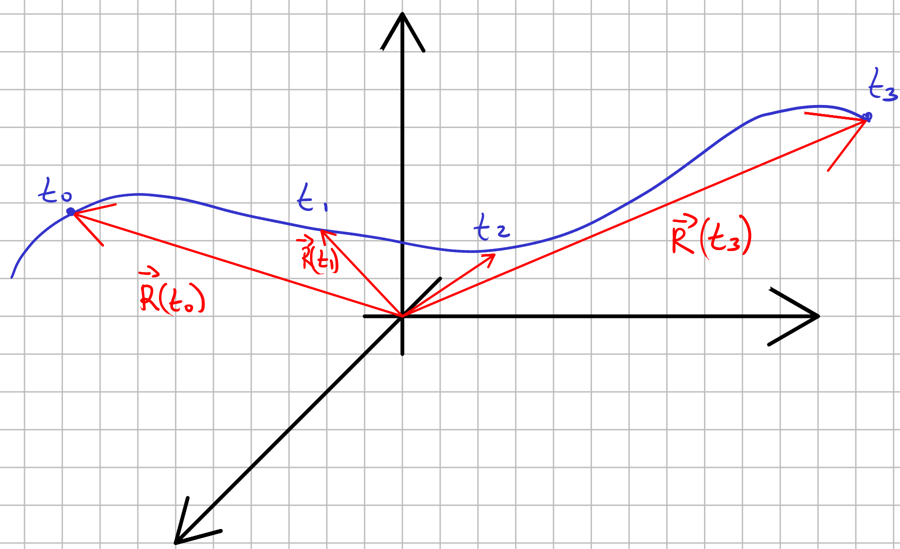
10.1 VECTOR VALUE FUNCTIONS: $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle \quad t \in D$

DEF. A vector value function is a rule that associates for any t in the domain D a unique vector $\vec{R}(t)$ in the range R.



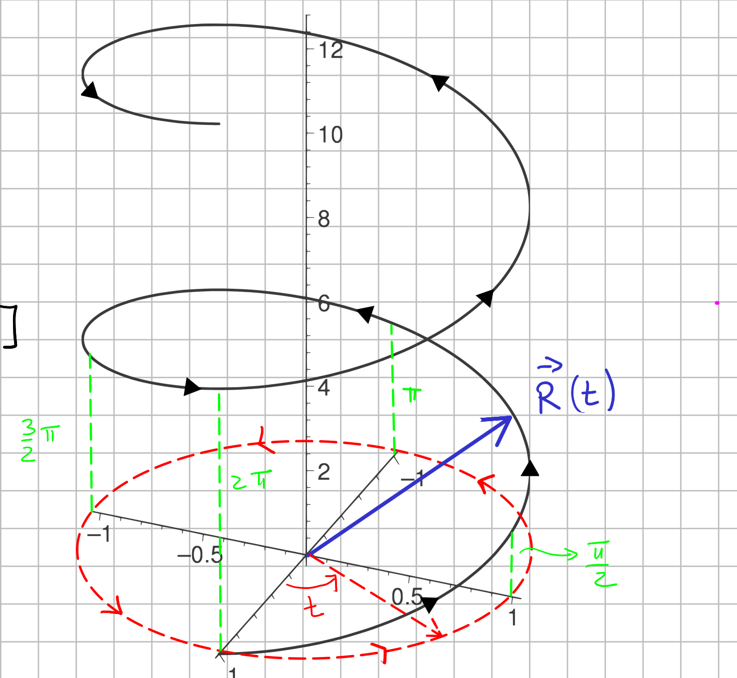
Duality between vector functions and parametric equations

$\vec{R}(t) = \langle f(t), g(t), h(t) \rangle \quad t \in D$

$\Updownarrow$

$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \\ z(t) = h(t) \end{cases} \leftarrow \text{Graph described by the vector function}$$

Ex. Helix $\vec{R}(t) = \langle \cos t, \sin t, t \rangle \quad t \in [0, 4\pi]$



If the domain D is not given we take the largest set for which each component is defined.

Ex. Find the domain of $\vec{R}(t) = \langle \ln t, \frac{1}{t^2-1}, \tan t \rangle$

D_1: $\ln t$ is defined on $t > 0$

D_2: $\frac{1}{t^2-1}$ is defined for $t \neq \pm 1$

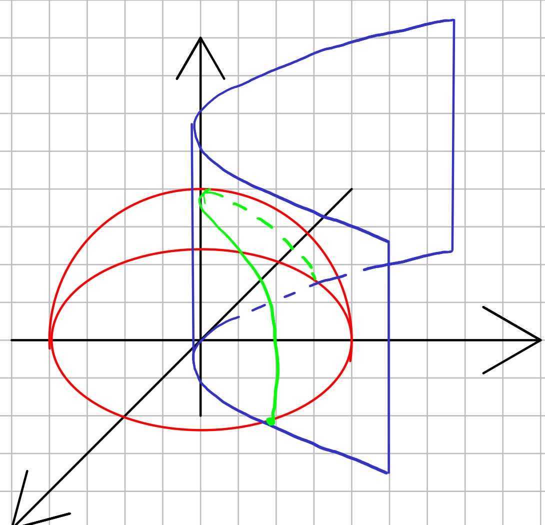
D_3: $\tan t$ is defined for $t \neq \frac{\pi}{2} \pm k\pi$, for k in $\mathbb{N}$

$\Rightarrow$ $D = D_1 \cap D_2 \cap D_3$:

$\left\{ \underline{t > 0} \setminus \{ \underline{t = 1 \cup t = \frac{\pi}{2} + k\pi \text{ for } k \in \mathbb{N}} \} \right\}$

Ex. Find a vector function that describes the intersection between

$z = \sqrt{4 - x^2 - y^2}$ and $y = x^2$



$$\begin{cases} x(t) = t \\ y(t) = x^2(t) = t^2 \\ z(t) = \sqrt{4 - x^2(t) - y^2(t)} = \sqrt{4 - t^2 - t^4} \end{cases}$$

$\vec{R}(t) = \langle t, t^2, \sqrt{4 - t^2 - t^4} \rangle$ for $4 - t^2 - t^4 \geq 0$

Algebra. We use the same algebra we used for vectors:

DEF.

Let $\vec{F}$ and $\vec{G}$ be vector functions and $f(t)$ be a regular function.

$$(f\vec{F})(t) = f(t)\,\vec{F}(t)$$

$$(\vec{F} \pm \vec{G})(t) = \vec{F}(t) \pm \vec{G}(t)$$

$$(\vec{F} \cdot \vec{G})(t) = \vec{F}(t) \cdot \vec{G}(t)$$

$$(\vec{F} \times \vec{G})(t) = \vec{F}(t) \times \vec{G}(t)$$

All this operations are defined on the intersection of the domains of the functions that compose the operation.

Ex. $\vec{F}(t) = \langle t^2, t, -\sin t\rangle$ $\quad \vec{G}(t) = \langle t, \frac{1}{t}, 5\rangle$. Find

$(\vec{F}+\vec{G})(t)$ $\quad (e^t\vec{F})(t)$ $\quad (\vec{F}\cdot\vec{G})(t)$ $\quad (\vec{F}\times\vec{G})(t)$

$$(\vec{F}+\vec{G})(t) = \langle t^2+t,\ t+\frac{1}{t},\ -\sin t+5\rangle \quad D_1 \cap D_2 \text{ for } t \neq 0$$

$$e^t\langle t^2, t, -\sin t\rangle = \langle e^t t^2, e^t t, e^t(-\sin t)\rangle \text{ for all } t \ (-\infty, \infty)$$

$$\boxed{(\vec{F}\cdot\vec{G})(t)} = t^2(t) + t\left(\frac{1}{t}\right) + (-\sin t)5 = \boxed{\underbrace{t^3+1-5\sin t}} \text{ for } t \neq 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & t & -\sin t \\ t & \frac{1}{t} & 5 \end{vmatrix} = \hat{i}\left(5t - (-\sin t)\frac{1}{t}\right) - \hat{j}\left(5t^2 - (-\sin t)\right) + \hat{k}\left(t^2\left(\frac{1}{t}\right) - t(t)\right)$$

$$= \langle 5t + \frac{\sin t}{t},\ -5t^2 - t\sin t,\ t - t^2\rangle \quad D_1 \cap D_2 : t \neq 0$$

Limit : $\lim_{t \to t_0} \vec{F}(t) = \lim_{t \to t_0} \langle F_1(t), F_2(t), F_3(t) \rangle = \langle \lim_{t \to t_0} F_1(t), \lim_{t \to t_0} F_2(t), \lim_{t \to t_0} F_3(t_0) \rangle$

The limit of the vector function exists if and only if the limit of each component exists.

Ex. Find the $\lim_{t \to 0} \left\langle e^t(1+t^2), \frac{1-\cos t}{t^2}, \frac{3t^4 - 4t^3}{7t^3 - t^2} \right\rangle = \langle L_1, L_2, L_3 \rangle = \langle 1, 1/2, 0 \rangle$

$$\lim_{t \to 0} e^t(1+t^2) = L_1 = e^0(1+0^2) = 1(1) = \boxed{1}$$

$$\lim_{t \to 0} \frac{1-\cos(t)}{t^2} = L_2 = \frac{1-1}{0^2} = \frac{0}{0}$$ undeterminate form

this answer is wrong, more work is needed

L'hospital's Rule:

$$\lim_{t \to 0} \frac{\sin t}{2t} = \frac{0}{0} \qquad \lim_{t \to 0} \frac{\cos t}{2} = \boxed{\frac{1}{2}}$$

$$\lim_{t \to 0} \frac{3t^4 - 4t^3}{7t^3 - t^2} = L_3 = \frac{0}{0} = \lim_{t \to 0} \frac{\cancel{t^2}(3t^2 - 4t)}{\cancel{t^2}(7t - 1)} = \frac{0}{-1} = \boxed{0}$$

Algebra of the limits

THEOREM

i) $$\lim_{t\to t_0}\left(\vec{F}(t)\pm\vec{G}(t)\right)=\lim_{t\to t_0}\vec{F}(t)\pm\lim_{t\to t_0}\vec{G}(t)$$

ii) $$\lim_{t\to t_0}\left(g(t)\,\vec{F}(t)\right)=\mathop{\text{limit}}_{t\to t_0} g(t)\ \mathop{\text{limit}}_{t\to t_0}\vec{F}(t)$$

iii) $$\lim_{t\to t_0}\vec{F}(t)\cdot\vec{G}(t)=\mathop{\text{limit}}_{t\to t_0}\vec{F}(t)\cdot\mathop{\text{limit}}_{t\to t_0}\vec{G}(t)$$

iv) $$\lim_{t\to t_0}\vec{F}(t)\times\vec{G}(t)=\mathop{\text{limit}}_{t\to t_0}\vec{F}(t)\times\mathop{\text{limit}}_{t\to t_0}\vec{G}(t)$$

Proof iii) (for the dot product)

$$\lim_{t\to t_0}\left(\vec{F}(t)\cdot\vec{G}(t)\right)=\lim_{t\to t_0}\left(f_1g_1+f_2g_2+f_3g_3\right)(t)=\lim_{t\to t_0}f_1\lim_{t\to t_0}g_1+\lim_{t\to t_0}f_2\lim_{t\to t_0}g_2+\lim_{t\to t_0}f_3\lim_{t\to t_0}g_3$$

$$=\left\langle\lim_{t\to t_0}f_1,\ \lim_{t\to t_0}f_2,\ \lim_{t\to t_0}f_3\right\rangle\cdot\left\langle\lim_{t\to t_0}g_1,\ \lim_{t\to t_0}g_2,\ \lim_{t\to t_0}g_3\right\rangle$$

$$\boxed{=\lim_{t\to t_0}\vec{F}(t)\cdot\lim_{t\to t_0}\vec{G}(t)}$$

Show that $\lim\limits_{t \to 1} \vec{F}(t) \times \vec{G}(t) = \lim\limits_{t \to 1} \vec{F}(t) \times \lim\limits_{t \to 1} \vec{G}(t)$ with

$$\vec{F}(t) = \langle t,\ 1-t,\ t^2 \rangle \qquad \vec{G}(t) = \langle e^t,\ 0,\ -3-e^t \rangle$$

$$L_1 = \langle 1, 0, 1 \rangle \qquad L_2 = \langle e^1, 0, -3-e^1 \rangle = \langle e, 0, -3-e \rangle$$

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & 1-t & t^2 \\ e^t & 0 & -3-e^t \end{vmatrix} = \hat{i}\left((1-t)-(-3-e^t)\right) - \hat{j}\left(t(3-e^t) - t^2 e^t\right) + \hat{k}\left(0-(1-t)e^t\right)$$

$$\lim_{t \to 1} \vec{F} \times \vec{G} = \langle 0, -(-3-e-e), 0 \rangle = \boxed{\langle 0, 3+2e, 0 \rangle}$$

Alternatively, you could use this strategy:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ e & 0 & -3-e \end{vmatrix} = \hat{i}(0) - \hat{j}\left((-3-e)-e\right) + \hat{k}(0) = \boxed{\langle 0, 3+2e, 0 \rangle}$$

Def. A vector function $\vec{F}$ is continuous at t_0 if t_0 is in the domain of $\vec{F}$ and $\lim_{t \to t_0} \vec{F}(t) = \vec{F}(t_0)$

A vector function is continuous at t_0 if and only if each component is continuous at t_0.

Ex. Find where the function $\vec{F} = \langle \underline{\sin} t, (1 \underline{-} t)^{-1}, \underline{\ln} t \rangle$ is continuous.

$\sin(t)$ D_1: for t

$\dfrac{1}{1-t}$ D_2 $t \neq 1$

$\ln t$ D_3 $t > 0$

"set minus"

$D_1 \cap D_2 \cap D_3 = \{ t > 0 \setminus t = 1 \}$

Is equivalent to:

$\{ t > 0 \text{ with } t \neq 1 \}$