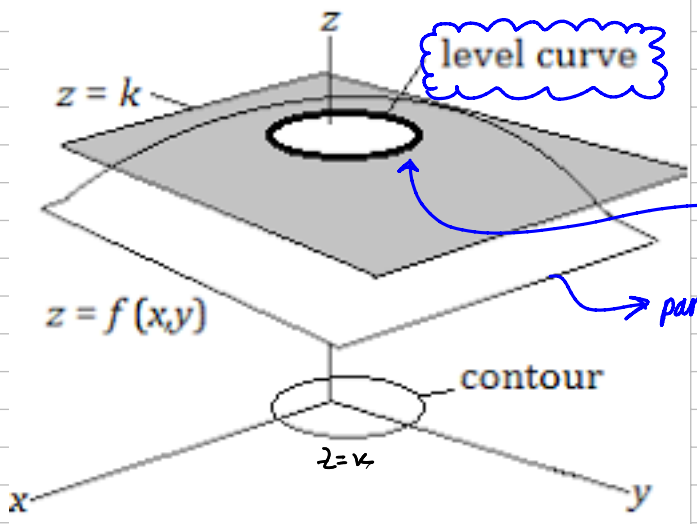
Operation. Let $F(x,y)$ a function defined on D_1 and $g(x,y)$ a function defined on D_2, then

$(f \pm g)(x,y) = F(x,y) \pm g(x,y)$ For all (x,y) in $D_1 \cap D_2$

$(Fg)(x,y) = F(x,y)\, g(x,y)$ For all (x,y) in $D_1 \cap D_2$

$\left(\frac{F}{g}\right)(x,y) = \frac{F(x,y)}{g(x,y)}$ For all (x,y) in $D_1 \cap D_2 \setminus (x,y)$ such that $g(x,y) = 0$

Level curves and Surfaces. $z = f(x,y)$

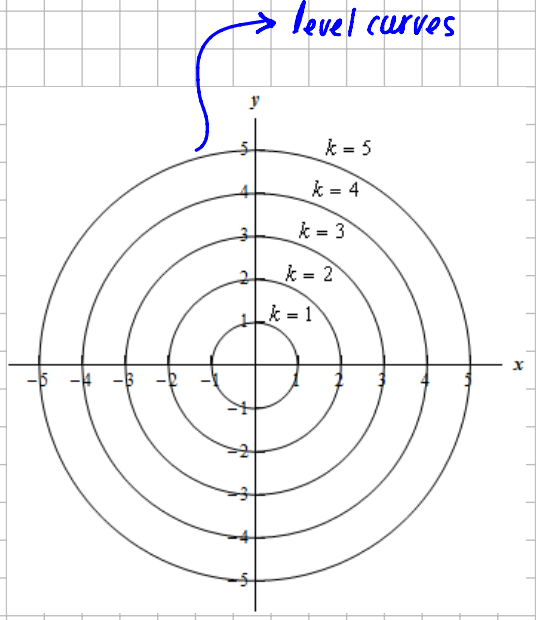


$z = f(x,y)$

$z = k$

$f(x,y) = k$

paraboloid



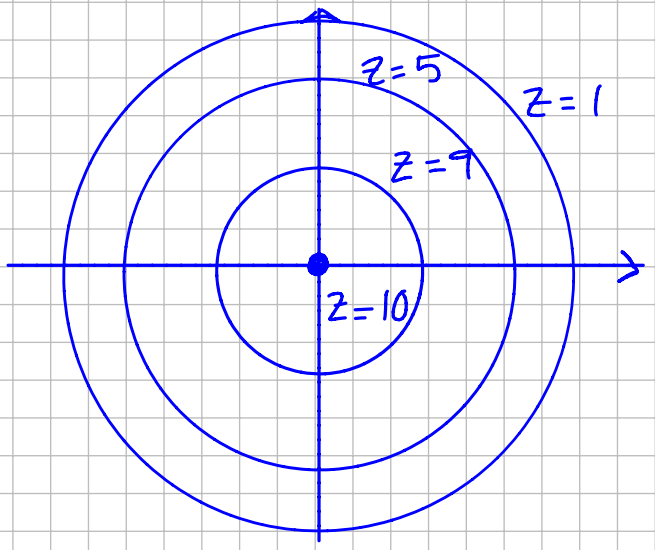
Ex. Find the level curves of $z = 10 - x^2 - y^2$

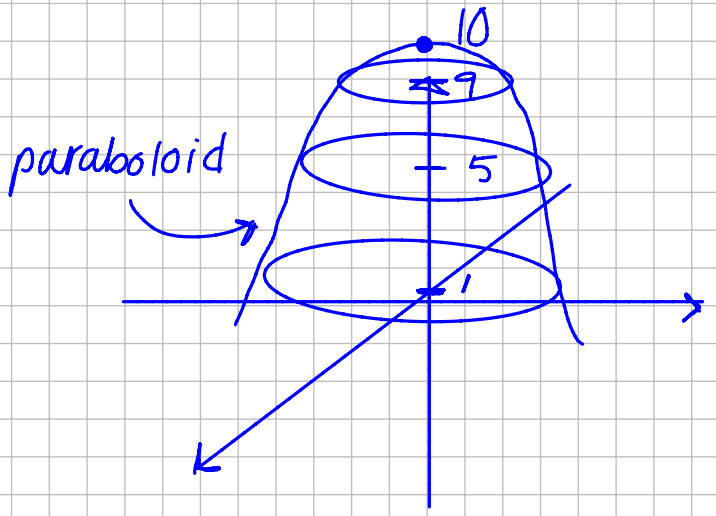
We already know this is a paraboloid, but let's pretend we have no idea what this is...

$$\begin{cases} z = 10 - x^2 - y^2 \\ z = 10 \end{cases} \qquad \begin{cases} x^2 + y^2 = 0 \quad (x,y) = (0,0) \\ z = 10 \end{cases}$$

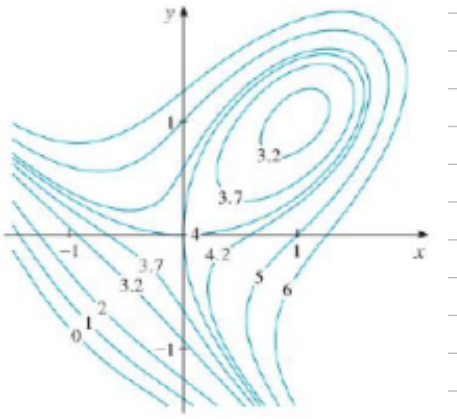
$$\begin{cases} z = 9 \\ x^2 + y^2 = 10 - 9 = 1 \end{cases} \qquad \begin{cases} z = 5 \\ x^2 + y^2 = 10 - 5 = 5 \end{cases}$$

$$\begin{cases} z = 1 \\ x^2 + y^2 = 10 - 9 = 1 \end{cases}$$

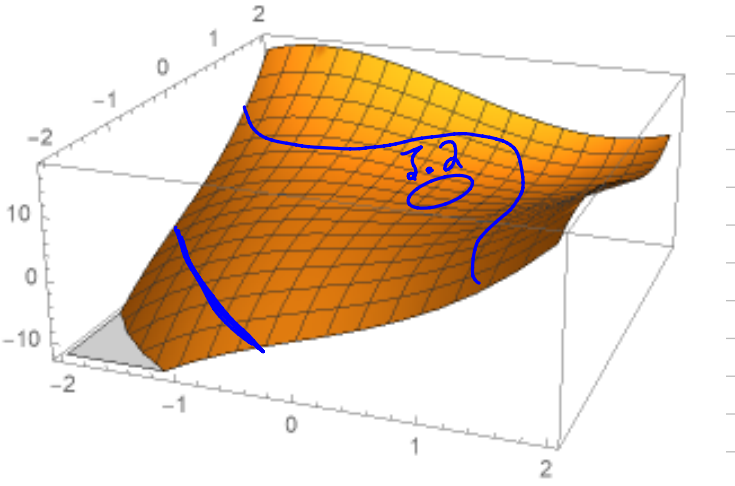




$$f(x,y) = 4 + x^3 + y^3 - 4xy$$



.

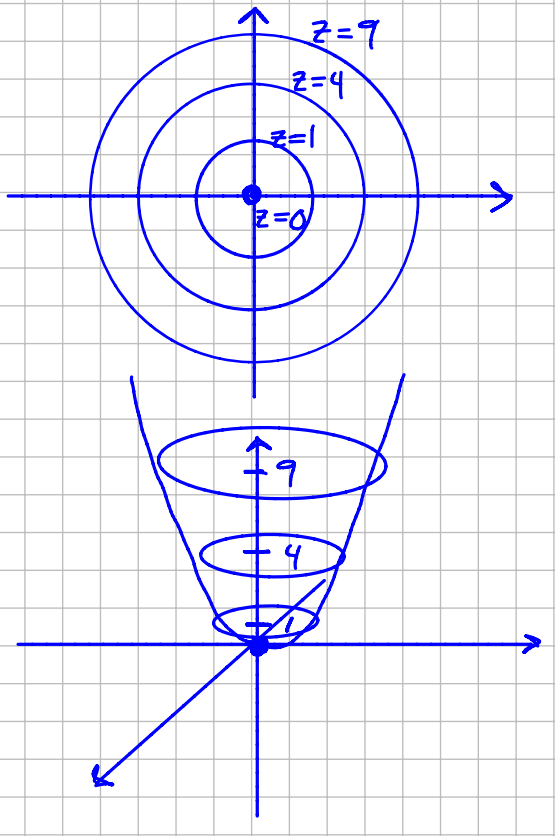


Ex. Use the level curves of $f(x,y) = x^2 + y^2$ to sketch the graph of f

(exactly the same problem as the one above!)

$$\begin{cases} z = 0 \\ x^2+y^2 = 0 \end{cases} \qquad \begin{cases} z = 1 \\ x^2+y^2 = 1 \end{cases} \qquad \begin{cases} z = 4 \\ x^2+y^2 = 4 \end{cases}$$

$$\begin{cases} z = 9 \\ x^2+y^2 = 9 \end{cases}$$



Function of 3 variables $T(x,y,z)$. Temperature of a 3D object

Function of 4 variables $T(x,y,z,t)$. Temperature of a 3D object in time.

Ex. Find the domain and the range of $T=\sqrt{9-x^2-y^2-z^2}$, and sketch 3 level surfaces.

$9-x^2-y^2-z^2=0$

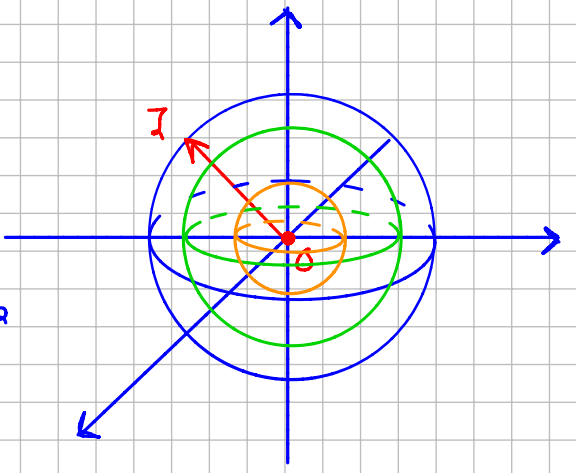
$x^2+y^2+z^2 \le 3^2$ ← ball

for $x^2+y^2+z^2=9 \longrightarrow T=0$

for $(x,y,z)=(0,0,0) \longrightarrow T=\sqrt{9}=3$

$\begin{cases} T=1 \\ 1^2=\sqrt{9-x^2-y^2-z^2} \end{cases}$

$\begin{cases} T=1 \\ x^2+y^2+z^2=8=(2\sqrt{2})^2 \end{cases}$



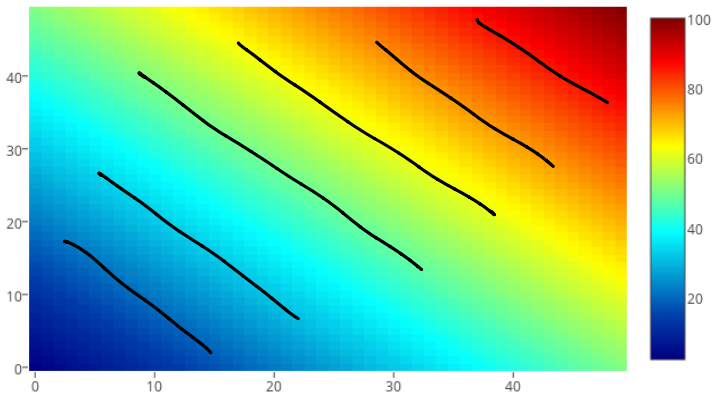
$T(x,0,0)=\sqrt{9-x^2}=g(x) \quad 0 \le x \le 3$ (continuous function)

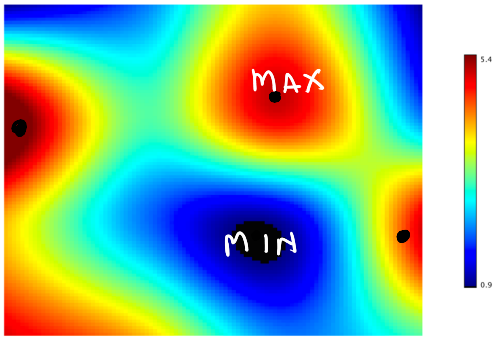
$R=[0,3]$

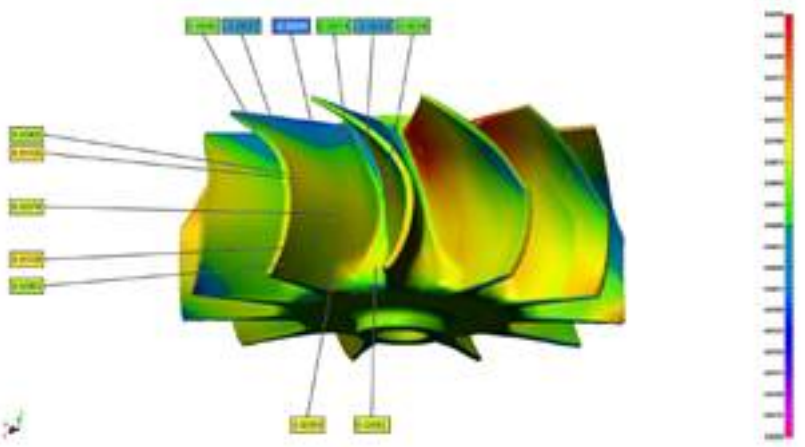
* Note: you cannot sketch in 4D when sketching level surfaces.

COLOR MAPS

PLANE







* Same color = same height

Do not confuse functions of several variables with implicit functions

Function

$T = 4 - x^2 - y^2 - z^2$

$f(x, y, z)$

hyperparaboloid

4D

vs

Implicit Function

$x^2 + y^2 + z^2 = 4$

$z = \pm\sqrt{4 - x^2 - y^2}$

or $z = f(x, y)$

surface

3D