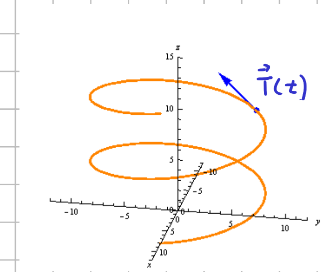
10.4

DEF. UNIT TAGENT VECTOR. Let $\vec{R}(t)$ be a smooth vector Field. $\vec{T} = \dfrac{\vec{R}'(t)}{\|\vec{R}'(t)\|}$ is the unit tangent vector of $\vec{R}(t)$.



Why $\vec{T}$ is unit?

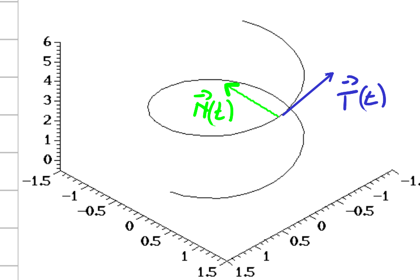
why $\vec{T}$ it Tangent to $\vec{R}$?

DEF. PRINCIPAL UNIT NORMAL VECTOR. Let $\vec{R}(t)$ and $\vec{T}(t)$ as before. Let $\vec{T}(t)$ be smooth. $\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$ is the principal unit normal vector.

if $\|\vec{F}\|$ is constant and smooth $\Rightarrow \vec{F} \perp \vec{F}'$

Why is $\vec{N} \perp \vec{T}$? $\quad \|\vec{T}\| = 1 \quad \vec{T}$ is smooth $\vec{T}' \perp \vec{T} \Rightarrow \vec{T}' \perp \vec{T}$

How many normal vector to $\vec{R}(t)$ there exist?



why $\vec{N}$ is principal?

Ex. Let $\vec{R}(t) = \langle a\cos t, a\sin t, bt\rangle$. Find $\vec{T}(t)$ and $\vec{N}(t)$.

$$\vec{T}' = \frac{\vec{R}}{\|\vec{R}\|}$$

$$\vec{R}' = \langle -a\sin(t), a\cos(t), b\rangle$$

$$\|\vec{R}'\| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + b^2} = \sqrt{a^2+b^2}$$

$$\vec{T} = \frac{\langle -a\sin t, a\cos t, b\rangle}{\sqrt{a^2+b^2}}$$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$$

$$\vec{T}' = \frac{1}{\sqrt{a^2+b^2}}\langle -a\cos t, -a\sin t, 0\rangle$$

$$\|\vec{T}'\| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \frac{\sqrt{a^2}}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\vec{N} = \langle -\cos t, -\sin t, 0\rangle$$

Ex. Let $\vec{R}(t) = \langle e^t\cos(2t),\ e^t\sin(2t),\ e^t \rangle$. Find $\vec{T}(t)$ and $\vec{N}(t)$

$$\vec{R}' = \langle e^t\cos(2t) + 2e^t(-\sin(2t)),\ e^t\sin(2t) + e^t 2\cos(2t),\ e^t \rangle$$

$$= e^t\langle \cos(2t) - \sin(2t),\ \sin(2t) + 2\cos(2t),\ 1 \rangle = f(t)\ \vec{F}(t)$$

$$\|\vec{R}'\| = |f(t)|\ \|\vec{F}(t)\| = |e^t|\sqrt{(\cos(2t) - 2\sin(2t))^2 + (\sin(2t) + 2\cos(2t))^2 + 1^2}$$

$$\cos^2(2t) - 4\cancel{\cos(2t)\sin(2t)} + 4\sin^2(2t) + \sin^2(2t) + 4\cancel{\sin(2t)\cos(2t)} + 4\cos^2(2t) + 1$$

$$= e^t\sqrt{1+4+1} = \sqrt{6}\,e^t$$

$$\vec{T} = \frac{1}{\sqrt{6}}\langle \cos(2t) - 2\sin(2t),\ \sin(2t) + 2\cos(2t),\ 1 \rangle$$

$$\vec{T}' = \frac{1}{\sqrt{6}}\langle -2\sin(2t) - 4\cos(2t),\ 2\cos(2t) - 4\sin(2t),\ 0 \rangle$$

$$\|\vec{T}'\| = \frac{1}{\sqrt{6}}\sqrt{(-2\sin(2t) - 4\cos(2t))^2 + (2\cos(2t) - 4\sin(2t))^2}$$

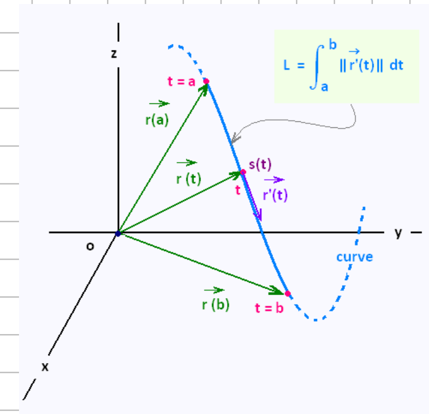
$$= \frac{1}{\sqrt{6}} \quad 4\sin^2(2t) + 16\cancel{\sin(2t) + \cos(2t)} + 16\cos^2(2t) + 4\cos^2(2t) - 16\cancel{\cos(2t)\sin(2t)} + 16\sin^2(2t)$$

$$= \frac{\sqrt{20}}{\sqrt{6}} = \frac{\sqrt{20}}{\sqrt{6}}$$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \frac{1}{\sqrt{20}}\langle -2\sin(2t) - 4\cos(2t),\ 2\cos(2t) - 4\sin(2t),\ 0 \rangle$$

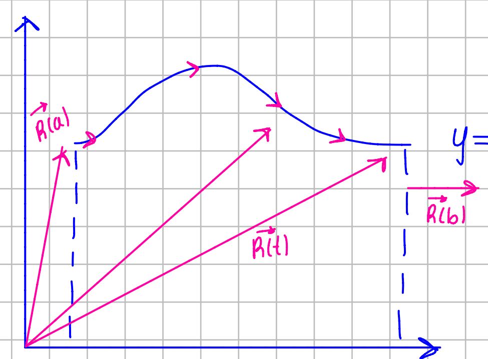
ARC LENGHT FORMULA. Let $\vec{R}(t)$ be smooth curve traversed only in one direction.

The arc lenght of $\vec{R}(t)$ from a to b is given by



$$s = \int_a^b \|\vec{R}'(t)\|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\; dt$$

Informal proof from calculus 2.



$$s = \int_c^d \sqrt{1 + (y')^2}\; dx \qquad \vec{R}(t) = \langle x(t), y(t) \rangle \quad t \in [a, b]$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right)^2}\, dx = \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\; dx$$

$$= \sqrt{(x'(t))^2 + (y'(t))^2}\; \frac{dt}{\cancel{dx}}\, \cancel{dx} \qquad s = \int_a^b \sqrt{(x')^2 + (y')^2}\; dt$$

Ex. Find the arclenght of $\vec{R} = \langle a \cos t, a \sin t, bt \rangle \quad t \in [0, 2\pi]$.

$$s = \int_0^{2\pi} \|\vec{R}'\|\, dt = \int_0^{2\pi} \sqrt{a^2 + b^2}\; dt = \sqrt{a^2 + b^2}\; t \Big|_0^{2\pi} = 2\pi \sqrt{a^2 + b^2}$$

$$\vec{R}' = \langle -a \sin t, a \cos t, b \rangle$$

$$\|\vec{R}'\| = \sqrt{a^2 + b^2}$$

$s(t) = \int_a^t \|R'(u)\|\, du$ describes the distance traveled on $\vec{R}(t)$ from a fixed point $\vec{R}(a)$

$$\frac{d\, s(t)}{dt} = \frac{d}{dt} \int_0^t \|\vec{R}'(u)\|\, dt = \|\vec{R}'(t)\| = s'(t)$$

Ex. Find the arc lengh parametrization of $\vec{R}(t) = \langle 5\cos t, 5\sin t \rangle$ from $t=0$

$$\vec{R}(s) \rightarrow s(t) \Rightarrow t(s) \rightarrow \vec{R}(t(s)) = \vec{R}(s)$$

$$s(t) = \int_0^t \|\vec{R}'(u)\|\, du = \int_0^t 5\, du = 5u \Big|_0^t = 5t \Leftrightarrow t = \left[\frac{s}{5}\right] \qquad \vec{R}\left(\frac{s}{5}\right) = \left\langle 5\cos\frac{s}{5}, 5\sin\frac{s}{5} \right\rangle$$

$$\vec{R}'(t) = \langle -5\sin(t), 5\cos t \rangle$$

$$\|\vec{R}'(t)\| = \sqrt{(-5\sin(t))^2 + (5\cos t)^2} = 5$$

Ex. Find the arc lenght parametrization of $\vec{R}(t) = \langle e^t \cos t, e^t \sin t \rangle$ from $t=0$.

$$s(t) = \int_0^t \|\vec{R}'(u)\|\, du = \int_0^t \sqrt{2}\, e^u\, du = \sqrt{2}\, e^u \Big|_0^t = \sqrt{2}\,(e^t - e^0) = \sqrt{2}\,(e^t - 1)$$

$$\vec{R}'(t) = \langle \cancel{e^t}\cos t - \cancel{e^t}\sin t, \cancel{e^t}\sin t + \cancel{e^t}\cos t \rangle e^t$$

$$\|\vec{R}'\| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2}\, e^t$$

$$\frac{s}{\sqrt{2}} = \cancel{\sqrt{2}}\,(e^t - 1) \qquad e^t = \frac{s}{\sqrt{2}} + 1 \longrightarrow \ln e^t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

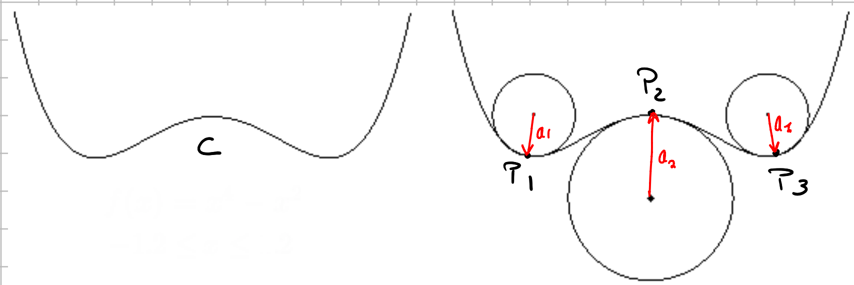
$$t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\vec{R}\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right) = \left\langle e^{\ln\left(\frac{s}{\sqrt{2}}+1\right)} \cos\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right), e^{\ln\left(\frac{s}{\sqrt{2}}+1\right)} \sin\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right) \right\rangle$$

$$= \left\langle \left(\tfrac{s}{\sqrt{2}}+1\right)\cos\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right), \left(\tfrac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right) \right\rangle$$

$$= \vec{R}(s)$$

The osculating circle of a curve C at one point P is the circle that best approximate C at P.



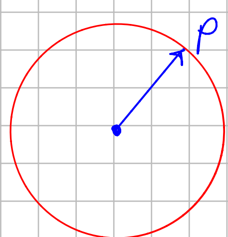
$K_1 = \frac{1}{a_1}$

$K_2 = \frac{1}{a_2}$

$K_3 = \frac{1}{a_3}$

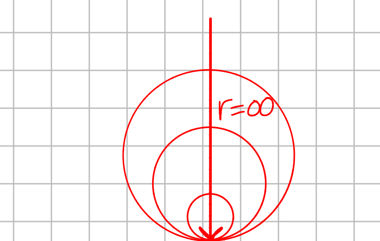
Curvature of a smooth curve. It is the iverse of the radius of the osculating circle.

What is the curvature of a circle of radius a?



$K = \frac{1}{a}$

What is the curvature of a straight line?



$K = \frac{1}{\infty} = 0$

Curvature of a smooth curve $\vec{R}(t)$. (or $\vec{R}(s)$)

$$\kappa = \|\vec{T}'(s)\| \quad \text{or} \quad \boxed{\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|} \quad \text{or} \quad \kappa = \frac{\|\vec{R}' \times \vec{R}''\|}{\|R'\|^3}}$$

Kappa

The formulas are all equivalent but we use either (2) or (3)

Ex. Find the curvature of $\vec{R}(t) = \langle a\cos t, a\sin t, bt \rangle$

$\vec{R}' = \langle -a\sin(t), a\cos(t), b \rangle \qquad \|\vec{R}'\| = \sqrt{a^2+b^2}$

$\vec{R}'' = \langle -a\cos(t), -a\sin(t), 0 \rangle$

$$\vec{R}' \times \vec{R}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \hat{i}\left(-(-ab)\sin t\right) - \hat{j}\left(-(-ab)\cos t\right) + \hat{k}\left(a^2\sin^2 t + a^2\cos^2 t\right)$$

$$= \langle ab\sin t, -ab\cos t, a^2 \rangle$$

$$\|\vec{R}' \times \vec{R}''\| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2+b^2}$$

$$\kappa = \frac{\|\vec{R}' \times \vec{R}''\|}{\|R'\|^3} = \frac{a\cancel{\sqrt{a^2+b^2}}}{(\sqrt{a^2+b^2})^{\cancel{3}\,2}} = \frac{a}{a^2+b^2}$$

Ex. Find the curvature of $\vec{R}(t) = \langle e^{-t}\cos 3t, e^{-t}\sin 3t\rangle$

$$\kappa = \frac{\|\vec{T}'\|}{\|\vec{R}'\|}$$

$$\vec{R}' = \langle -\cancel{e^{-t}}\cos 3t - 3\cancel{e^{-t}}\sin 3t, -\cancel{e^{-t}}\sin 3t + 3\cancel{e^{-t}}\cos 3t\rangle e^{-t}$$

$$\|\vec{R}'\| = e^{-t}\sqrt{(-\cos(3t)-3\sin(3t))^2+(-\sin(3t)+3\cos(3t))^2} = \sqrt{10}\,e^{-t}$$

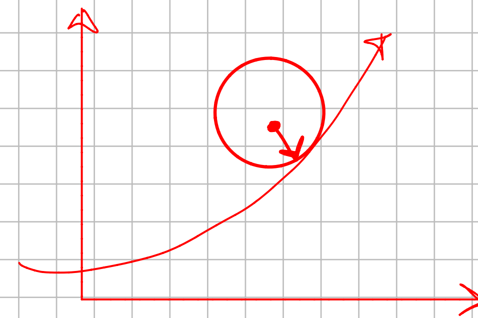
$$\vec{T} = \frac{\vec{R}'}{\|\vec{R}\|} = \frac{1}{\sqrt{10}}\langle -\cos(3t)-3\sin(3t), -\sin(3t)+3\cos(3t)\rangle$$

$$\vec{T}' = \frac{\cancel{1}\,3}{\sqrt{10}}\langle \cancel{+3}\sin(3t) - \overset{3}{\cancel{9}}\cos(3t), -\cancel{3}\cos(3t) - \overset{3}{\cancel{9}}\sin(3t)\rangle$$

$$\|\vec{T}'\| = \frac{3}{\sqrt{10}}\sqrt{(\sin(3t)-3\cos(3t))^2+(-\cos(3t)-3\sin(3t))^2} = 3\,\frac{\cancel{\sqrt{10}}}{\cancel{\sqrt{10}}} = 3$$

$$\kappa = \frac{\|\vec{T}'\|}{\|\vec{R}\|} = \boxed{\frac{3}{\sqrt{10}\,e^{-t}}}$$

Ex. Find the curvature $y(x) = e^{2x}$



$$\vec{R}(t) = \langle t, y(t), 0 \rangle$$

$$\vec{R}'(t) = \langle 1, y', 0 \rangle \qquad \|\vec{R}'\| = \sqrt{1 + (y')^2}$$

$$\vec{R}''(t) \ \langle 0, y'', 0 \rangle$$

$$\vec{R}' \times \vec{R}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & y' & 0 \\ 0 & y'' & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(y'') = \langle 0, 0, y'' \rangle$$

$$\|\vec{R}' \times \vec{R}''\| = \sqrt{0^2 + 0^2 + (y'')^2} = |y''|$$

$$\boxed{\kappa = \frac{\|\vec{R}' \times \vec{R}''\|}{\|\vec{R}'\|^3} = \frac{|y''(x)|}{(1 + (y'(x))^2)^{3/2}}}$$

$$y = e^{2x}$$

$$y' = 2e^{2x} \qquad \kappa(x) = \frac{|4e^{2x}|}{(1 + (2e^{2x})^2)^{3/2}} = \boxed{\frac{4e^{2x}}{(1 + (2e^{2x})^2)^{3/2}}}$$

$$y'' = 4e^{2x}$$