Calc III w/applications

9.1 – 9.4 Vectors in Plane + Space

9.1 Vectors in Space

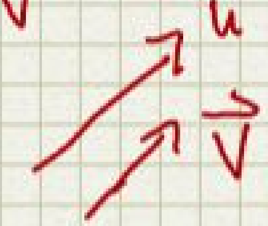
Vectors in $\mathbb{R}^2$

$\vec{v}$ (vectors) have magnitude + diretion
-magnitude of vetor $\|V\|$

$\vec{u} = -\vec{v}$; u + v are parallel, have equal magnitude but opposite directions

$\vec{u} = m\vec{v}$; m is the coefficient

$\vec{u} = 2\vec{v}$ $\quad$ $\vec{u}$ is twice as long as $\vec{v}$

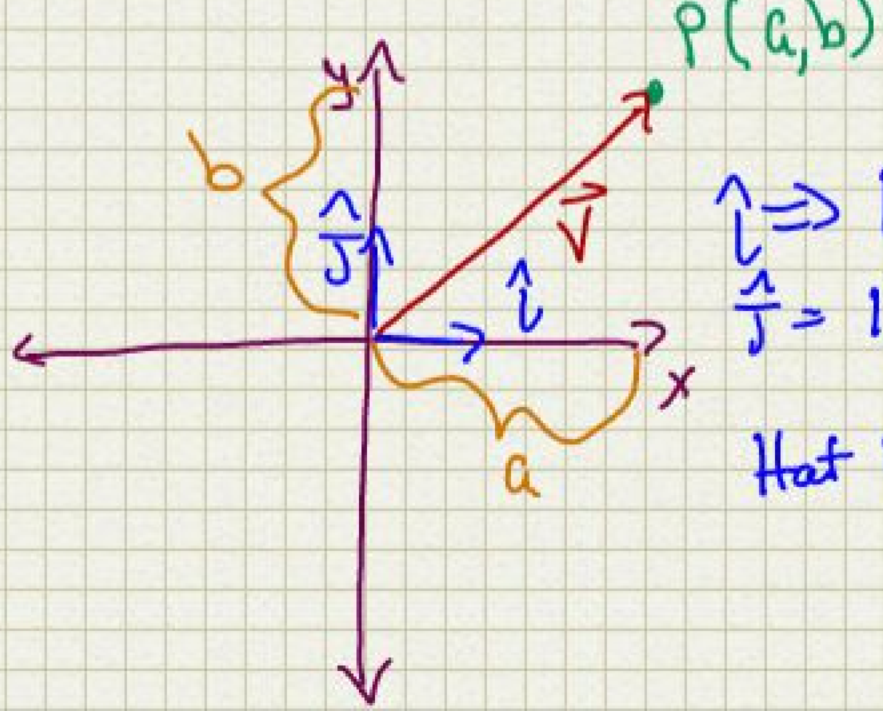


$\vec{u}$ w/same direction of $\vec{v}$; $\vec{v} = 1$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ $\quad$ $\frac{\vec{v}}{\|\vec{v}\|} = 1$

$\vec{z} = \vec{u} - \vec{v}$

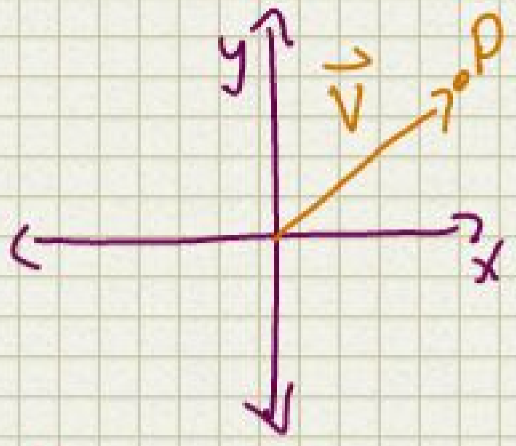
$\vec{u}$: $\quad$ $\vec{v}$: $\quad$ $\vec{u} - \vec{v} = \vec{w}$



$\hat{i} \Rightarrow \|i\| = (1, 0)$

$\hat{j} = \|j\| = (0, 1)$

Hat means magnitude of 1



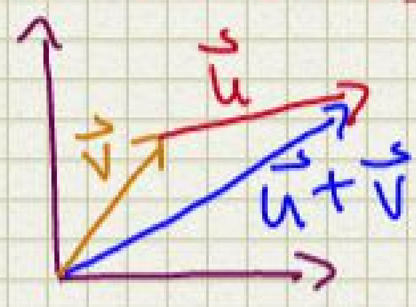
$P(3,2)$ $\quad \vec{v} = 3\hat{i} + 2\hat{j} \neq P \equiv (a,b)$

$\vec{v} = \langle a,b \rangle = a\hat{i} + b\hat{j}$

$\vec{v} = \langle 1,2 \rangle = \hat{i} + 2\hat{j}$

$\vec{u} = \langle 3,1 \rangle = 3\hat{i} + \hat{j}$

$\vec{u} + \vec{v} = (1+3)\hat{i} + (2+1)\hat{j} = \langle 4,3 \rangle$



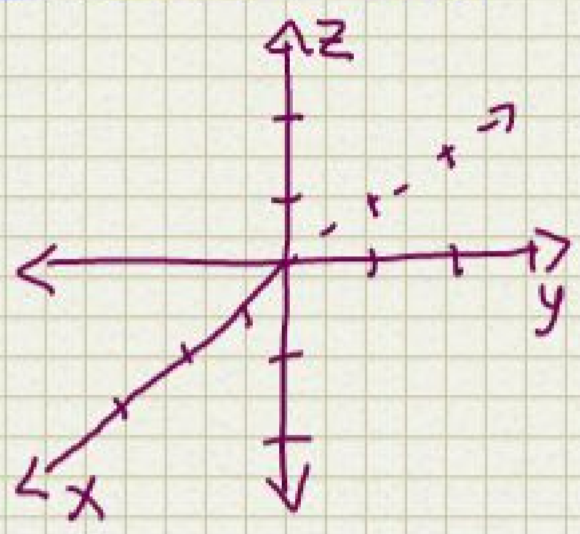
$\|\vec{v}\| = \sqrt{a^2 + b^2}$

$\vec{v} = \langle 1,2 \rangle \quad \|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

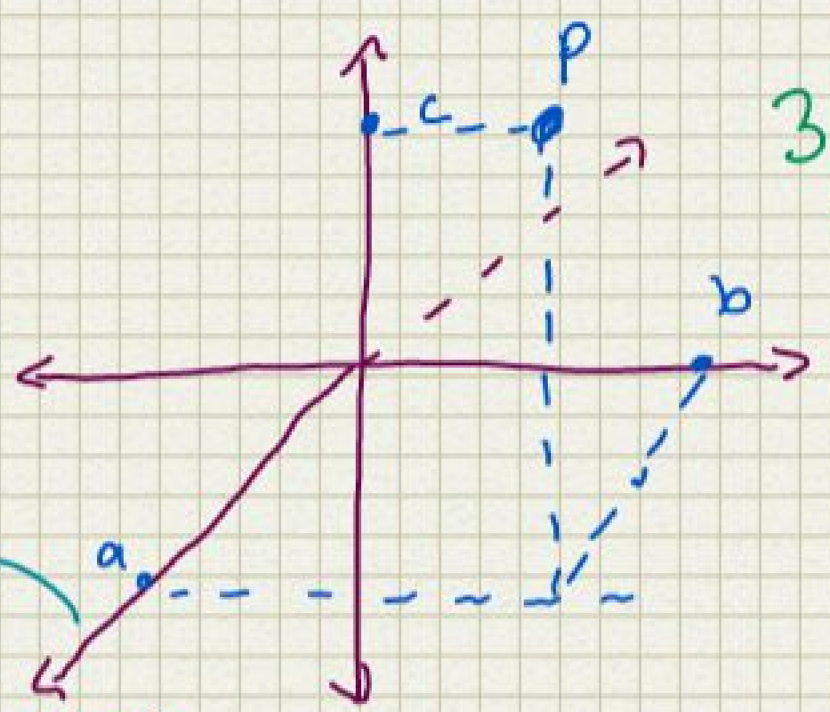
$\vec{u} = \langle 3,1 \rangle$

$\hat{a} \parallel \vec{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 3,1 \rangle}{\sqrt{10}} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle = \frac{3}{\sqrt{10}} i + \frac{1}{\sqrt{10}} j$

Vectors in $\mathbb{R}^3$



$P(a, b, c)$



$R(x_r, \emptyset, \emptyset)$

3 Reference Planes

x-y Plane

y-z Plane

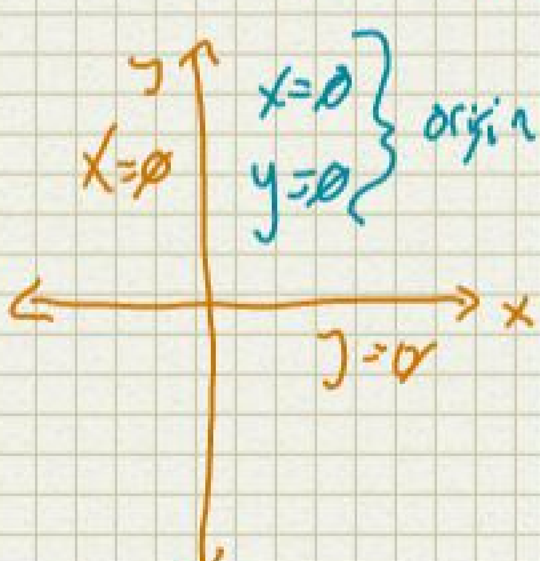
x-z Plane

x-y Plane Equation

$q(x_q, z_q, \emptyset)$

y-z Plane Eq

$r(\emptyset, y_r, z_r)$

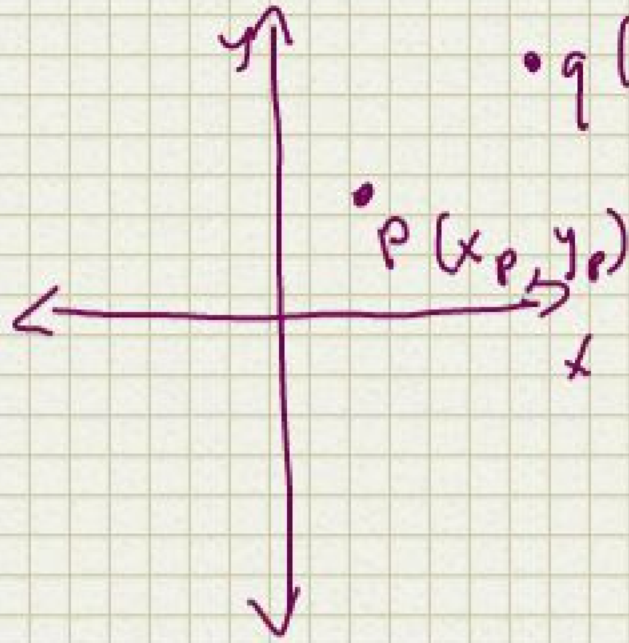


2-D : $x = \emptyset \Rightarrow$ line

3D : $x = \emptyset \Rightarrow$ plane

3D: $\begin{cases} x \text{ axis} \begin{cases} y = \emptyset \\ z = \emptyset \end{cases} \\ y \text{ axis} \begin{cases} x = \emptyset \\ z = \emptyset \end{cases} \\ z\text{-axis} \begin{cases} x = \emptyset \\ y = \emptyset \end{cases} \end{cases}$

3D Origin $\begin{cases} x = \emptyset \\ y = \emptyset \\ z = \emptyset \end{cases}$



$d = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$

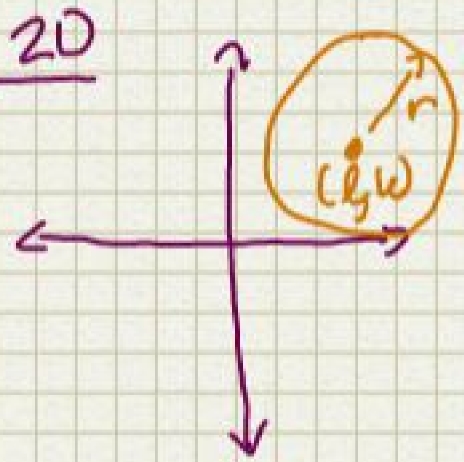
3D Distance: $d = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$

Sphere in 3D

Eq w/center at (l, k, m) & radius r

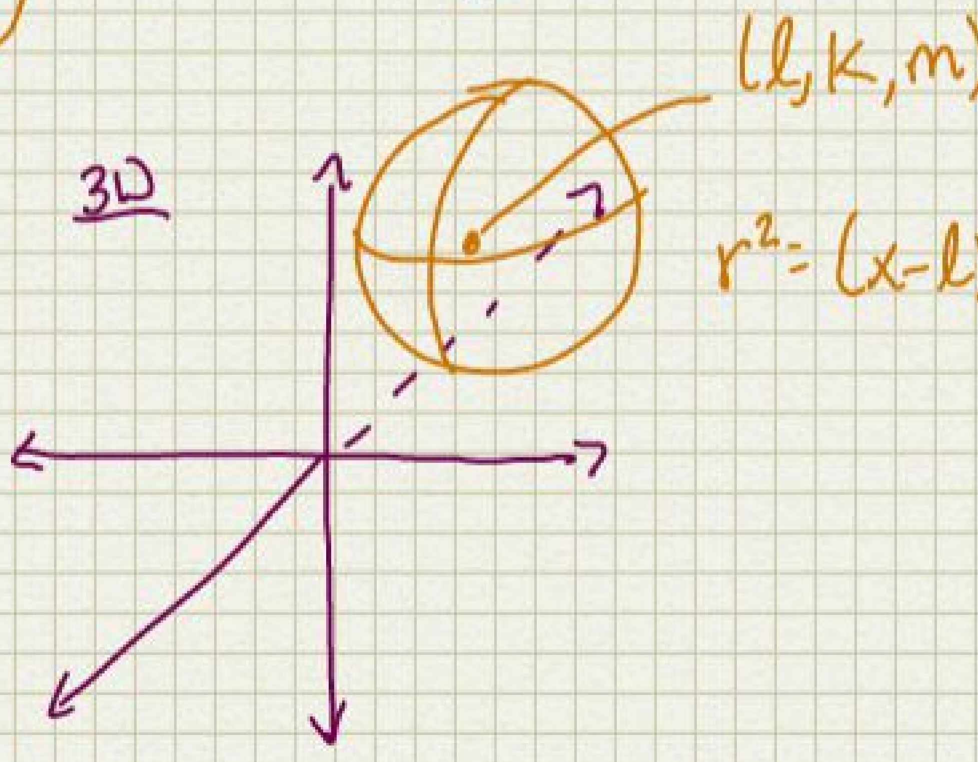
$(x-l)^2 + (y-k)^2 + (z-m)^2 = r^2$

2D



$(x-l)^2 + (y-k)^2 = r^2$

3D



$r^2 = (x-l)^2 + (y-k)^2 + (z-m)^2$

3D Sphere

$$r^2 = (x-l)^2 + (y-k)^2 + (z-m)^2$$

Ex: $x^2 + y^2 + z^2 + 4x - 6y - 3 = 0$

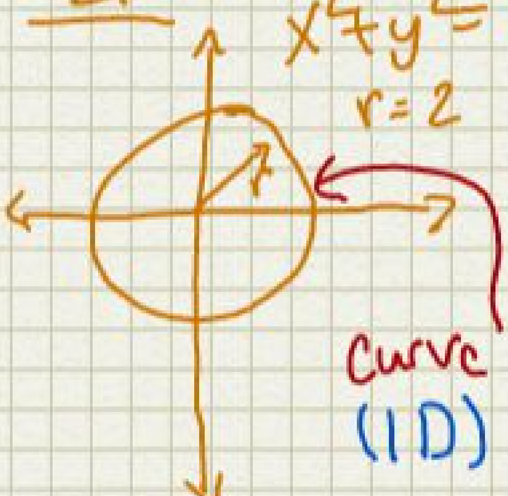
$$(x^2 + 4x) + (y^2 - 6y) + z^2 = 3$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + z^2 = 16$$

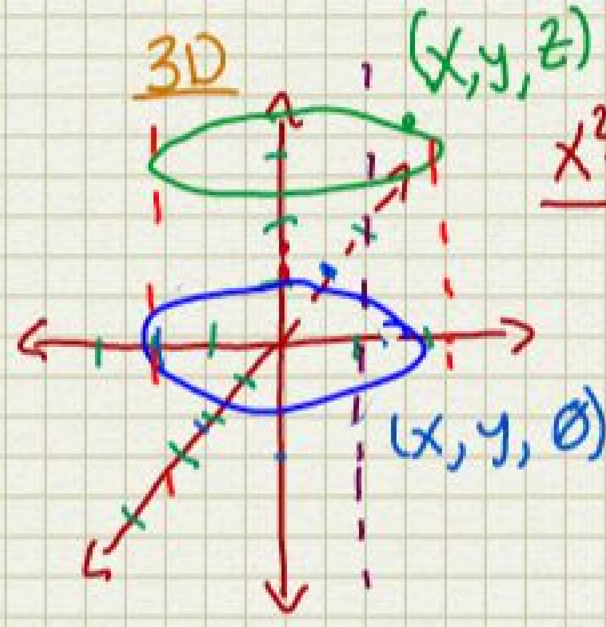
$$(x+2)^2 + (y-3)^2 + z^2 = 16$$

$$\boxed{\begin{array}{l}\text{Center: } (-2, 3, 0) \\ \text{Radius} = 4\end{array}}$$

2D



3D

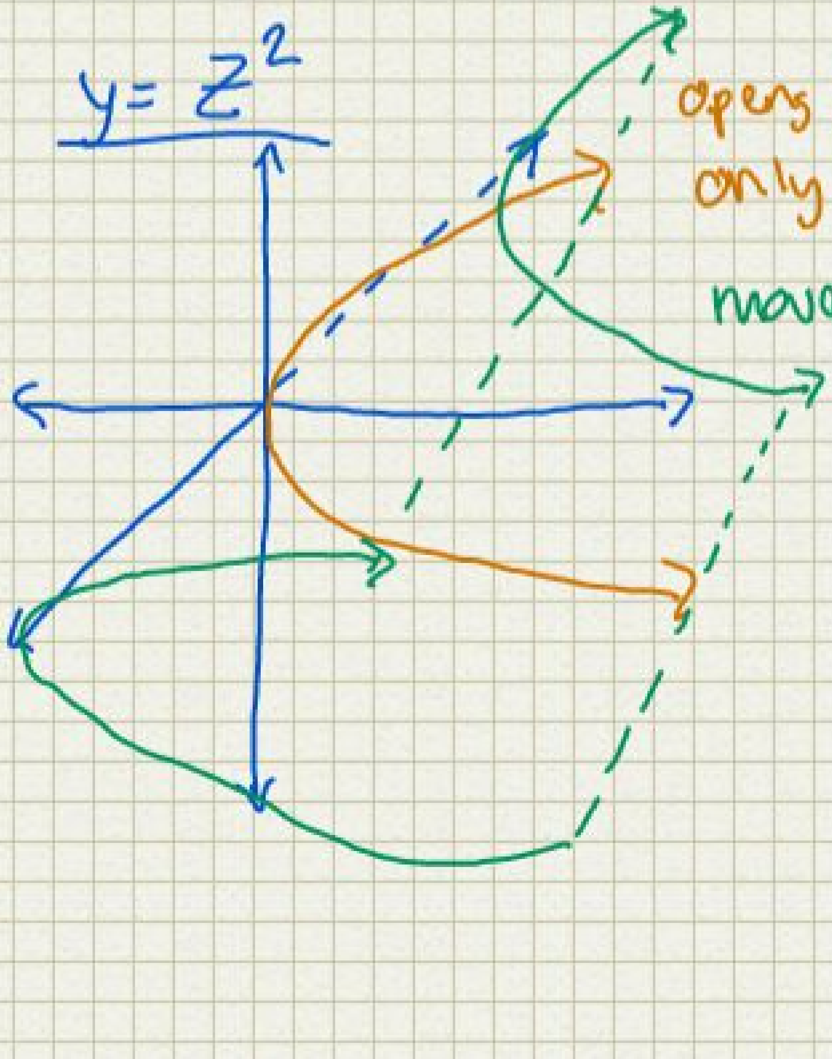


surface (2D)

$x^2+y^2=4$

$z=0$ for all z

$y = z^2$



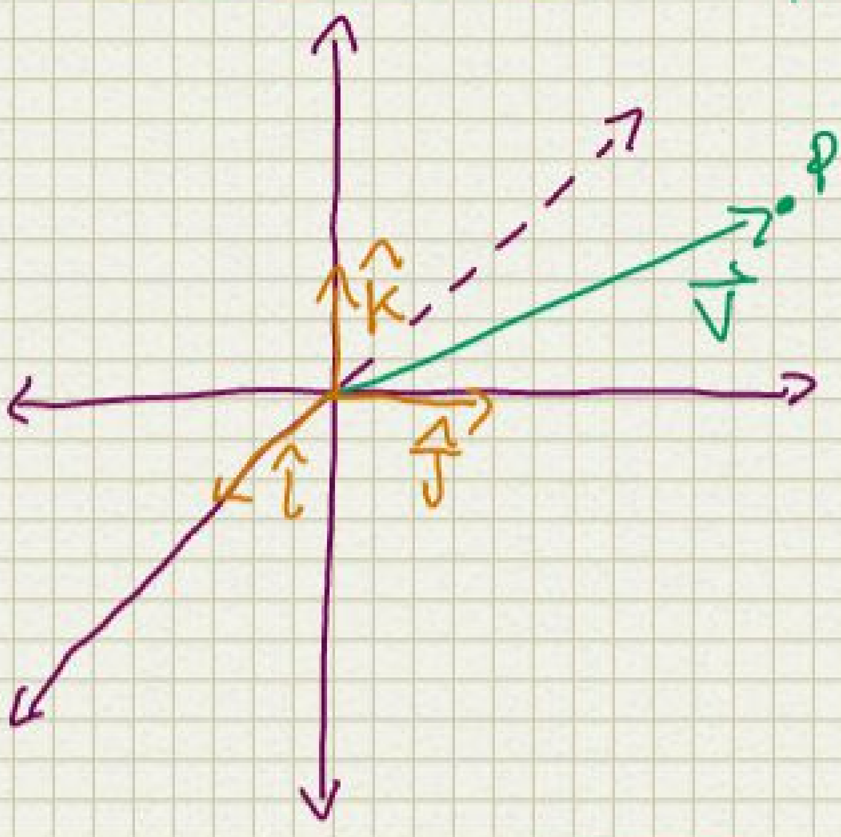
opens toward positive y direction

only on yz plane on $x=0$

move equation parallel to x-axis

Vector in 3D

$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k} = \langle a, b, c \rangle$



$\vec{u} = \langle 2, 3, -2)$

$\vec{v} = \langle 1, 7, 1 \rangle$

$\vec{u} + 2\vec{v} = \langle 2, 3, -2 \rangle + 2\langle 1, 7, 1 \rangle$

$= \langle 2, 3, -2 \rangle + \langle 2, 14, 2 \rangle$

$= \langle 4, 17, 0 \rangle$

$\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{2^2 + 3^2 + (-2)^2}$

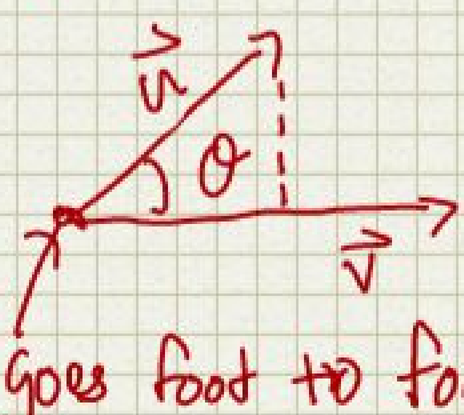
find $\|\vec{a}\|$ w/ same direction as $\vec{u}$

$\vec{a} = \dfrac{\vec{u}}{\|\vec{v}\|} = \dfrac{\langle 2, 3, -2 \rangle}{\sqrt{17}} = \left\langle \dfrac{2}{\sqrt{17}}, \dfrac{3}{\sqrt{17}}, \dfrac{-2}{\sqrt{17}} \right\rangle$

$\|\vec{a}\| = \sqrt{\dfrac{4}{17} + \dfrac{3}{17} + \dfrac{4}{17}} = 1$ $\boxed{\text{Check!}}$

Dot Product (scalar product)

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$ (θ is the smallest angle between the vectors $0 \le \theta \le \pi$)



Goes foot to foot!

* $\|\vec{u}\| \cos(\theta)$ is the projection of $\vec{u}$ on $\vec{v}$

or 

$\|\vec{v}\| \cos(\theta)$ is the projection of $\vec{v}$ on $\vec{u}$

$\vec{u} = \langle u_1, u_2, u_3 \rangle = \langle 2, 3, -2 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle 1, 7, 1 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ * Scalar don't put in brackets

$= 2 + 21 - 2 = 21$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$

- if answer is positive $\theta < 90°$
- if answer is negative $\theta > 90°$

Zero vector?

$\vec{u} \cdot \vec{v} > 0 \Rightarrow \theta < 90°$

$\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta > 90°$

$\vec{u} \cdot \vec{v} = 0 \Rightarrow \theta = 90°$

or if both $\vec{u}, \vec{v} \neq \vec{0}$

Scalar Projection of $\vec{u}$ on $\vec{v}$

- $\|\vec{u}\| \cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$

$\vec{v}$ on $\vec{u}$

- $\|\vec{v}\| \cos(\theta) = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$

Vector Projection

$\vec{u}$ on $\vec{v}$: $\|\vec{u}\| \cos(\theta) \dfrac{\vec{v}}{\|\vec{v}\|} = (u \cdot \vec{v}) \dfrac{\vec{v}}{(\|\vec{v}\|)^2}$

$= \|\vec{v}\| \cos(\theta) \left(\dfrac{\vec{u}}{\|u\|} \right) = \boxed{(\vec{u} \cdot \vec{v}) \dfrac{\vec{u}}{\|u\|^2}}$

Two Vectors Parrallel in Same Direction

$\uparrow\uparrow$ $\theta = 0$

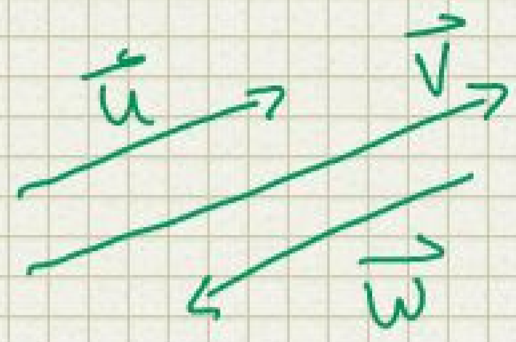
$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\|$

Opposite Dir

$\downarrow\uparrow$ $\theta = \pi$ $\vec{u} \cdot \vec{v} = -\|\vec{u}\| \|\vec{v}\|$

$\theta = 90$

$\vec{u} \cdot \vec{v} = 0$



$\vec{v} = m\vec{u}$

Ex: $\vec{u} = \langle 3, 2, -1 \rangle$ Divide $\vec{v}$ values by $\vec{u}$

$\vec{v} = \langle -6, -4, 2 \rangle$ parrallel if quotients all same

$\vec{v} = -2\langle 3, 2, 1 \rangle$

$m = -2$; oppisite direction + stretched by 2

$\vec{u} \cdot \vec{v} = \vec{v} + \vec{u}$

$u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3$

* only for Scalar!

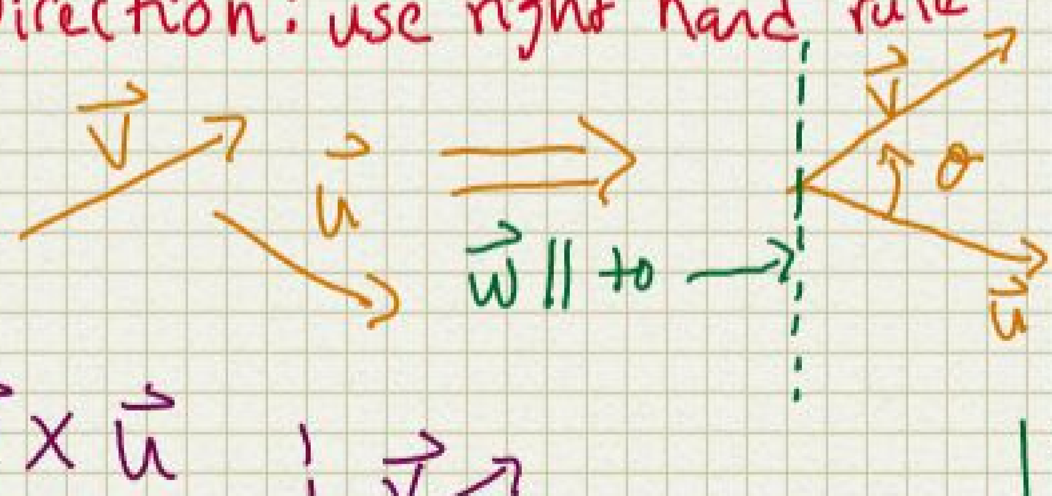
Cross Product (Vector Product)

-only in 3D!

$\vec{u} \cdot \vec{v} = \vec{w}$; $\|\vec{w}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta) \geq 0$

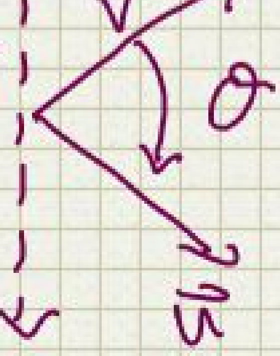
$\vec{w} \perp \vec{u}$ & $\vec{v}$

Direction: use right hand rule



$\vec{v} \times \vec{u}$

$= -\vec{w} = -\vec{u} \times \vec{v}$



$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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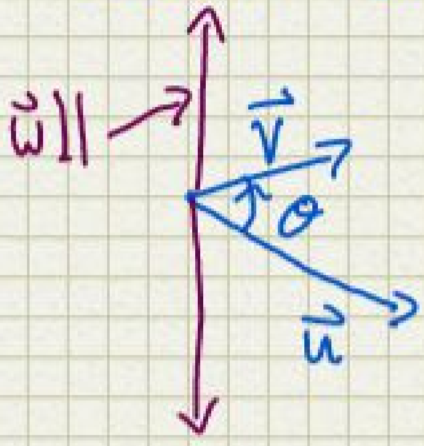
Cross (vector) product

$\vec{u} \times \vec{v} = \vec{w}$ (product is always a vector)



$\|\vec{w}\| = \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$

exception: $\vec{u}, \vec{v} \neq \emptyset$ (Zero Vector)

- the geometric interpretation is lost w/a $\emptyset$ vector. The crossproduct can still be solved.
- $\vec{u} \uparrow\uparrow \vec{v}$ or $\vec{u} \uparrow\downarrow \vec{v} = \vec{\emptyset}$ (when $\theta = \emptyset$ or π) but cross product can still be solved.

$\vec{v} \times \vec{u} = -\vec{w} = -(\vec{u} \times \vec{v})$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\hat{\imath} - (u_1 v_3 - u_3 v_1)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{k}$$

Ex 1.

$\vec{u} = \langle 1, 2, 3 \rangle$

$\vec{v} = \langle -1, 3, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \vec{w}$$

$$= (2-9)\hat{\imath} - (1+3)\hat{\jmath} + (3+2)\hat{k}$$

$$= -7\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$$

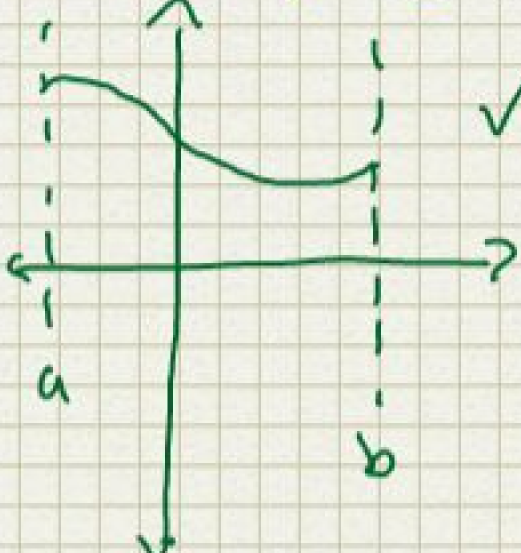
$\vec{u} \cdot \vec{w} = \emptyset$

- because $\vec{u} \perp \vec{w}$, the angle between the vectors in $90°$ or $\pi/2 \Longrightarrow$ by def. of cosine $= \emptyset$

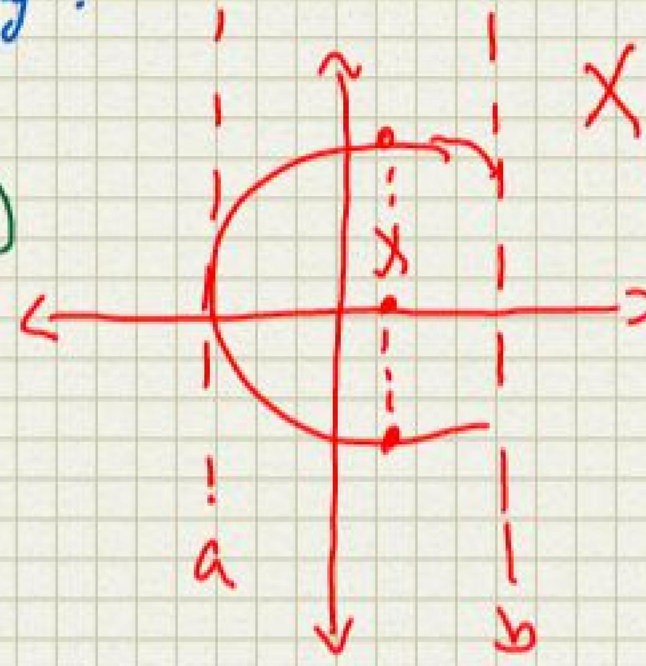
9.5 Parametric Functions

$y = f(x)$

- a function is a rule that associates for any x in the domain, a unique value in the range.



✓ (yes a function)



X (not a function because x can have two values)

$x^2 + y^2 = r^2$ (not function) ← equation

$y = x^2$ (function!)

Parametric Functions in 2D

2D $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ 3 variables

- t - independent variable in domain,
- (x, y) is the dependent variable in the range of function

- A parametric function is a rule that associates for any (t) in the domain in a unique Point (x, y) in the range.

Side Notes

$$\begin{cases} x = 2t \\ y = t^2 \end{cases}$$

* x & y are functions of t

* find domain of functions

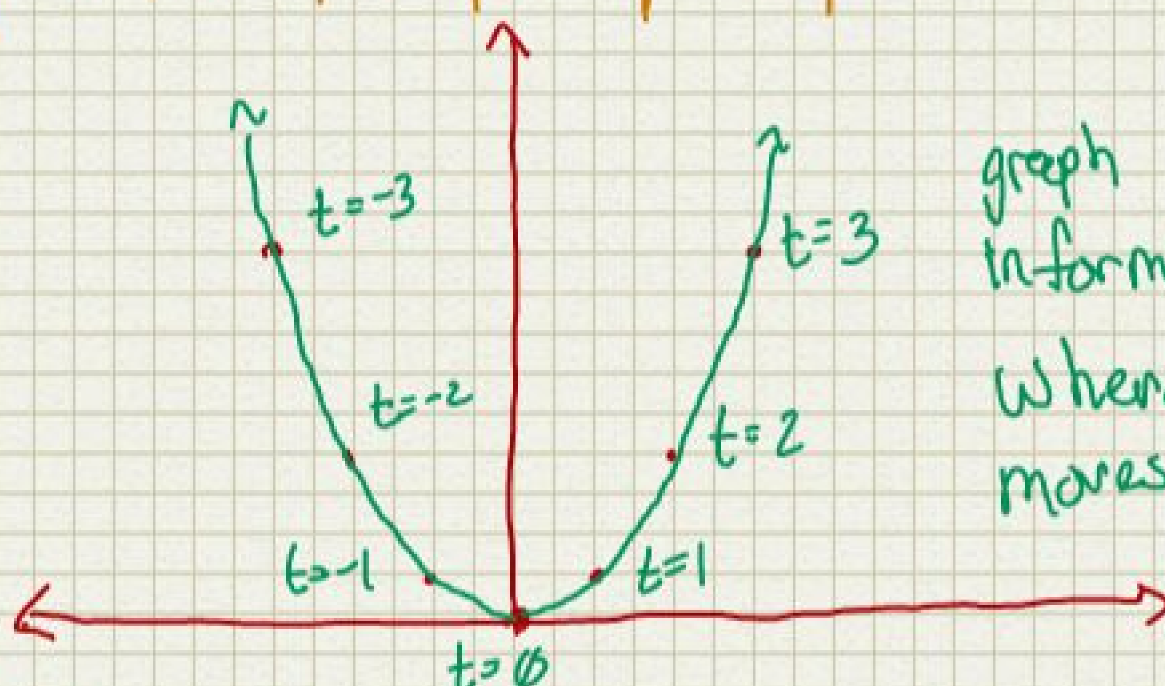
$D_x = ?$ and $D_y = ?$ > the intersection = ?

$D_x = \mathbb{R}$

$D_y = \mathbb{R}$

$\wedge \quad D = \mathbb{R}$

t	-3	-2	-1	Ø	1	2	3
x	-6	-4	-2	Ø	2	4	6
y	9	4	1	Ø	1	4	9



graph has extra information: tells where an object moves according to time

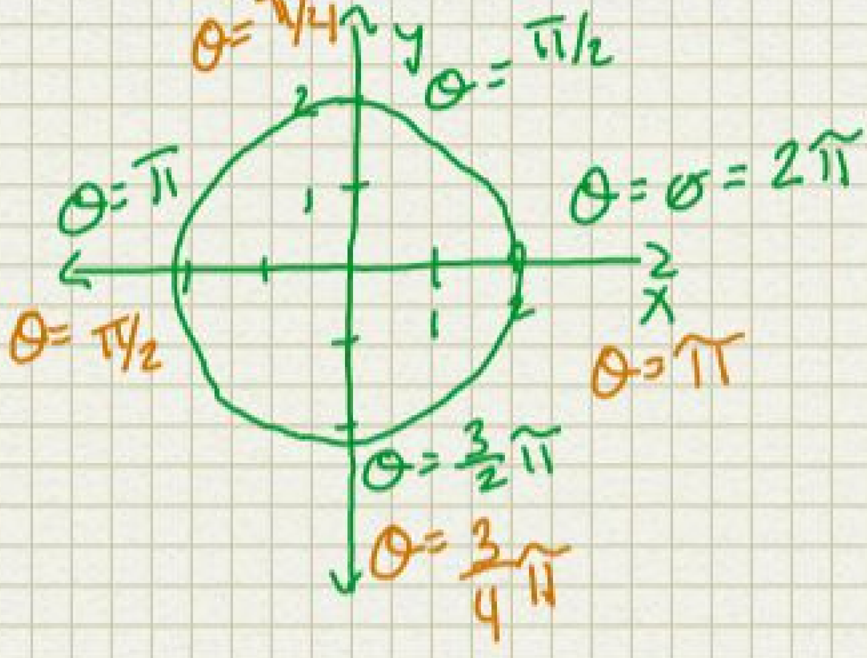
$y = \frac{x^2}{4}$: just an equation, gives an idea how the object moves w/no respect to time. just a line

$$\begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) \end{cases}$$

* Parameter is (θ)
circle

$$\begin{cases} x^2 = 4\cos^2(\theta) \\ y^2 = 4\sin^2(\theta) \end{cases}$$

$$x^2 + y^2 = 4\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{=1}$$
$$= 4$$



$$\begin{cases} x = 2\cos(2\theta) \\ y = 2\sin(2\theta) \end{cases}$$

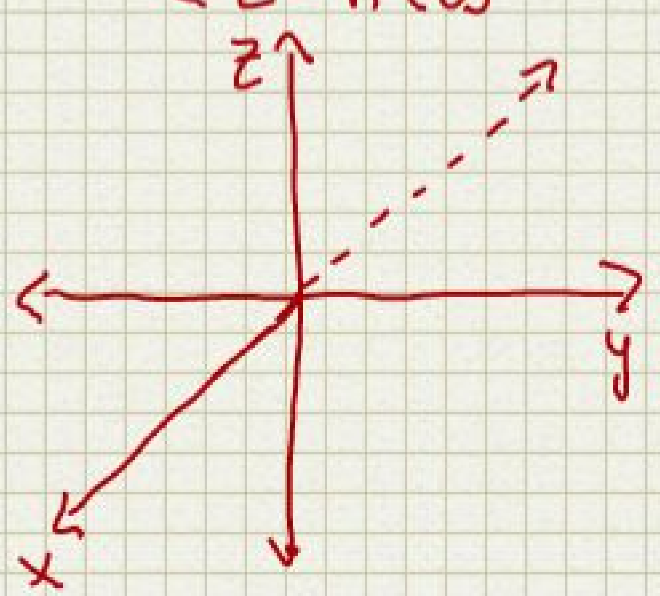
* angle moves faster

$$\begin{cases} x = 2\cos(2\theta) \\ y = 2\sin(2\theta) \end{cases} \neq \begin{cases} x = 2\cos(\theta) \\ y = 2\sin(\theta) \end{cases}$$

In 3D

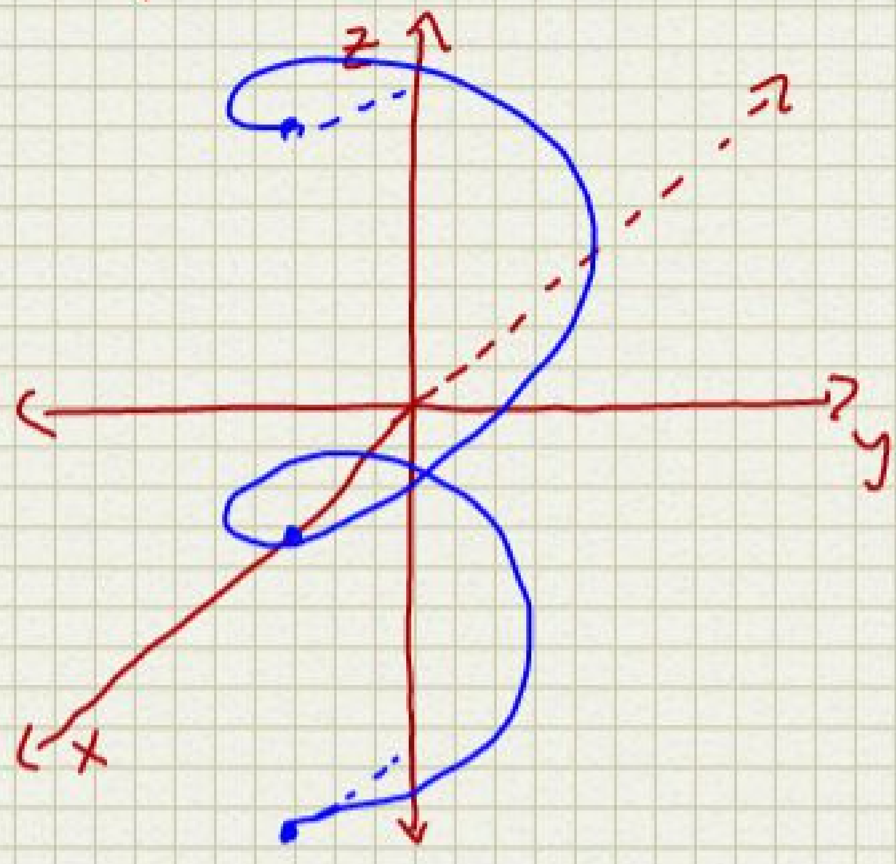
$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$

t is independent variable in domain
(x, y, z) are dependent in Range.



$$\begin{cases} x=\cos(\theta) \\ y=\sin(\theta) \\ z=2(\theta) \end{cases}$$

* Parametric in 3D (defined for any θ)

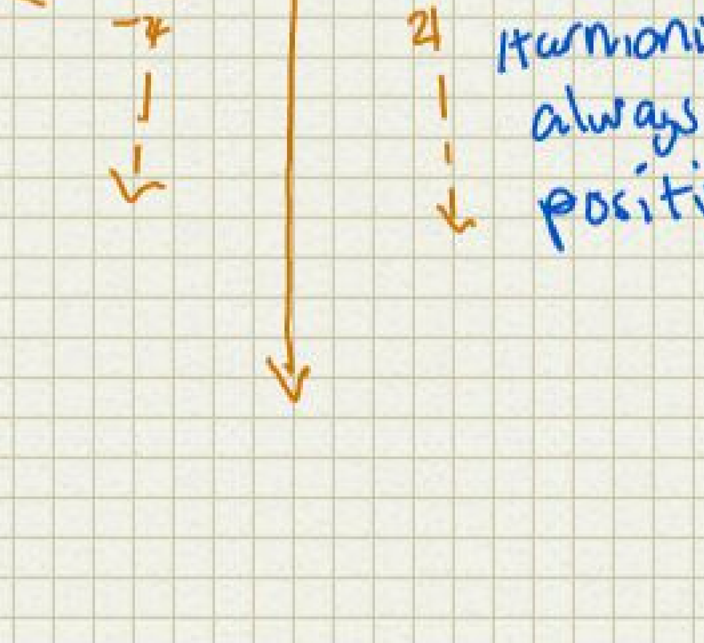


Ex. 2

$$\begin{cases} x=2\sin(\theta) \\ y=4\cos^2(\theta) \end{cases} \Longrightarrow \begin{cases} x^2=4\sin^2(\theta) \\ y=4\cos^2(\theta) \end{cases}$$

$$x^2+y=4(\sin^2(\theta)+\cos^2(\theta))$$

$$y=4-x^2$$

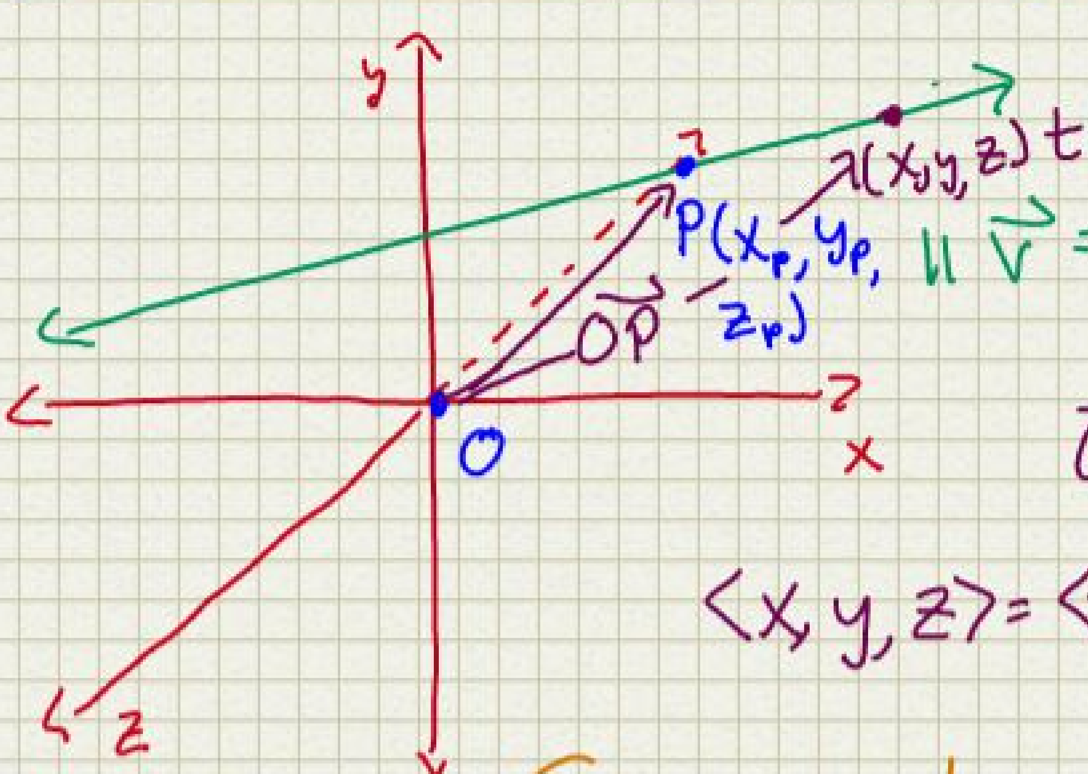


Harmonic always stays positive

$y=4-x^2$

* not parametric function

Parametric Function of a Straight Line in 3D



$\parallel \vec{v} = \langle v_1, v_2, v_3 \rangle$

$\overrightarrow{OP} = \langle x, y, z \rangle$

$\langle x, y, z \rangle = \langle x_p, y_p, z_p \rangle + t \langle v_1, v_2, v_3 \rangle$

$$\begin{cases} x = x_p + t v_1 \\ y = y_p + t v_2 \\ z = z_p + t v_3 \end{cases}$$

standard representation of a line passing through $P(x_p, y_p, z_p) \parallel$ to $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$P(1, 2, 3)$
$\vec{v} = \langle 2, 1, -2 \rangle \Longrightarrow \begin{cases} x = 1 + 2t \\ y = 2 + t \\ z = 3 - 2t \end{cases}$

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$$\begin{cases} \dfrac{x - x_p}{V_1} = \dfrac{y - y_p}{V_2} \\ \dfrac{y - y_p}{V_2} = \dfrac{z - z_p}{V_3} \end{cases}$$

$P(2,1,-1) \parallel \vec{v} = \langle 1,2,3 \rangle$

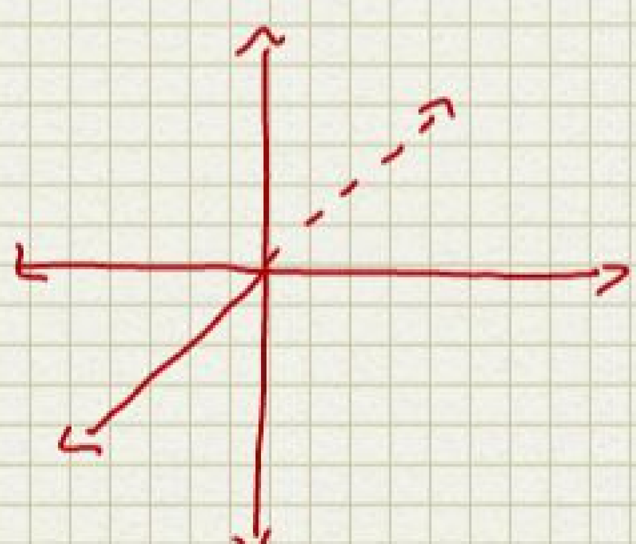
$$\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-(-1)}{3}$$

$$\begin{cases} 2x-4 = y-1 \\ 3y-3 = 2z+2 \end{cases} \quad \begin{cases} 2x-y-3 = \varnothing \\ 3y-2z-5 = \varnothing \end{cases}$$

Find Intersection Between 2 lines

$L_1: \dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-2}{4}$

$L_2: \dfrac{x+2}{4} = \dfrac{y}{-3} = \dfrac{z-\frac{1}{2}}{-1}$



$$L_1 = \begin{cases} x(t) = 1+2t \\ y(t) = -1+t \\ z(t) = 2+4t \end{cases}$$

$$\frac{x-x_p}{V_1} = \frac{y-y_p}{V_2} = \frac{z-z_p}{V_3}$$

$$L_2 \begin{cases} x(s) = -2+4s \\ y(s) = \varnothing - 3s \\ z(s) = \frac{1}{2} - s \end{cases}$$

$x = x_p + V_1 t$

$y = y_p + V_2 t$

$z = z_p + V_3 t$

$$\begin{cases} 1+2t = -2+4s \\ -1+t = -3s \\ 2+4t = \frac{1}{2} - s \end{cases}$$

* in order to find intersection point set $L_1 = L_2$ equations

$$\begin{cases} 2t - 4s = -3 \\ -2\cdot\left(t + 3s = 1\right) \end{cases}$$

$$\emptyset - 10s = -5 \qquad s = 1/2$$

$$t = -1/2$$

$$t + 3\left(\tfrac{1}{2}\right) = 1$$

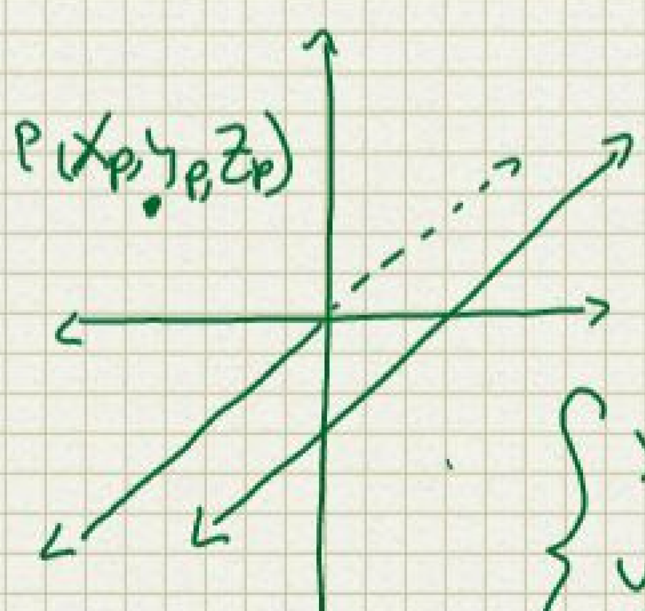
$$t = 1 - \tfrac{3}{2} = -1/2$$

*Plug into third equation

$$2 + 4(-1/2) = \tfrac{1}{2} - \tfrac{1}{2}$$

$\emptyset = \emptyset$ yes, the lines intersect

$$\begin{cases} x(-1/2) = 1 + 2(-1/2) = \emptyset \\ y(-1/2) = -1 - 1/2 = -3/2 \\ z(-1/2) = 2 + 4(-1/2) = \emptyset \end{cases}$$



same

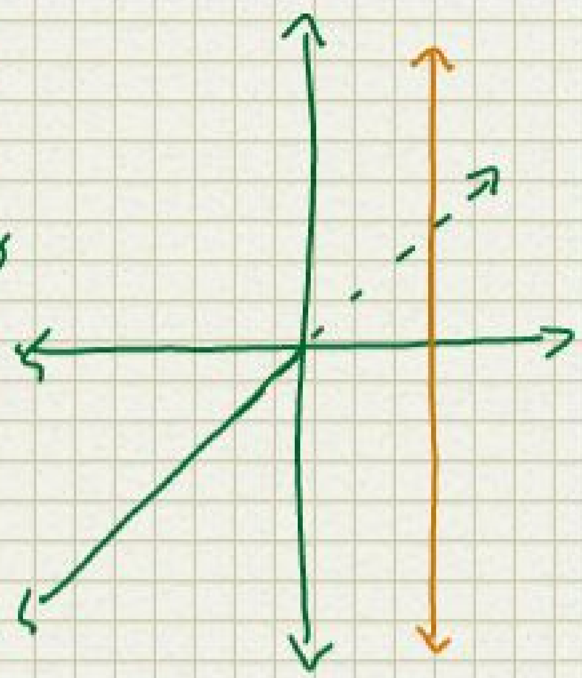
$$\begin{cases} x = x_p \\ \dfrac{y - y_p}{V_2} = \dfrac{z - z_p}{V_3} \end{cases}$$

$$x - x_p = V_1\left(\frac{y - y_p}{V_2}\right) = V_1\left(\frac{z - z_p}{V_3}\right)$$

$$V_1 = \emptyset$$

$$x - x_p = \emptyset$$

Assume $V_1 = \emptyset$
$V_2 = \emptyset$
$V_3 \neq \emptyset$

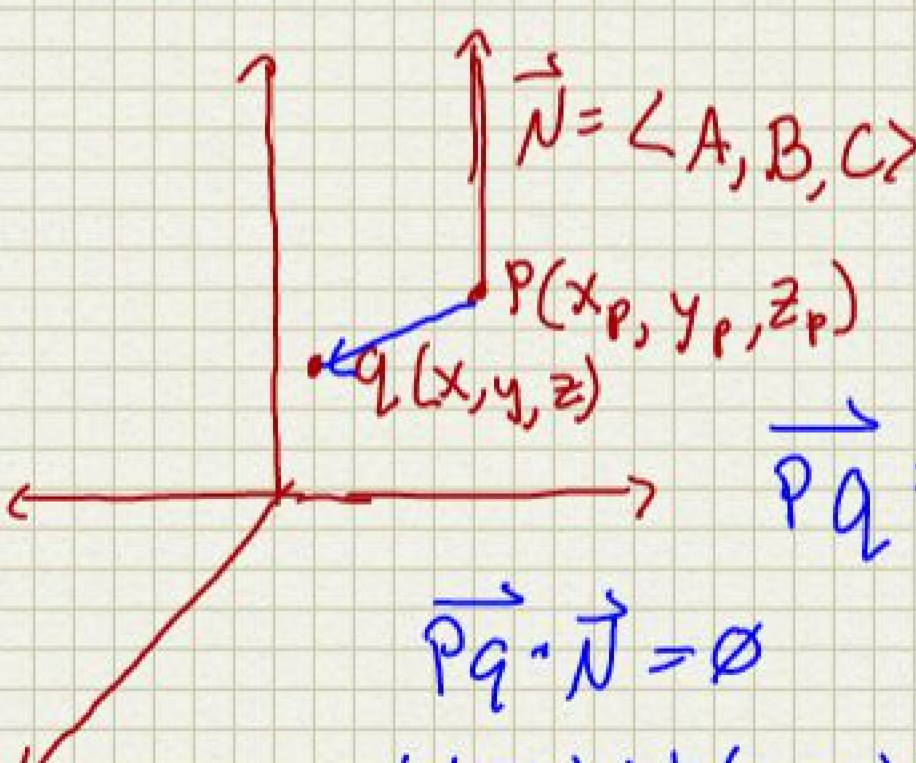


*Line is parallel to z axis.

x + y are fixed

$$\begin{cases} x = x_p \\ y = y_p \end{cases}$$ the point of intersection of x_p + y_p

9.6 Planes



$$\overrightarrow{Pq} = \langle x - x_p, y - y_p, z - z_p \rangle$$

$$\overrightarrow{Pq} \cdot \vec{N} = \emptyset$$

$$(x - x_p)A + (y - y_p)B + (z - z_p)C = \emptyset$$

$$Ax + By + Cz = \underbrace{Ax_p + By_p + Cz_p}_{\emptyset}$$

$P(2,1,3) \quad \perp \vec{N} = \langle -1, 4, 5 \rangle$

$\quad x_p \; y_p \; z_p \qquad\qquad\qquad A \; B \; C$

$(-1)x + 4y + 5C = -1(2) + 4(1) + 5(3)$

$\underline{-x + 4y + 5C = 17}$; Linear Equation in x, y, z

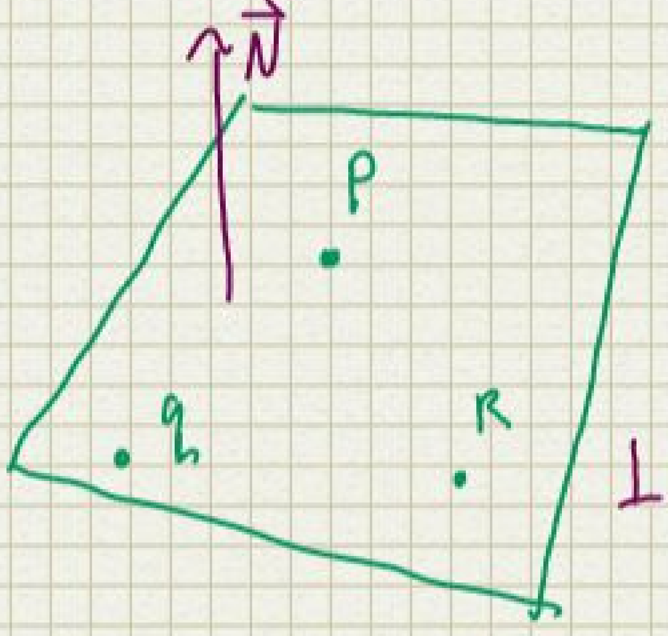
- An equation in 3D w/ x, y & z components is the equation of a plane

Example

$P(-1,2,1)$

$q(0,3,2)$

$R(1,1,-4)$



$Ax + By + Cz = D$

$\perp \vec{N}$ to plane

$\vec{Pq} \times \vec{PR} = \vec{N}$

$\vec{Pq} = \langle 1, -5, 1 \rangle$

$\vec{PR} = \langle 2, -1, -5 \rangle$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -5 & 1 \\ 2 & -1 & -5 \end{vmatrix} = \langle 26, 7, 9 \rangle$$

$26x + 7y + 9z = 26(-1) + 7(2) + 9(1) = -3$