4/15/2014

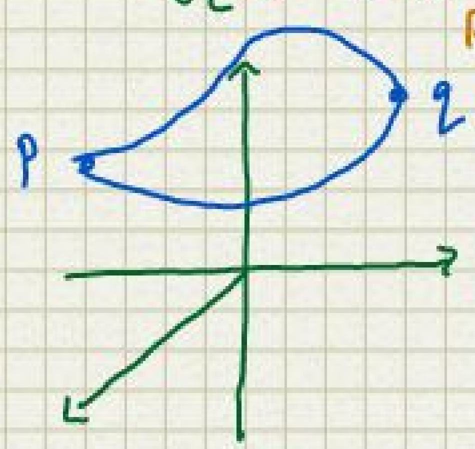
$\vec{F} = \nabla f(x,y,z)$

$\vec{F} = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$\vec{F}$ is conservative and we call f the scalar potential of $\vec{F}$

- Theorm for a conservative vector field $\vec{F}$

$\int_C \vec{F} \cdot d\vec{R} = f(q) - f(p)$



$\Updownarrow$

$\int_C \vec{F} \cdot d\vec{R}$ is path independent $= \int_{C_i} \vec{F} \cdot d\vec{R}$

$\Updownarrow$

$\oint_C \vec{F} \cdot d\vec{R} = 0$

2D

$\vec{F} = \langle f_1(x,y), f_2(x,y) \rangle$

Test?

$f_{1y} = f_{2x} \Longrightarrow \vec{F}$ is conservative

$f_x = f_1 \Longrightarrow f = \int f_1\, dx + h(y)$

$f_y = f_2 \qquad f = \int f_2\, dy + \ell(x)$

$f = \int f_1\, dx + \ldots$

$\cup$

$f = \int f_2\, dx + \ldots$

3D

$\text{curl}\,\vec{F} = \vec{0}$
$\nabla \times \vec{F} = \vec{0}$ $\Longrightarrow$ $\vec{F}$ is conservative

$f = \int f_1\,dx + \ldots$

$\cup$

$f = \int f_2\,dy + \ldots$

$\cup$

$f = \int f_3\,dz + \ldots$

$$\int_c f_1\,dx + f_2\,dy + f_3\,dz$$

$\vec{F}$ $\qquad$ $d\vec{R}$

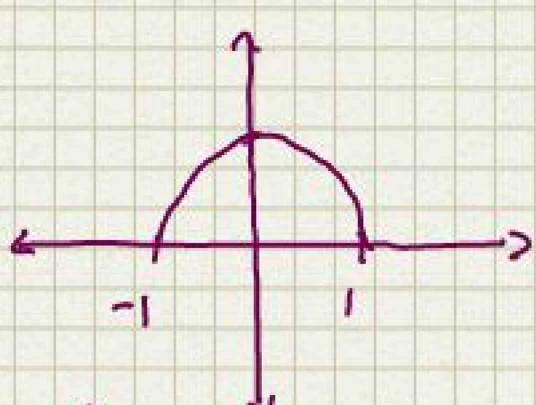
$$= \int_c \langle f_1, f_2, f_3 \rangle \cdot \langle dx, dy, dz \rangle$$

$$= \int_c \vec{F} \cdot d\vec{R}$$

Ex

$$\int_c (3x+2y)\,dx + (2x+3y)\,dy$$

conservative?



$= \int_c \vec{F} \cdot d\vec{R}$ $\quad \vec{F} = \langle \underbrace{3x+2y}_{f_1}, \underbrace{2x+3y}_{f_2} \rangle$

$f_{1y} = 2 \quad f_{2x} = 2 \therefore$ conservative

$f = \int (3x+2y)\,dx + \ldots$

$f = \int (2x+3y)\,dy + \ldots$

$= \frac{3}{2}x^2 + 2xy + \ldots$ $\qquad f = \frac{3}{2}x^2 + 2xy + \frac{3}{2}y^2$

$= 2xy + \frac{3}{2}y^2 + \ldots$

$$\int_c \vec{F} \cdot d\vec{R} = f(-1,0) - f(1,0) = \frac{3}{2}(-1)^2 - \frac{3}{2} = 0$$

$\vec{F}(x,y,z) = \langle 3x^2y^2z,\ 2x^3yz,\ x^3y^2 - e^{-z} \rangle$

$\int_C \vec{F} \cdot d\vec{R} = ?$ $\quad A(1,0,-1)$ $\quad B(0,-1,1)$

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2y^2z & 2x^3yz & x^3y^2 - e^{-z} \end{vmatrix} = \hat{\imath}(2x^3y - 2x^3y) - \hat{\jmath}(3x^2y^2 - 3x^2y^2) + \hat{k}(6x^2yz - 6x^2yz) = \vec{0}$$

$f = \int 3x^2y^2z\,dx = x^3y^2z$

$f = \int 2x^3yz\,dy = x^3y^2z$ $\qquad f = x^3y^2z + e^{-z}$

$f = \int x^3y^2 - e^{-z}\,dz = x^3y^2z + e^{-z}$

$\int_C \vec{F} \cdot d\vec{R} = -(e^2 + 0) + e^{-z} = -e^1 + e^{-1}$

13.4 Green's Theorm

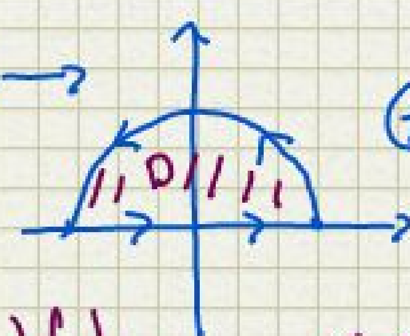
- def Positively oriented Jordan Curve
  - Jordan curve is a closed curve that doesn't self intersect

- Let C be a positive oriented Jordan Curve.

Let $\vec{F}(x,y)$ be def inside C

Then: $\oint_C \vec{F} \cdot d\vec{R} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$

Ex. $\oint_C (-y\,dx + x\,dy)$ $C \longrightarrow$



(+) Jordan Curve

$$\oint_C \vec{F}\cdot d\vec{R} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$

$C_1 : \langle \cos(\theta), \sin(\theta) \rangle \quad 0 \le \theta \le \pi$

$C_2 : \langle t, 0 \rangle \quad -1 \le t \le 1$

$$\int_{C_1} \vec{F}\cdot d\vec{R}_1 = \int_0^{\pi} \vec{F}(\vec{R},\theta)\cdot \vec{R}_1'(\theta)\,d\theta$$

$\vec{R}_1'(t) = \langle -\sin(\theta), \cos(\theta) \rangle$

$\vec{R}_2'(t) = \langle 1, 0 \rangle$

$$= \int_0^{\pi} \langle -\sin(\theta), \cos(\theta)\rangle \cdot \langle -\sin(\theta), \cos(\theta) \rangle\, d\theta$$

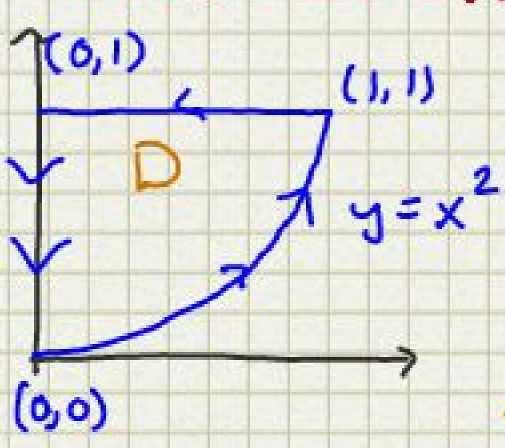
$$= \int_0^{\pi} (\sin^2(\theta) + \cos^2(\theta))\,d\theta = \pi$$

$$\int_{C_2} \vec{F}\cdot d\vec{R}_2 = \int_{-1}^{1} \langle t, 0 \rangle \cdot \langle 1, 0 \rangle\, dt = 0$$

$$= \iint_D 1 - (-1)\,dA = 2\iint_D dA = \pi$$

Ex. Find work done by force $\vec{F}$

$\vec{F}(x,y) = \langle x + xy^2, 2(x^2y - y^2\sin(y)) \rangle$



$$\oint_C \vec{F}\cdot d\vec{R} = \iint_D \frac{\partial}{\partial x}(2x^2y - 2y^2\sin(y)) - \frac{\partial}{\partial y}(x + xy^2)\,dA$$

$$= \iint_D (4xy - (0) - 2xy)\,dA = \iint_D 2xy\,dA$$

$$= \int_0^1 \int_{x^2}^1 2xy\,dy\,dx = 2\int_0^1 \frac{x}{2}(y^2)\Big|_{x^2}^{1} = 2\int_0^1 \frac{x}{2}(1 - x^4)\,dx$$

$$= 2\left(\frac{x^2}{4} - \frac{x^6}{12}\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{12} = \frac{1}{3}$$