10.2 DIFFERENTIATION AND INTEGRATION OF A VECTOR FUNCTION

Def. The derivative of a vector function at t_0 is defined by

$$\vec{F}'(t_0) = \lim_{t \to t_0} \frac{\vec{F}(t) - \vec{F}(t_0)}{t - t_0},$$ wherever the limit exists.

If $\vec{F}'(t_0)$ exists we say that $\vec{F}$ is differentiable at t_0.

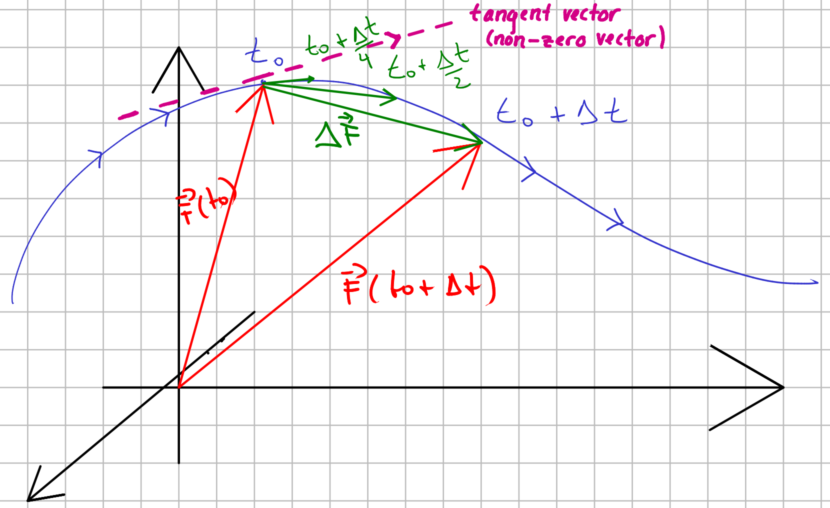
Leibniz notation $\frac{d\vec{F}(t)}{dt}$

The derivative of a vector function exists if and only if the derivative of each component exists.

Ex. Find the derivative of $\vec{F}(t) = \langle \underline{\ln t}, \sin t, \frac{1}{\underline{t-1}} \rangle$, where it is defined.

$$\vec{F}'(t) = \left\langle \frac{1}{t}, \cos(t), -\frac{1}{(t-1)^2} \right\rangle \quad t > 0 \text{ and } t \neq 1$$

Tangent vector: $\boxed{\vec{F}'(t_0)} = \lim_{\Delta t \to 0} \frac{\vec{F}(t_0+\Delta t) - \vec{F}(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \vec{F}(t_0, \Delta t)}{\Delta t}$



If $\vec{F}'(t_0) \neq \vec{0}$, it is a vector tangent to the graph that points in the direction of motion.

Ex. Find the tangent vector to $\vec{F}(t) = \langle e^{2t}, t^2 - t, \ln t \rangle$ at $t = 0.2$

$$\vec{T}(t_0) = \vec{F}'(t_0)$$

$$\vec{F}'(t) = \left\langle 2e^{2t}, 2t - 1, \frac{1}{t} \right\rangle$$

$$\vec{F}'(0.2) = \left\langle 2e^{2(0.2)}, 2(0.2) - 1, \frac{1}{0.2} \right\rangle = \langle a, b, c \rangle$$

Smooth curve: The graph of a vector Function $\vec{F}(t)$ is said to be smooth at t_0 iF and only if $\vec{F}'(t_0)$ exists and $\vec{F}'(t_0) \neq \vec{0}$

A vector function is smooth where the tangent vector is deFined.

Ex. Find the interval where $\vec{F}(t) = \langle t^2+1, \cos t, e^t + e^{-t} \rangle$ is smooth

$$\vec{F}'(t) = \langle 2t, -\sin(t), e^t - e^{-t} \rangle \neq \vec{0}$$

$$\begin{cases} 2t = 0 \\ -\sin(t) = 0 \\ e^t - e^{-t} = 0 \end{cases} \qquad \begin{cases} t = 0 \\ -\sin(0) = 0 \\ e^0 - e^{-0} = 1 - 1 = 0 \end{cases} \qquad \boxed{t = 0}$$

$\vec{F}$ is smooth for all $t \neq 0$.

Higher order derivatives:

$$\vec{F}''(t) = \frac{d}{dt}\left(\vec{F}'(t)\right)$$

$$\vec{F}^{(n)}(t) = \frac{d}{dt}\left(\vec{F}^{(n-1)}(t)\right)$$

Ex. Find $\vec{F}''(t)$ For $\vec{F}(t) = \langle t^2, \cos t, e^t \rangle$

$$\vec{F}'(t) = \langle 2t, -\sin(t), e^t \rangle$$

$$\vec{F}''(t) = \frac{d}{dt}\left(\vec{F}'(t)\right) = \langle 2, -\cos(t), e^t \rangle$$

Rules of differentiation

$$\left(a\vec{F}(t) + b\vec{G}(t)\right)' = a\vec{F}'(t) + b\vec{G}'(t) \longrightarrow \text{Linearily Rule}$$

→ $$\left(g(t)\vec{F}(t)\right)' = g'(t)\vec{F}(t) + g(t)\vec{F}'(t)$$

$$\left(\frac{\vec{F}(t)}{g(t)}\right)' = \frac{\vec{F}'(t)\,g(t) - \vec{F}(t)\,g'(t)}{(g(t))^2} \qquad \text{where } g(t) \neq 0$$

→ $$\left(\vec{F}(t) \cdot G(t)\right)' = \vec{F}' \cdot \vec{G} + \vec{F} \cdot \vec{G}'$$

→ $$\left(\vec{F}(t) \times G(t)\right)' = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$$

$$\left(\vec{F}(g(t))\right)' = g'(t)\,\frac{d\vec{F}}{dy}\,(g(t))$$

Ex. Let $\vec{F}(t) = \langle 1, t, t^2 \rangle$ and $\vec{G}(t) = \langle t, e^t, 3 \rangle$. Verify that $(\vec{F} \times \vec{G})' = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$

$$\vec{F} \times \vec{G} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & t^2 \\ t & e^t & 3 \end{vmatrix} = \hat{i}(3t - t^2e^t) - \hat{j}(3 - t^3) + \hat{k}(e^t - t^2) = H \text{ (a vector function)}$$

$$\vec{H}'(t) = \langle 3 - 2te^t - t^2e^t, 0 + 3t^2, e^t - 2t \rangle$$

Alternatively...

$$\vec{F}' = \langle 0, 1, 2t \rangle \quad \vec{G}' = \langle 1, e^t, 0 \rangle$$

$$\vec{F}' \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2t \\ t & e & 3 \end{vmatrix} = \hat{i}(3 - 2te^t) - \hat{j}(0 - 2t^2) + \hat{k}(0 - t)$$

take these two vectors and add them up to see if they equal $\vec{H}'$

$$\vec{F} \times \vec{G}' \begin{vmatrix} \hat{i} & \hat{j} & K \\ 1 & t & t^2 \\ 1 & e^t & 0 \end{vmatrix} = \hat{i}(0 - t^2e^t) - \hat{j}(0 - t^2) + \hat{k}(e^t - t)$$

$$(\vec{F}' \times \vec{G}) + (\vec{F} \times \vec{G}') = \vec{H}'$$

So you can choose which method to use.

Theorem. Let $\vec{F}(t)$ be a smooth function and $\|\vec{F}(t)\| = \text{const}$, then $\vec{F} \perp \vec{F}'$ for all t in D.

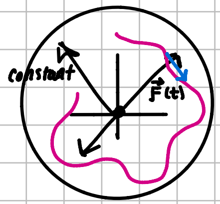
Proof.

$$\|\vec{F}(t)\| = \sqrt{f_1^2 + f_2^2 + f_3^2} = \text{constant}$$

$$f_1^2 + f_2^2 + f_3^2 = \text{constant}^2$$

$$\langle f_1, f_2, f_3 \rangle \cdot \langle f_1, f_2, f_3 \rangle = \text{constant}^2$$

$$(\vec{F} \cdot \vec{F})' = (\text{constant}^2)'$$



$$\vec{F}' \cdot \vec{F} + \vec{F} \cdot \vec{F}' = 0$$

$$2\vec{F}' \cdot \vec{F} = 0 \Rightarrow \underline{\underline{\vec{F}' + \vec{F}(t)}}$$

Modeling trajectories in space.

DEF. Let $\vec{R}(t)$ describe the trajectory of an object in space as function of t.

Then

i) $\vec{V}(t) = \dfrac{d\vec{R}(t)}{dt}$ is the velocity

ii) $\|\vec{v}(t)\|$ is the speed

iii) $\vec{A}(t) = \dfrac{d\vec{V}(t)}{dt}$ is the acceleration

always positive

Ex. Find velocity, speed and acceleration for $\vec{R}(t) = \langle \cos t, \sin t, t^3 \rangle$

$$\vec{V} = \frac{d\vec{R}}{dt} = \langle -\sin(t), \cos(t), 3t^2 \rangle$$

$$\|\vec{V}\| = \sqrt{(-\sin(t))^2 + (\cos t)^2 + (3t^2)} = \sqrt{1 + 9t^4}$$

$$\vec{A} = \frac{d\vec{v}}{dt} = \langle -\cos t, -\sin t, 6t \rangle$$

Vector integral. The indefinite integral of the vector Function $\vec{F}(t) = \langle F_1(t), F_2(t), F_3(t) \rangle$ is given by

$$\int \vec{F}(t) = \left\langle \overset{C_1}{\int F_1(t)\,dt}, \overset{C_2}{\int F_2(t)\,dt}, \overset{C_3}{\int F_3\,dt} \right\rangle + \underline{\underline{\vec{C}}}$$

The definite integral is given by

$$\int_a^b \vec{F}(t)\,dt = \left\langle \int_a^b F_1(t)\,dt, \int_a^b F_2(t)\,dt, \int_a^b F_3(t)\,dt \right\rangle$$

Ex. Find $\int_0^\pi \langle t, 3, -\sin t \rangle\,dt = \left\langle \frac{t^2}{2}, 3t, \cos(t) \right\rangle \Big|_0^\pi = \left\langle \frac{\pi^2}{2}, 3\pi, \overset{-1}{\cos(\pi)} \right\rangle - \left\langle 0, 0, \overset{1}{\cos(0)} \right\rangle$

$= \boxed{\left\langle \frac{\pi^2}{2}, 3\pi, -2 \right\rangle}$

Ex. The velocity of an object is $\vec{v}(t) = \langle e^t, t^2, \cos(2t) \rangle$. Find its trajectory $\vec{R}(t)$ if $\vec{R}(0) = \langle 2, 1, -1 \rangle$.

$$\vec{V}(t) = \frac{d\vec{R}}{dt} \Rightarrow \vec{R}(t) = \underbrace{\int \vec{v}(t)\,dt + \underline{\underline{\vec{C}}}}_{\text{indefinite integral}}$$

$\vec{R}(t) = \left\langle e^t, \frac{t^3}{3}, \frac{\sin(2t)}{2} \right\rangle + \vec{C}$

↳ use $\vec{R}(0)$ to find this

$\vec{R}(0) = \langle 2, 1, -1 \rangle = \left\langle e^0, \frac{0^3}{3}, \frac{\sin(0)}{2} \right\rangle + \vec{C} = \langle 1, 0, 0 \rangle + \vec{C}$

$\vec{C} = \langle 2, 1, -1 \rangle - \langle 1, 0, 0 \rangle = \langle 1, 1, -1 \rangle$

$$\boxed{\vec{R}(t) = \left\langle e^t, \frac{t^3}{3}, \frac{\sin(2t)}{2} \right\rangle + \langle 1, 1, -1 \rangle}$$

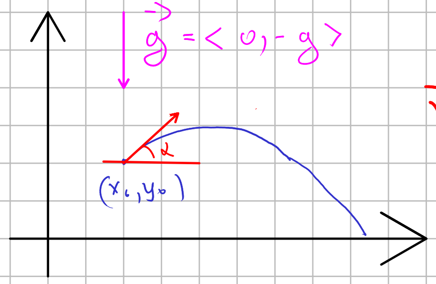
Ex (10.3) Motion of a projectile in a vacum.

(x_0, y_0) initial position

V_0 speed

α inclination

$\vec{g}$ gravity



initial velocity

$\vec{V}_0 = \langle V_0 \cos\alpha, V_0 \sin\alpha \rangle$

$m\vec{a} = \vec{F} = m\vec{g}$ ← Newton's law

$\vec{a} = \vec{g} = \langle 0, -g \rangle$ $\qquad \vec{a} = \frac{d\vec{v}}{dt}$

$$\vec{V} = \int \vec{a}\,dt + \vec{C} = \int \langle 0, -g \rangle\, dt + \vec{C} = \langle 0, -gt \rangle + \vec{C} = \langle 0, -gt \rangle + \langle V_0 \cos\alpha, V_0 \sin\alpha \rangle$$

$$\vec{V}(0) = \langle V_0 \cos\alpha, V_0 \sin\alpha \rangle = \langle 0, -g(0) \rangle + \vec{C} = \vec{C}$$

$$\vec{V}(t) = \langle V_0 \cos\alpha, V_0 \sin\alpha - gt \rangle$$

$$\vec{r}(t) = \int \vec{V}(t) + \vec{D} = \langle V_0 \cos\alpha, V_0 \sin\alpha\, t - \frac{gt^2}{2} \rangle + \vec{D}$$

$$\vec{r}(0) = \langle x_0, y_0 \rangle = \langle 0, 0 \rangle + \vec{D} \qquad \vec{D} = \langle x_0, y_0 \rangle$$

$$\boxed{\vec{r}(t) = \langle x_0 + V_0 \cos\alpha\, t,\ y_0 + V_0 \sin\alpha\, t - \frac{1}{2} g t^2 \rangle}$$